REMARKS ON RANDOM WALK PROBLEMS

LAJOS TAKÁCS

Introduction

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent random variables which take on the values $+1$ and -1 with probability $1/2$. Put $\eta_0 = 0$ and $\eta_n = \xi_1 + \xi_2 + \cdots + \xi_n$ $(n = 1, 2, \ldots)$. The sequence of random variables $\{\eta_n\}$ describes the motion of a free particle on a straight line. The particle starts at $x = 0$ and in each step it can move either a unit distance to the right or a unit distance to the left with probability $1/2$. The displacements are independent of each other. We say that the sequence $\{\eta_n\}$ describes an *ordinary random walk.*

Next by the aid of the above random variables ξ_n $(n = 1, 2, \ldots)$ let us define the sequence of random variables $\{\eta_n^*\}$ as follows: $\eta_0^* = 0$ and for $n = 1, 2, \ldots$

$$\eta_n^* = \begin{cases} \eta_{n-1}^* + \xi_n & \text{if } \eta_{n-1}^* \neq a \text{ or } -b \\ \eta_{n-1}^* & \text{if } \eta_{n-1}^* = a \text{ or } -b \end{cases}$$

where a and b are fixed positive integers. As it can be easily seen the sequence of random variables $\{\eta_n^*\}$ describes a *random walk with two absorbing barriers*. The particle starts at $x = 0$ and moves as above but if the particle reaches one of the points $x = a$ and $x = -b$ (absorbing barriers) its motion terminates.

In what follows we shall prove some limiting theorems concerning the above random walk problems. These theorems play an important rôle in the theory of order statistics. Though these results are not new the following proofs are very simple and the theorems are expressed in a new form.

§ 1. The ordinary random walk problem

Define $\delta_n^+ = \max(\eta_0, \eta_1, \ldots, \eta_n)$, $\delta_n^- = -\min(\eta_0, \eta_1, \ldots, \eta_n)$ and $\delta_n = \delta_n^+ + \delta_n^-$. We shall prove the following results.

Theorem 1. *If $z > 0$ and $y > 0$ then we have*

$$\lim_{n\to\infty} \mathbf{P}\{\delta_n^+ < n^{1/2} z, \delta_n^- < n^{1/2} y\} = F(z, y) \tag{1}$$

where

$$(2) \qquad F(z, y) = \sum_{k=-\infty}^{\infty} (-1)^k [\Phi(k(z+y)+z) - \Phi(k(z+y)-y)]$$

or

$$(3) \qquad F(z, y) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} e^{-\frac{(2j+1)^2\pi^2}{2(z+y)^2}} \sin \frac{(2j+1)\pi z}{2(z+y)}.$$

Here as usually

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

This theorem was proved by A. A. ANIS [1] in the form (3). The function (3) is well known as the solution of the heat conduction equation.

Theorem 2. *If* $z > 0$ *and* $y > 0$ *then we have*

$$(4) \qquad \lim_{n\to\infty} \mathbf{P}\{\delta_{2n}^{+} < (2n)^{1/2} z, \delta_{2n}^{-} < (2n)^{1/2} y \mid \eta_{2n} = 0\} = K(z, y)$$

where

$$(5) \qquad K(z, y) = \sum_{k=-\infty}^{\infty} \left(e^{-2(z+y)^2k^2} - e^{-2((z+y)k+z)^2}\right)$$

or

$$(6) \qquad K(z, y) = \frac{\sqrt{2\pi}}{z+y} \sum_{j=1}^{\infty} e^{-\frac{j^2\pi^2}{2(z+y)^2}} \sin^2 \frac{j\pi z}{z+y}.$$

This theorem in the form (5) was proved earlier by B. V. GNEDENKO [5].

Theorem 3. *If* $x > 0$ *then we have*

$$(7) \qquad \lim_{n\to\infty} \mathbf{P}\{\delta_n < n^{1/2} x\} = F^*(x)$$

where

$$(8) \qquad F^*(x) = 2 \sum_{k=0}^{\infty} (-1)^k (2k+1) [\Phi((k+1)x) - \Phi(kx)]$$

or

$$(9) \qquad F^*(x) = \frac{8}{\pi} \sum_{k=0}^{\infty} e^{-\frac{(2k+1)^2\pi^2}{2x^2}} \left[\frac{(2k+1)\pi^2}{x^3} + \frac{1}{(2k+1)^2\pi}\right].$$

This theorem was proved earlier by W. FELLER [4] and A. A. ANIS [1]. They determined the density function $F^{*\prime}(x)$, corresponding to the formulae (8) resp. (9).

