ON A GENERALIZATION OF THE RENEWAL THEORY

LAJOS TAKÁCS

Introduction

The renewal theory, as it is well known, deals with the following problem: Suppose that a machine is working *continuously*. Let us consider a certain piece of equipment of the machine. This piece was placed in the machine at the instant $t = 0$. As soon as the piece fails, it is replaced by a new piece of the same kind. Proceeding similarly if the n-th piece fails it will be replaced by an $n + 1$-st piece ($n = 1, 2, \ldots$). Denote by $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ the life spans of the 1-st, 2-nd, ..., n-th, ... piece respectively. It is supposed that $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ are independent identically distributed non-negative random variables with $\mathbf{P}\{\zeta_n \leqq x\} = K(x)$. Denote by ν_t the number of replacements in the time interval $(0, t]$. The renewal theory deals with the investigation of the behaviour of the random variable ν_t.

At present we shall deal with a more general problem. Suppose that the machine is working *intermittently*. Assume that the machine does not work at the instant $t = 0$. Denote by $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ the lengths of the successive idle periods and working periods respectively. We suppose that they are non-negative independent random variables with $\mathbf{P}\{\xi_n < x\} = G(x)$ and $\mathbf{P}\{\eta_n \leqq x\} = H(x)$ $(n=1, 2, \ldots)$, further that these variables are independent of the variables ζ_n too. Denote by $\beta(t)$ the measure of the set consisting of those points of the interval $(0, t]$ for which the machine is working. As the number of replacements in the time interval $(0, t]$ depends only on $\beta(t)$ we shall denote it by $\nu_{\beta(t)}$. In the following we shall determine the asymptotic distribution of $\nu_{\beta(t)}$ as $t \to \infty$, under different assumptions.

We remark that the necessity of the determination of the asymptotic distribution of such a random variable as $\nu_{\beta(t)}$ arose for instance in connection with the storage problem treated by I. PALÁSTI, A. RÉNYI, T. SZENTMÁRTONY and the author [4] and by M. ZIERMANN [8].

1. §. The distribution of the random variable $\nu_{\beta(t)}$

Denote by $G_n(x)$, $H_n(x)$ and $K_n(x)$ respectively the n-fold convolution of $G(x)$, $H(x)$ and $K(x)$ with itself and put $G_0(x) \equiv H_0(x) \equiv K_0(x) = 1$ if $x \geqq 0$ and $= 0$ if $x < 0$. It is easy to see that

$$\mathbf{P}\{\nu_{\beta(t)} < n \mid \beta(t) = x\} = 1 - K_n(x) .$$

We have shown in [5] that

$$\mathsf{P}\{\beta(t) \leq x\} = \Omega(t, x) = \sum_{n=0}^{\infty} H_n(x)\,[G_n(t-x) - G_{n+1}(t-x)] \ .$$

By the total probability theorem we have

$$\mathsf{P}\{\nu_{\beta(t)} < n\} = 1 - \int_0^t K_n(x)\, d_x\, \Omega(t, x) \ .$$

2. §. The asymptotic distribution of $\nu_{\beta(t)}$

It is plausible that $\nu_{\beta(t)}$ has an asymptotic distribution if and only if $\xi_1 + \xi_2 + \ldots + \xi_n$, $\eta_1 + \eta_2 + \ldots + \eta_n$ and $\zeta_1 + \zeta_2 + \ldots + \zeta_n$ have asymptotic distributions as $n \to \infty$. A necessary and sufficient condition for the existence of these asymptotic distributions was given by W. Doeblin [2] (cf. W. Feller [3]). Considering Doeblin's conditions and simplifying the situation we shall assume that the distribution functions $G(x)$, $H(x)$ and $K(x)$ satisfy one of the assumptions $(g_1), (g_2), (g_3)$; $(h_1), (h_2), (h_3)$ and $(k_1), (k_2), (k_3)$, respectively which will be defined below. In this case there exists always a limiting distribution.

Introduce the following notations

$$\alpha = \int_0^{\infty} x\, d\, G(x) \ , \quad \sigma_\alpha^2 = \int_0^{\infty} (x-\alpha)^2\, d\, G(x) \ ,$$

$$\beta = \int_0^{\infty} x\, dH(x) \ , \quad \sigma_\beta^2 = \int_0^{\infty} (x-\beta)^2\, dH(x) \ ,$$

and

$$\tau = \int_0^{\infty} x\, dK(x) \ , \quad \sigma_\tau^2 = \int_0^{\infty} (x-\tau)^2\, dK(x) \ .$$

Further denote by $F_\gamma(x)$ $(0 < \gamma < 2, \gamma \neq 1)$ the stable distribution function whose characteristic function is given by

$$\varphi_\gamma(z) = \exp\left\{- |z|^\gamma \left(\cos\frac{\pi\gamma}{2} - i \sin\frac{\pi\gamma}{2}\, \mathrm{sign}\, z\right) \Gamma(1-\gamma)\right\}$$

and put

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du \ .$$

Finally let A, B, C be arbitrary finite positive numbers.