Theorem 4. *If* $x > 0$ *then we have*

$$(10) \qquad \lim_{n\to\infty} \mathbf{P}\{\delta_{2n} < (2n)^{1/2} x \mid \eta_{2n} = 0\} = K^*(x)$$

where

$$(11) \qquad K^*(x) = 1 - 2 \sum_{k=1}^{\infty} (4k^2x^2 - 1) e^{-2k^2x^2}$$

or

(12) $$K^*(x)=\frac{\pi^{3/2}}{2^{1/2}x^2}\sum_{j=1}^{\infty} j^2 e^{-\frac{j^2\pi^2}{2x^2}}.$$

This theorem was proved earlier by B. V. GNEDENKO [5]. He showed that

$$\frac{dK^*(x)}{dx}=8x\sum_{k=1}^{\infty}(4k^4x^2-3k^2)e^{-2k^2x^2}.$$

The proofs of these theorems are based on the following theorem concerning the random walk with absorbing barriers.

§ 2. Random walk with absorbing barriers

Theorem 5. *If* $-b<x<a$, *then we have*

(13) $$\mathbf{P}\{\eta_n^*=x\}=\sum_{k=-\infty}^{\infty}[\mathbf{P}\{\eta_n=2(a+b)k+x\}-\mathbf{P}\{\eta_n=2(a+b)k+2a-x\}]$$

or

(14) $$\mathbf{P}\{\eta_n^*=x\}=\frac{2}{a+b}\sum_{k=0}^{a+b}\left(\cos\frac{k\pi}{a+b}\right)^n \sin\frac{k\pi a}{a+b}\sin\frac{k\pi(a-x)}{a+b}.$$

The formula (13) as usual can be proved by the method of images The formula (14) can be obtained by the calculus of finite differences (cf. R. E. ELLIS [2], JORDAN K. [6]) or by the methods of Markov chains (cf. W. FELLER [3], A. A. ANIS [1]). In what follows we shall give a simple proof of this theorem.

Proof of (13). Denote by A the set of the numbers $\{2(a+b)k+x\}$ and B the set of the numbers $\{2(a+b)k+2a-x\}$ $(k=0, \pm 1, \pm 2, \ldots)$. Define by A and B the events $\eta_n \in \mathrm{A}$ and $\eta_n \in \mathrm{B}$, respectively. Further denote by A_0 the simultaneous occurrence of the events $\eta_n=x$ and $-b<\eta_i<a$ $(i=1, 2, \ldots, n)$. Now we can write $\mathbf{P}\{A\}=\mathbf{P}\{AA_0\}+\mathbf{P}\{A\bar{A}_0\}$. Here first $A_0\subset A$ and consequently $\mathbf{P}\{AA_0\}=\mathbf{P}\{A_0\}$. Secondly $\mathbf{P}\{A\bar{A}_0\}=\mathbf{P}\{B\}$. Namely the event $A\bar{A}_0$ denotes that the particle at the n-th step will be in the set A and during the first n steps reaches at least one of the points $x=a$ and $x=-b$. If the particle reaches first either of the points $x=a$ and $x=-b$ then let us change the direction of the further displacements of the particle into opposite. Thus we have a path leading by n steps into the set B instead of A. Conversely if we proceed similarly we can correspond to all pathes leading by n steps into the set B such a path which leads by n steps into the set A and which reaches at least one of the points $x=a$ and $x=-b$. This is one-to-one correspondence and the pathes have the same probabilities. Consequently $\mathbf{P}\{A\bar{A}_0\}=\mathbf{P}\{B\}$. So we have $\mathbf{P}\{A\}=\mathbf{P}\{A_0\}+\mathbf{P}\{B\}$. Since $\eta_n^*=x$ and A_0 are the same events, we obtain $\mathbf{P}\{\eta_n^*=x\}=\mathbf{P}\{A_0\}=\mathbf{P}\{A\}-\mathbf{P}\{B\}$ what was to be proved.

Proof of (14). It is well known that if $r<l$ then

$$(15)\quad \binom{n}{r}+\binom{n}{r+l}+\binom{n}{r+2l}+\cdots=\frac{1}{l}\sum_{k=0}^{l-1}\left(2\cos\frac{k\pi}{l}\right)^n\cos\frac{k(n-2r)\pi}{l}$$

(cf. E. NETTO [7] p. 20). Since

$$\mathbf{P}\{\eta_n=x\}=\begin{cases}\binom{n}{\frac{n+x}{2}}\frac{1}{2^n} & \text{if } n+x \text{ is even}\\ 0 & \text{if } n+x \text{ is odd,}\end{cases}$$

we obtain by (15) that

$$\sum_k \mathbf{P}\{\eta_n=2(a+b)k+x\}=\frac{1}{a+b}\sum_{k=0}^{a+b-1}\left(\cos\frac{k\pi}{a+b}\right)^n\cos\frac{k\pi x}{a+b}$$

if $n+x$ is even and similarly

$$\sum_k \mathbf{P}\{\eta_n=2(a+b)k+2a-x\}=\frac{1}{a+b}\sum_{k=0}^{a+b-1}\left(\cos\frac{k\pi}{a+b}\right)^n\cos\frac{k\pi(2a-x)}{a+b}.$$

if $n+x$ is even. Consequently by (13) we have

$$\mathbf{P}\{\eta_n^*=x\}=\frac{1}{a+b}\sum_{k=0}^{a+b-1}\left(\cos\frac{k\pi}{a+b}\right)^n\left(\cos\frac{k\pi x}{a+b}-\cos\frac{k\pi(2a-x)}{a+b}\right),$$

if $n+x$ is even. But this formula is valid also for every x $(-b<x<a)$. Namely if $n+x$ is odd then both side of this formula is zero. This can be easily seen. If we substitute $k=a+b-j$ then we obtain that $\mathbf{P}\{\eta_n^*=x\}=$ $=(-1)^{n+x}\mathbf{P}\{\eta_n^*=x\}$.

§ 3. Proof of Theorems 1.—4.

Proof of (1)—(2). Evidently

$$\mathbf{P}\{\delta_n^+<a,\delta_n^-<b\}=\mathbf{P}\{-b<\eta_n^*<a\}.$$

By (13) we can write

$$\mathbf{P}\{-b<\eta_n^*<a\}=\sum_{k=-\infty}^{\infty}(-1)^k\mathbf{P}\{k(a+b)-b<\eta_n<k(a+b)+a\}.$$

Now suppose that a and b depend on n, that is $a=a_n$ and $b=b_n$ where

$$(16)\qquad \lim_{n\to\infty}\frac{a_n}{n^{1/2}}=z>0,\quad \lim_{n\to\infty}\frac{b_n}{n^{1/2}}=y>0.$$

By applying the central limit theorem, we have

$$\lim_{n\to\infty}\mathbf{P}\{-b_n<\eta_n^*<a_n\}=\sum_{k=-\infty}^{\infty}(-1)^k[\Phi(k(z+y)+z)-\Phi(k(z+y)-y)].$$

The limit may be formed term by term the sum being uniformely convergent. As the limiting distribution is continuous in z and y, we have

$$\lim_{n\to\infty} \mathbf{P}\{-n^{1/2}y<\eta_n^*<n^{1/2}z\}=\lim_{n\to\infty}\mathbf{P}\{-b_n<\eta_n^*<a_n\}$$

which proves (2).

Proof of (1)—(3). Now by (14) we have

$$\mathbf{P}\{-b<\eta_n^*<a\}=\frac{2}{a+b}\sum_{k=0}^{a+b}\left(\cos\frac{k\pi}{a+b}\right)^n \sin\frac{k\pi a}{a+b}\sum_{x=-b}^{a}\sin\frac{k\pi(a-x)}{a+b}.$$

Here

$$\sum_{x=-b}^{a}\sin\frac{k\pi(a-x)}{a+b}=\frac{1-(-1)^k}{2}\,\frac{\cos\dfrac{k\pi}{a+b}}{\sin\dfrac{k\pi}{a+b}}.$$