Assumptions for $G(x)$:

(g_1) $\sigma_\alpha < \infty$.

(g_2) $\lim_{x\to\infty} [1 - G(x)]\, x^{\gamma_1} = A$ where $1 < \gamma_1 < 2$.

(g_3) $\lim_{x\to\infty} [1 - G(x)]\, x^{\gamma_1} = A$ where $0 < \gamma_1 < 1$.

Assumptions for $H(x)$:

(h_1) $\sigma_\beta < \infty$.

(h_2) $\lim_{x\to\infty} [1 - H(x)]\, x^{\gamma_2} = B$ where $1 < \gamma_2 < 2$.

(h_3) $\lim_{x\to\infty} [1 - H(x)]\, x^{\gamma_2} = B$ where $0 < \gamma_2 < 1$.

Assumptions for $K(x)$:

(k_1) $\sigma_\tau < \infty$.

(k_2) $\lim_{x\to\infty} [1 - K(x)]\, x^{\gamma_3} = C$ where $1 < \gamma_3 < 2$.

(k_3) $\lim_{x\to\infty} [1 - K(x)]\, x^{\gamma_3} = C$ where $0 < \gamma_3 < 1$.

The determination of the asymptotic distribution of ν_t for fixed t can be reduced to that of $\zeta_1 + \zeta_2 + \ldots + \zeta_n$, namely

$$\mathbf{P}\{\nu_t < n\} = \mathbf{P}\{\zeta_1 + \zeta_2 + \ldots + \zeta_n > t\} .$$

Applying a method given by W. FELLER [3] we obtain the following asymptotic distributions for ν_t.

In case (k_1)

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_t - \frac{t}{\tau}}{\left(\frac{\sigma_\tau^2 t}{\tau^3}\right)^{1/2}} \leq x\right\} = \Phi(x) ,$$

in case (k_2)

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_t - \frac{t}{\tau}}{\left(\frac{Ct}{\tau^{1+\gamma_3}}\right)^{1/\gamma_3}} \leq x\right\} = 1 - F_{\gamma_3}(-x) ,$$

and in case (k_3)

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_t C}{t^{\gamma_3}} \leq x\right\} = 1 - F_{\gamma_3}(x^{-\frac{1}{\gamma_3}}) .$$

In our papers [6] and [7] we have proved the asymptotic distributions of $\beta(t)$ given in Table I., where ξ and η are independent random variables with distribution functions $\mathbf{P}\{\xi \leq x\} = F_{\gamma_1}(x)$ and $\mathbf{P}\{\eta \leq x\} = F_{\gamma_2}(x)$.

Table I.

	$H(x)$	$G(x)$	γ	Ω_t	$\lim\limits_{t\to\infty} \Omega_t$	x
1.	h_1	g_1		$\Omega\left(t, \frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{1/2}\right)$	$\Phi\left(\frac{(\alpha+\beta)\,x}{\sqrt{\beta^2\sigma_\alpha^2 + \alpha^2\sigma_\beta^2}}\right)$	$(-\infty, \infty)$
2.	h_1	g_2		$\Omega\left(t, \frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{1/\gamma_1}\right)$	$1 - F_{\gamma_1}\left(\frac{-(\alpha+\beta)\,x}{\beta A^{1/\gamma_1}}\right)$	$(-\infty, \infty)$
3.	h_1	g_3		$\Omega(t, x\,t^{\gamma_1})$	$1 - F_{\gamma_1}\left(\left(\frac{\beta}{A x}\right)^{\frac{1}{\gamma_1}}\right)$	$(0, \infty)$
4.	h_2	g_1		$\Omega\left(t, \frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}\right)$	$F_{\gamma_2}\left(\frac{(\alpha+\beta)\,x}{\alpha B^{1/\gamma_2}}\right)$	$(-\infty, \infty)$
5.	h_2	g_2	$\gamma_2 > \gamma_1$	$\Omega\left(t, \frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma_1}}\right)$	$1 - F_{\gamma_1}\left(\frac{-(\alpha+\beta)\,x}{\beta A^{1/\gamma_1}}\right)$	$(-\infty, \infty)$
6.	h_2	g_2	$\gamma_i = \gamma$ $(i = 1, 2)$	$\Omega\left(t, \frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma}}\right)$	$\mathbf{P}\left\{\frac{\alpha B^{1/\gamma}\eta - \beta A^{1/\gamma}\xi}{\alpha+\beta} \leqq x\right\}$	$(-\infty, \infty)$
7.	h_2	g_2	$\gamma_2 < \gamma_1$	$\Omega\left(t, \frac{\beta t}{\alpha+\beta} + x\left(\frac{t}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}\right)$	$F_{\gamma_2}\left(\frac{(\alpha+\beta)\,x}{\alpha B^{1/\gamma_2}}\right)$	$(-\infty, \infty)$
8.	h_2	g_3		$\Omega(t, x\,t^{\gamma_1})$	$1 - F_{\gamma_1}\left(\left(\frac{\beta}{A x}\right)^{\frac{1}{\gamma_1}}\right)$	$(0, \infty)$
9.	h_3	g_1		$\Omega(t, t + x\,t^{\gamma_2})$	$F_{\gamma_2}\left(\left(\frac{-\alpha}{B x}\right)^{\frac{1}{\gamma_2}}\right)$	$(-\infty, 0)$
10.	h_3	g_2		$\Omega(t, t + x\,t^{\gamma_2})$	$F_{\gamma_2}\left(\left(\frac{-\alpha}{B x}\right)^{\frac{1}{\gamma_2}}\right)$	$(-\infty, 0)$
11.	h_3	g_3	$\gamma_2 > \gamma_1$	$\Omega\left(t, x\,t^{\frac{\gamma_1}{\gamma_2}}\right)$	$P\left\{\frac{\eta^{\gamma_2}}{\xi^{\gamma_1}} \leqq \frac{A\,x^{\gamma_2}}{B}\right\}$	$(0, \infty)$
12.	h_3	g_3	$\gamma_i = \gamma$ $(i = 1, 2)$	$\Omega(t, x\,t)$	$\mathbf{P}\left\{\frac{\eta}{\xi} \leqq \left(\frac{A}{B}\right)^{\frac{1}{\gamma}} \frac{x}{1-x}\right\}$	$(0, 1)$
13.	h_3	g_3	$\gamma_2 < \gamma_1$	$\Omega\left(t, t + x\,t^{\frac{\gamma_2}{\gamma_1}}\right)$	$\mathbf{P}\left\{\frac{\eta^{\gamma_2}}{\xi^{\gamma_1}} \leqq \frac{A}{B(-x)^{\gamma_1}}\right\}$	$(-\infty, 0)$

If $K(x)$ satisfies one of the assumptions (k_1), (k_2), (k_3) and the asymptotic distribution of $\beta(t)$ agrees with one of the above expressions 1—13., then taking into consideration that the random variables $\{\nu_t\}$ are independent of the random variables $\{\beta(t)\}$, we can apply a theorem of R. L. DOBRUSHIN [1] to prove the existence of the following limiting distribution

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_{\beta(t)} - Tt^{\lambda}}{St^{\mu}} \leq x\right\} = \Psi(x)$$

with appropriate distribution function $\Psi(x)$ and appropriate parameters T, λ, S, μ. The unknowns $\Psi(x)$, T, λ, S and μ may be determined by the aid of DOBRUSHIN's theorem. So we obtain 59 different cases and the corresponding theorems are stated in Table II. where ξ_0, ξ, η and ζ are independent random variables with distribution functions $\mathbf{P}\{\xi_0 \leq x\} = \Phi(x)$, $\mathbf{P}\{\xi \leq x\} = F_{\gamma_1}(x)$, $\mathbf{P}\{\eta \leq x\} = F_{\gamma_2}(x)$ and $\mathbf{P}\{\zeta \leq x\} = F_{\gamma_3}(x)$.

Table II. (cf. pp. 96—101.)

In particular we have by Theorem 1. the following important case

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_{\beta(t)} - \dfrac{\beta t}{(\alpha+\beta)\tau}}{\sqrt{\left(\dfrac{\beta^2\sigma_a^2}{(\alpha+\beta)^3\tau^2} + \dfrac{\alpha^2\sigma_\beta^2}{(\alpha+\beta)^3\tau^2} + \dfrac{\beta\sigma_\tau^2}{(\alpha+\beta)\tau^3}\right)t}} \leq x\right\} = \Phi(x)\ ,$$

if $\sigma_a^2 + \sigma_\beta^2 + \sigma_\tau^2 < \infty$,

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Table II.