If $a=a_n$ and $b=b_n$ correspond to (16) and $n\to\infty$ we obtain

$$\lim_{n\to\infty}\mathbf{P}\{-b_n<\eta_n^*<a_n\}=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}e^{-\frac{(2j+1)^2\pi^2}{2(z+y)^2}}\sin\frac{(2j+1)\pi z}{z+y}.$$

Namely,

$$\lim_{n\to\infty}\left(\cos\frac{k\pi}{a_n+b_n}\right)^n=e^{-\frac{k^2\pi^2}{2(z+y)^2}}$$

and

$$\lim_{n\to\infty}\frac{1}{a_n+b_n}\sum_{x=-b_n}^{a_n}\sin\frac{k\pi(a_n-x)}{a_n+b_n}=\begin{cases}\dfrac{1}{(2j+1)\pi} & \text{if } k=2j+1\\[2ex] 0 & \text{if } k=2j.\end{cases}$$

(3) follows similarly as above.

Proof of (4)—(5). Evidently

$$\mathbf{P}\{\delta_{2n}^+<a,\,\delta_{2n}^-<b\,|\,\eta_{2n}=0\}=\mathbf{P}\{\eta_{2n}^*=0\,|\,\eta_{2n}=0\}.$$

Now by (13) we have

$$\mathbf{P}\{\eta_{2n}^*=0\}=\sum_{k=-\infty}^{\infty}[\mathbf{P}\{\eta_{2n}=2(a+b)k\}-\mathbf{P}\{\eta_{2n}=2(a+b)k+2a\}].$$

Putting $a=a_{2n}$ and $b=b_{2n}$, according to (16), we obtain

$$\lim_{n\to\infty}\mathbf{P}\{\eta_{2n}^*=0\,|\,\eta_{2n}=0\}=\lim_{n\to\infty}\frac{\mathbf{P}\{\eta_{2n}^*=0\}}{\mathbf{P}\{\eta_{2n}=0\}}=\sum_{k=-\infty}^{\infty}[e^{-2(z+y)^2k^2}-e^{-2((z+y)k+z)^2}].$$

Namely by Moivre-Laplace theorem

$$\lim_{n\to\infty}\frac{\mathbf{P}\{\eta_{2n}=2(a_n+b_n)k\}}{\mathbf{P}\{\eta_{2n}=0\}}=e^{-2(z+y)^2k^2}$$

and

$$\lim_{n\to\infty} \frac{\mathbf{P}\{\eta_{2n}=2(a_n+b_n)k+2a_n\}}{\mathbf{P}\{\eta_{2n}=0\}} = e^{-2((z+y)k+z)^2}$$

Further it is easy to see that the limit may be form term by term. So we have proved (5).

Proof of (4)—(6). We proceed similarly as above. Now by (14) we have

$$\mathbf{P}\{\eta_{2n}^*=0\} = \frac{2}{a+b} \sum_{k=0}^{a+b} \left(\cos\frac{k\pi}{a+b}\right)^{2n} \sin^2\frac{k\pi a}{a+b}.$$

Putting $a=a_{2n}$ and $b=b_{2n}$, we obtain

$$\lim_{n\to\infty} \mathbf{P}\{\eta_{2n}^*=0 \mid \eta_{2n}=0\} = \frac{\sqrt{2\pi}}{z+y} \sum_{j=0}^{\infty} e^{-\frac{k^2\pi^2}{2(z+y)^2}} \sin^2\frac{k\pi z}{z+y}$$

as

$$\mathbf{P}\{\eta_{2n}=0\} \sim \frac{1}{\sqrt{n\pi}} \qquad (n\to\infty)$$

This proves (6).

Proof of Theorem 3. It can be easily seen by the theorem of total probability that

$$F^*(x) = \int_0^x \left(\frac{\partial F(z,y)}{\partial z}\right)_{y=x-z} dz.$$

Carrying out the calculation with $F(z,y)$ defined by (2) or (3) we obtain (8) and (9) respectively.