	$K(x)$	$H(x)$	$G(x)$	γ	T	λ	S	μ	$\Psi(x)$
1.	k_1	h_1	g_1		$\frac{\beta}{(\alpha+\beta)\tau}$	1	1	$\frac{1}{2}$	$\Phi\left(\frac{(\alpha+\beta)^{3/2}\tau^{3/2}x}{\sqrt{\tau\beta^2\sigma_\alpha^2+\tau\alpha^2\sigma_\beta^2+\beta(\alpha+\beta)^2\sigma_\tau^2}}\right)$
2.	k_1	h_1	g_2		$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma_1}}$	$\frac{1}{\gamma_1}$	$1-F_{\gamma_1}\left(\frac{-(\alpha+\beta)x}{\beta A^{1/\gamma_1}}\right)$
3.	k_1	h_1	g_3		0	—	$\frac{1}{\tau}$	γ_1	$1-F_{\gamma_1}\left(\left(\frac{\beta}{Ax}\right)^{\frac{1}{\gamma_1}}\right)$
4.	k_1	h_2	g_1		$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}$	$\frac{1}{\gamma_2}$	$F_{\gamma_2}\left(\frac{(\alpha+\beta)x}{\alpha B^{1/\gamma_2}}\right)$
5.	k_1	h_2	g_2	$\gamma_2>\gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma_1}}$	$\frac{1}{\gamma_1}$	$1-F_{\gamma_1}\left(\frac{-(\alpha+\beta)x}{\beta A^{1/\gamma_1}}\right)$
6.	k_1	h_2	g_2	$\gamma_i=\gamma$ $(i=1,2)$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma}}$	$\frac{1}{\gamma}$	$\mathbf{P}\left\{\frac{\alpha B^{\frac{1}{\gamma}}\eta-\beta A^{\frac{1}{\gamma}}\xi}{(\alpha+\beta)}\leqq x\right\}$
7.	k_1	h_2	g_2	$\gamma_2<\gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}$	$\frac{1}{\gamma_2}$	$F_{\gamma_2}\left(\frac{(\alpha+\beta)x}{\alpha B^{1/\gamma_2}}\right)$
8.	k_1	h_2	g_3		0	—	$\frac{1}{\tau}$	γ_1	$1-F_{\gamma_1}\left(\left(\frac{\beta}{Ax}\right)^{\frac{1}{\gamma_1}}\right)$
9.	k_1	h_3	g_1	$\gamma_2>\frac{1}{2}$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}$	γ_2	$F_{\gamma_2}\left(\left(\frac{-\alpha}{Bx}\right)^{1/\gamma_2}\right)$

10.	k_1	h_3	g_1	$\gamma_2 = \frac{1}{2}$	$\frac{1}{\tau}$	1	1	$\frac{1}{2}$	$\mathbf{P}\left\{\left(\frac{\sigma_\tau^2}{\tau^3}\right)^{\frac{1}{2}} \xi_0 - \frac{\alpha\eta^{-\frac{1}{2}}}{B\tau} \leqq x\right\}$
11.	k_1	h_3	g_1	$\gamma_2 < \frac{1}{2}$	$\frac{1}{\tau}$	1	$\left(\frac{\sigma_\tau^2}{\tau^3}\right)^{\frac{1}{2}}$	$\frac{1}{2}$	$\Phi(x)$
12.	k_1	h_3	g_2	$\gamma_2 > \frac{1}{2}$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}$	γ_2	$F_{\gamma_2}\left(\left(\frac{-\alpha}{Bx}\right)^{1/\gamma_2}\right)$
13.	k_1	h_3	g_2	$\gamma_2 = \frac{1}{2}$	$\frac{1}{\tau}$	1	1	$\frac{1}{2}$	$\mathbf{P}\left\{\left(\frac{\sigma_\tau^2}{\tau^3}\right)^{\frac{1}{2}} \xi_0 - \frac{\alpha\eta^{-\frac{1}{2}}}{B\tau} \leqq x\right\}$
14.	k_1	h_3	g_2	$\gamma_2 < \frac{1}{2}$	$\frac{1}{\tau}$	1	$\left(\frac{\sigma_\tau^2}{\tau^3}\right)^{\frac{1}{2}}$	$\frac{1}{2}$	$\Phi(x)$
15,	k_1	h_3	g_3	$\gamma_2 > \gamma_1$	0	—	$\frac{1}{\tau}$	$\frac{\gamma_1}{\gamma_2}$	$\mathbf{P}\left\{\frac{\eta^{\gamma_2}}{\xi^{\gamma_1}} \leqq \frac{Ax^{\gamma_2}}{B}\right\}$
16.	k_1	h_3	g_3	$\gamma_i = \gamma$ $(i = 1, 2)$	0	—	$\frac{1}{\tau}$	1	$\mathbf{P}\left\{\frac{\eta}{\xi} \leqq \left(\frac{A}{B}\right)^{\frac{1}{\gamma}} \frac{x}{1-x}\right\}$
17.	k_1	h_3	g_3	$\gamma_1 > \gamma_2 > \frac{\gamma_1}{2}$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}$	$\frac{\gamma_2}{\gamma_1}$	$\mathbf{P}\left\{\frac{\eta^{\gamma_2}}{\xi^{\gamma_1}} \leqq \frac{A}{B(-x)^{\gamma_1}}\right\}$
18.	k_1	h_3	g_3	$\gamma_2 = \frac{\gamma_1}{2}$	$\frac{1}{\tau}$	1	1	$\frac{1}{2}$	$\mathbf{P}\left\{\left(\frac{\sigma_\tau^2}{\tau^3}\right)^{1/2} \xi_0 - \frac{1}{\tau}\left(\frac{A}{B}\right)^{\frac{1}{\gamma_1}} \frac{\xi}{\eta^{1/2}} \leqq x\right\}$
19.	k_1	h_3	g_3	$\gamma_2 < \frac{\gamma_1}{2}$	$\frac{1}{\tau}$	1	$\left(\frac{\sigma_\tau^2}{\tau^3}\right)^{\frac{1}{2}}$	$\frac{1}{2}$	$\Phi(x)$

Table II. (continued)

	$K(x)$	$H(x)$	$G(x)$	γ	T	λ	S	μ	$\Psi(x)$
20.	k_2	h_1	g_1		$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{\beta C}{(\alpha+\beta)\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
21.	k_2	h_1	g_2	$\gamma_3 > \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma_1}}$	$\frac{1}{\gamma_1}$	$1 - F_{\gamma_1}\left(\frac{-(\alpha+\beta)x}{\beta A^{1/\gamma_1}}\right)$
22.	k_2	h_1	g_2	$\gamma_3 = \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	1	$\frac{1}{\gamma_1}$	$\mathbf{P}\left\{\frac{\beta}{(\alpha+\beta)\tau}\left(\frac{A}{\alpha+\beta}\right)^{1/\gamma_1}\xi + \frac{1}{\tau}\left(\frac{C\beta}{(\alpha+\beta)\tau}\right)^{1/\gamma_1}\zeta \geqq -x\right\}$
23.	k_2	h_1	g_2	$\gamma_3 < \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{\beta C}{(\alpha+\beta)\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
24.	k_2	h_1	g_3		0	—	$\frac{1}{\tau}$	γ_1	$1 - F_{\gamma_1}\left(\left(\frac{\beta}{Ax}\right)^{\frac{1}{\gamma_1}}\right)$
25.	k_2	h_2	g_1	$\gamma_3 > \gamma_2$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}$	$\frac{1}{\gamma_2}$	$F_{\gamma_2}\left(\frac{(\alpha+\beta)x}{\alpha B^{1/\gamma_2}}\right)$
26.	k_2	h_2	g_1	$\gamma_3 = \gamma_2$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	1	$\frac{1}{\gamma_2}$	$\mathbf{P}\left\{\frac{\alpha}{(\alpha+\beta)\tau}\left(\frac{B}{\alpha+\beta}\right)^{1/\gamma_2}\eta - \frac{1}{\tau}\left(\frac{C\beta}{(\alpha+\beta)\tau}\right)^{1/\gamma_2}\zeta \leqq x\right\}$
27.	k_2	h_2	g_1	$\gamma_3 < \gamma_2$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{\beta C}{(\alpha+\beta)\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
28.	k_2	h_2	g_2	$\gamma_3 > \gamma_1$ $\gamma_2 > \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{(\alpha+\beta)}\right)^{\frac{1}{\gamma_1}}$	$\frac{1}{\gamma_1}$	$1 - F_{\gamma_1}\left(\frac{-(\alpha+\beta)x}{\beta A^{1/\gamma_1}}\right)$

29.	k_2	h_2	g_2	$\gamma_3 = \gamma_1$ $\gamma_2 > \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	1	$\frac{1}{\gamma_1}$	$\mathbf{P}\left\{\frac{\beta}{(\alpha+\beta)\tau}\left(\frac{A}{\alpha+\beta}\right)^{1/\gamma_1}\xi + \frac{1}{\tau}\left(\frac{C\beta}{(\alpha+\beta)\tau}\right)^{1/\gamma_1}\zeta \geqq -x\right\}$
30.	k_2	h_2	g_2	$\gamma_3 < \gamma_1$ $\gamma_2 > \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{\beta C}{(\alpha+\beta)\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
31.	k_2	h_2	g_2	$\gamma_3 > \gamma$ $\gamma i = \gamma$ $(i = 1, 2)$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma}}$	$\frac{1}{\gamma}$	$\mathbf{P}\left\{\frac{\alpha B^{1/\gamma}\eta - \beta A^{1/\gamma}\xi}{\alpha+\beta} \leqq x\right\}$
32.	k_2	h_2	g_2	$\gamma i = \gamma$ $(i = 1, 2, 3)$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	1	$\frac{1}{\gamma}$	$\mathbf{P}\left\{\frac{\alpha}{(\alpha+\beta)\tau}\left(\frac{B}{\alpha+\beta}\right)^{1/\gamma}\eta - \frac{\beta}{(\alpha+\beta)\tau}\left(\frac{A}{\alpha+\beta}\right)^{1/\gamma}\xi - \frac{1}{\tau}\left(\frac{\beta C}{(\alpha+\beta)\tau}\right)^{1/\gamma}\zeta \leqq x\right\}$
33.	k_2	h_2	g_2	$\gamma_3 < \gamma$ $\gamma i = \gamma$ $(i = 1, 2)$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{\beta C}{(\alpha+\beta)\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
34.	k_2	h_2	g_2	$\gamma_3 > \gamma_2$ $\gamma_2 < \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{1}{\alpha+\beta}\right)^{\frac{1}{\gamma_2}}$	$\frac{1}{\gamma_2}$	$F_{\gamma_2}\left(\frac{(\alpha+\beta)x}{\alpha B^{1/\gamma_2}}\right)$
35.	k_2	h_2	g_2	$\gamma_3 = \gamma_2$ $\gamma_2 < \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	1	$\frac{1}{\gamma_2}$	$\mathbf{P}\left\{\frac{1}{\tau}\left(\frac{B}{\alpha+\beta}\right)^{1/\gamma_2}\frac{\alpha}{\alpha+\beta}\eta - \frac{1}{\tau}\left(\frac{C\beta}{(\alpha+\beta)\tau}\right)^{1/\gamma_2}\zeta \leqq x\right\}$
36.	k_2	h_2	g_2	$\gamma_3 < \gamma_2$ $\gamma_2 < \gamma_1$	$\frac{\beta}{(\alpha+\beta)\tau}$	1	$\frac{1}{\tau}\left(\frac{\beta C}{(\alpha+\beta)\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
37.	k_2	h_2	g_3		0	—	$\frac{1}{\tau}$	γ_1	$1 - F_{\gamma_1}\left(\left(\frac{\beta}{Ax}\right)^{\frac{1}{\gamma_1}}\right)$
38.	k_2	h_3	g_1	$\gamma_3\gamma_3 > 1$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}$	γ_2	$F_{\gamma_2}\left(\left(\frac{-\alpha}{Bx}\right)^{1/\gamma_2}\right)$