Remark. By (8) we have

$$\frac{dF^*(x)}{dx} = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-\frac{k^2x^2}{2}} \qquad (0<x<\infty).$$

Denote by M_s $(s=1,2,\ldots)$ the s-th moment of $F^*(x)$. Then we have

$$M_s = \int_0^\infty x^s dF^*(x) = s\int_0^\infty x^{s-1}[1-F^*(x)]dx$$

and specifically

$$M_1 = \sqrt{\frac{8}{\pi}}$$

and

$$M_s = \frac{2^{\frac{s+4}{2}} \Gamma\left(\frac{s+1}{2}\right)}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{s-1}}$$

if $s=2,3,\ldots$

Proof of Theorem 4. Similarly as above we have

$$K^*(x) = \int_0^x \left(\frac{\partial K(z, y)}{\partial z} \right)_{y=x-z} dz$$

what proves (11) and (12) if $K(z, y)$ is defined by (5) and (6) respectively.

Remark. The moments M_s' $(s = 1, 2, \ldots)$ of $K^*(x)$ can be expressed as follows

$$M_1' = \sqrt{\frac{\pi}{2}},$$

and

$$M_s' = \frac{4(s-1)\,\Gamma\left(\frac{s+2}{2}\right)}{2^{s/2}} \sum_{k=1}^{\infty} \frac{1}{k^s}$$

if $s = 2, 3, \ldots$

(Received: 21. VIII. 1957.)

REFERENCES

[1] Anis, A. A.: „On the distribution of the range of partial sums of independent random variables". *Proceedings of the Mathematical and Physical Society of Egypt* **1** (1954) 83—89.

[2] Ellis, R. E.: „On the solution of equations in finite differences". *Cambridge Mathematical Journal* **4** (1844) 182.

[3] Feller, W.: *An introduction to probability theory and its applications.* Wiley, New-York, 1950.

[4] Feller, W.: „The asymptotic distribution of the range of sums of independent random variables". *Annals of Mathematical Statistics* **22** (1951) 427—432.

[5] Гнеденко, Б. В.: „Проверка неизменности распределения вероятностей в двух независимых выборках". *Mathematische Nachrichten* **12** (1954) 29—66.

[6] Jordan K.: *Fejezetek a klasszikus valószínűségszámításból.* Akadémiai Kiadó, Budapest, 1956.

[7] Netto, E.: *Lehrbuch der Combinatorik.* Teubner, Leipzig, 1901.

[8] Takács L.: „Bolyongási feladatokról". *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **2** (1957) 81—90.

NÉHÁNY MEGJEGYZÉS BOLYONGÁSI FELADATOKKAL KAPCSOLATBAN

TAKÁCS LAJOS

Kivonat

Legyenek $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ kölcsönösen független valószínűségi változók, amelyekre $\mathbf{P}\{\xi_n = 1\} = \mathbf{P}\{\xi_n = -1\} = \frac{1}{2}$ $(n = 1, 2, 3, \ldots)$. Továbbá legyen $\eta_0 = 0$ és $\eta_n = \xi_1 + \xi_2 + \cdots + \xi_n$ $(n = 1, 2, 3, \ldots)$. A dolgozat a $\delta_n^+ =$

$=\max(\eta_0, \eta_1, \ldots, \eta_n)$ és a $\delta_n^- = -\min(\eta_0, \eta_1, \ldots, \eta_n)$ változók együttes eloszlásának és a $\delta_n = \delta_n^+ + \delta_n^-$ változó eloszlásának aszimptotikus viselkedésével foglalkozik.

НЕСКОЛЬКО ЗАМЕЧАНИЙ В СВЯЗИ С ЗАДАЧАМИ БЛУЖДАНИЯ

L. TAKÁCS

Резюме

Пусть $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ взаимно независимые случайные величины, для которых $\mathbf{P}\{\xi_n = 1\} = \mathbf{P}\{\xi_n = -1\} = \frac{1}{2}$ $(n = 1, 2, 3, \ldots)$. Пусть $\eta_0 = 0$ и $\eta_n = \xi_1 + \xi_2 + \cdots + \xi_n$ $(n = 1, 2, 3, \ldots)$. Работа изучает асимптотическое поведение совместного распределения величин $\delta_n^+ = \max(\eta_0, \eta_1, \ldots, \eta_n)$ и $\delta_n^- = -\min(\eta_0, \eta_1, \ldots, \eta_n)$ и распределения $\delta_n = \delta_n^+ + \delta_n^-$.