Table II. (continued)

	$K(x)$	$H(x)$	$G(x)$	γ	T	λ	S	μ	$\Psi(x)$
39.	k_2	h_3	g_1	$\gamma_2\gamma_3 = 1$	$\frac{1}{\tau}$	1	1	$\frac{1}{\gamma_3}$	$\mathbf{P}\left\{\frac{\alpha\eta^{-\gamma_2}}{\tau B} + \frac{1}{\tau}\left(\frac{C}{\tau}\right)^{1/\gamma_3}\zeta \geqq -x\right\}$
40.	k_2	h_3	g_1	$\gamma_2\gamma_3 < 1$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}\left(\frac{C}{\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
41.	k_2	h_3	g_2	$\gamma_2\gamma_3 > 1$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}$	γ_2	$F_{\gamma_2}\left(\left(\frac{-\alpha}{Bx}\right)^{1/\gamma_2}\right)$
42.	k_2	h_3	g_2	$\gamma_2\gamma_3 = 1$	$\frac{1}{\tau}$	1	1	$\frac{1}{\gamma_3}$	$\mathbf{P}\left\{\frac{\alpha\eta^{-\gamma_2}}{B\tau} + \frac{1}{\tau}\left(\frac{C}{\tau}\right)^{1/\gamma_3}\zeta \geqq -x\right\}$
43.	k_2	h_3	g_2	$\gamma_2\gamma_3 < 1$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}\left(\frac{C}{\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$
44.	k_2	h_3	g_3	$\gamma_2 > \gamma_1$	0	—	$\frac{1}{\tau}$	$\frac{\gamma_1}{\gamma_2}$	$\mathbf{P}\left\{\frac{\eta^{\gamma_2}}{\xi^{\gamma_1}} \leqq \frac{Ax^{\gamma_2}}{B}\right\}$
45.	k_2	h_3	g_3	$\gamma_i = \gamma$ $(i = 1, 2)$	0	—	$\frac{1}{\tau}$	1	$\mathbf{P}\left\{\frac{\eta}{\xi} \leqq \left(\frac{A}{B}\right)^{\frac{1}{\gamma}}\frac{x}{1-x}\right\}$
46.	k_2	h_3	g_3	$\gamma_2 < \gamma_1$ $\gamma_2\gamma_3 > \gamma_1$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}$	$\frac{\gamma_2}{\gamma_1}$	$\mathbf{P}\left\{\frac{\eta^{\gamma_2}}{\xi^{\gamma_1}} \leqq \frac{A}{B(-x)^{\gamma_1}}\right\}$
47.	k_2	h_3	g_3	$\gamma_2 < \gamma_1$ $\gamma_2\gamma_3 = \gamma_1$	$\frac{1}{\tau}$	1	1	$\frac{1}{\gamma_3}$	$\mathbf{P}\left\{\frac{1}{\tau}\left(\frac{A}{B}\right)^{\frac{1}{\gamma_1}}\frac{\xi}{\eta^{\gamma_2/\gamma_1}} + \frac{1}{\tau}\left(\frac{C}{\tau}\right)^{\frac{1}{\gamma_3}}\zeta \geqq -x\right\}$
48.	k_2	h_3	g_3	$\gamma_2 < \gamma_1$ $\gamma_2\gamma_3 < \gamma_1$	$\frac{1}{\tau}$	1	$\frac{1}{\tau}\left(\frac{C}{\tau}\right)^{\frac{1}{\gamma_3}}$	$\frac{1}{\gamma_3}$	$1 - F_{\gamma_3}(-x)$

49.	k_3	h_1	g_1		0	—	$\frac{1}{C}\left(\frac{\beta}{\alpha+\beta}\right)^{\gamma_3}$	γ_3	$1 - F_{\gamma_3}(x^{-1/\gamma_3})$
50.	k_3	h_1	g_2		0	—	$\frac{1}{C}\left(\frac{\beta}{\alpha+\beta}\right)^{\gamma_3}$	γ_3	$1 - F_{\gamma_3}(x^{-1/\gamma_3})$
51.	k_3	h_1	g_3		0	—	$\frac{1}{C}$	$\gamma_3\gamma_1$	$\mathbf{P}\left\{\left(\frac{\beta}{A\xi^{\gamma_1}\zeta}\right)^{\gamma_3} \leqq x\right\}$
52.	k_3	h_2	g_1		0	—	$\frac{1}{C}\left(\frac{\beta}{\alpha+\beta}\right)^{\gamma_3}$	γ_3	$1 - F_{\gamma_3}(x^{-1/\gamma_3})$
53.	k_3	h_2	g_2		0	—	$\frac{1}{C}\left(\frac{\beta}{\alpha+\beta}\right)^{\gamma_3}$	γ_3	$1 - F_{\gamma_3}(x^{-1/\gamma_3})$
54.	k_3	h_2	g_3		0	—	$\frac{1}{C}$	$\gamma_3\gamma_1$	$\mathbf{P}\left\{\left(\frac{\beta}{A\xi^{\gamma_1}\zeta}\right)^{\gamma_3} \leqq x\right\}$
55.	k_3	h_3	g_1		0	—	$\frac{1}{C}$	γ_3	$1 - F_{\gamma_3}(x^{-1/\gamma_3})$
56.	k_3	h_3	g_2		0	—	$\frac{1}{C}$	γ_3	$1 - F_{\gamma_3}(x^{-1/\gamma_3})$
57.	k_3	h_3	g_3	$\gamma_2 > \gamma_1$	0	—	$\frac{1}{C}$	$\frac{\gamma_1\gamma_3}{\gamma_2}$	$\mathbf{P}\left\{\left(\frac{B}{A}\right)^{\frac{\gamma_3}{\gamma_2}} \frac{\eta^{\gamma_3}}{\xi^{\gamma_1\gamma_3/\gamma_2}\zeta^{\gamma_3}} \leqq x\right\}$
58.	k_3	h_3	g_3	$\gamma_i = \gamma$ $(i = 1, 2)$	0	—	$\frac{1}{C}$	γ_3	$\mathbf{P}\left\{\left(\frac{B^{1/\gamma}\eta}{A^{1/\gamma}\xi + B^{1/\gamma}\eta}\right)^{\gamma} \frac{1}{\zeta^{\gamma_3}} \leqq x\right\}$
59.	k_3	h_3	g_3	$\gamma_2 < \gamma_1$	0	—	$\frac{1}{C}$	γ_3	$1 - F_{\gamma_3}(x^{-1/\gamma_3})$

A FELÚJÍTÁS-ELMÉLET ÁLTALÁNOSÍTÁSÁRÓL

TAKÁCS L.

Kivonat

Tekintsünk egy szakaszosan működő gépet. Feltesszük, hogy a gép a $t = 0$ időpontban áll, és az egymást követő állási idők és működési idők azonos eloszlású független valószínűségi változók. Jelölje az állási idők eloszlásfüggvényét $G(x)$, a működési idők eloszlásfüggvényét pedig $H(x)$. Vizsgáljuk valamilyen gépalkatrész cseréinek a számát. Feltesszük, hogy a $t = 0$ időpontban egy új alkatrészt állítunk be. Ha az alkatrész tönkremegy, akkor azonnal hasonló új alkatrésszel helyettesítjük. Feltesszük, hogy az egyes alkatrészek élettartamai azonos eloszlású független valószínűségi változók, $K(x)$ eloszlásfüggvénnyel. Jelölje a $(0, t]$ időközben történő cserék számát $\nu_{\beta(t)}$. E dolgozatban különböző feltételek mellett meghatározzuk a $\nu_{\beta(t)}$ valószínűségi változó aszimptotikus eloszlását, midőn $t \to \infty$.

A fenti probléma folytonosan működő gép esetében a közönséges felújítás-elméletre redukálódik. A jelenleg vizsgált eset annyival általánosabb, hogy megengedjük, hogy a gép működésében véletlen szünetek forduljanak elő.

A következő alternatív feltevésekkel élünk a $G(x)$, $H(x)$ és $K(x)$ eloszlásfüggvényekre vonatkozóan:

A $G(x)$ eloszlásfüggvényre vonatkozó feltevések:

(g_1) $G(x)$ szórása véges;

(g_2) $\lim_{x\to\infty} [1 - G(x)]\, x^{\gamma_1} = A$, ahol A pozitív állandó és $1 < \gamma_1 < 2$;

(g_3) $\lim_{x\to\infty} [1 - G(x)]\, x^{\gamma_1} = A$, ahol A pozitív állandó és $0 < \gamma_1 < 1$.

A $H(x)$ eloszlásfüggvényre vonatkozó feltevések:

(h_1) $H(x)$ szórása véges;

(h_2) $\lim_{x\to\infty} [1 - H(x)]\, x^{\gamma_2} = B$, ahol B pozitív állandó és $1 < \gamma_2 < 2$;

(h_3) $\lim_{x\to\infty} [1 - H(x)]\, x^{\gamma_2} = B$, ahol B pozitív állandó és $0 < \gamma_2 < 1$.

A $K(x)$ eloszlásfüggvényre vonatkozó feltevések:

(k_1) $K(x)$ szórása véges;

(k_2) $\lim_{x\to\infty} [1 - K(x)]\, x^{\gamma_3} = C$, ahol C pozitív állandó és $1 < \gamma_3 < 2$;

(k_3) $\lim_{x\to\infty} [1 - K(x)]\, x^{\gamma_3} = C$, ahol C pozitív állandó és $0 < \gamma_3 < 1$.

Kimutatjuk, hogy a fenti feltevések mellett a $\nu_{\beta(t)}$ változónak létezik aszimptotikus eloszlása. A lehetséges határeloszlások száma 59.

A bizonyítás a szerző [6] és [7] dolgozatán, valamint R. L. DOBRUSIN [1] tételén alapszik.

ОБ ОБОБЩЕНИИ ТЕОРИИ ВОССТАНОВЛЕНИЯ

L. TAKÁCS

Резюме

Рассмотрим периодически действующий станок. Предположим, что во времени $t = 0$ станок не действует, дальнее, что времена простоя и времена действия отдельно одинаково распределённые случайные величины. Пусть $G(x)$ означает функцию распределения времён простоя и $H(x)$ функцию распределения времён действия. Будем исследовать число обменов некоторой детали станка. Предположим, что в точке времени $t = 0$ мы положили новую деталь. Если деталь станка выходит из строя, то мы сейчас заменяем с одной новой подобной деталью. Будем предполагать, что сроки службы деталей одинаково распределённые случайные величины с функцией распределения $K(x)$. Пусть $\nu_{\beta(t)}$ означает число обменов в интервале $(0, t]$. В этой работе мы определяем асимптотическое распределение случайной величины $\nu_{\beta(t)}$ в случае $t \to \infty$ при разных условиях.

Вышеупомянутая проблема в случае непрерывно действующего станка редуцируется к общей теории восстановления. Вышеупомянутая проблема более общая, потому что мы допустим, что в действии станка могут находиться паузы.

Будем использовать следующими алтернативными предположениями:

Предположения касающиеся функции распределения $G(x)$:

(g_1) дисперсия функции $G(x)$ конечна,

(g_2) $\lim\limits_{x\to\infty} [1-G(x)]\, x^{\gamma_1} = A$, где A положительное постоянное и $1<\gamma_1<2$.

(g_3) $\lim\limits_{x\to\infty} [1-G(x)]\, x^{\gamma_1} = A$, где A положительное постоянное и $0<\gamma_1<1$.

Предположения, касающиеся функции распределения $H(x)$:

(h_1) дисперсия функции $H(x)$ конечна

(h_2) $\lim\limits_{x\to\infty} [1-H(x)]\, x^{\gamma_2} = B$, где B положительное постоянное и $1 < \gamma_2 < 2$.

(h_3) $\lim\limits_{x\to\infty} [1-H(x)]\, x^{\gamma_2} = B$, где B положительное постоянное и $0 < \gamma_2 < 1$

Предположения, касающиеся функции распределения $K(x)$:

(k_1) дисперсия функции $K(x)$ конечна

(k_1) $\lim\limits_{x\to\infty} [1 - K(x)]\, x^{\gamma_3} = C$, где C положительное постоянное и $1<\gamma_3<2$.

(k_2) $\lim\limits_{x\to\infty} [1 - K(x)]\, x^{\gamma_3} = C$, где C положительное постоянное и $0<\gamma_3<1$.

Покажем, что при этих предположениях существует асимптотическое распределение случайной величины $\nu_{\beta(t)}$. Число возможных предельных распределений 59.

Доказательство основывается на работах автора [6] и [7] и на теореме Р. Л. Добрушина [1].