EXTENSION OF CERTAIN THEOREMS OF THE STURMIAN TYPE TO NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

by

IMRE BIHARI[1)]

Introduction

M. BÔCHER [1], [2] (pp. 44—52) added a series of new results to those of CH. STURM [3] concerning the linear combination of a solution and its derivative, of a second order linear differential equation. The present paper gives some new contributions to these investigations concerning certain nonlinear equations.

I thank J. CZIPSZER for his valuable criticism and remarks.

§ 1

In a preceding paper [4] of the author the equation

$$\frac{d}{dx}\big(P(x)\,v'\big) + Q(x,\lambda)\,f(v,v') = 0$$

was discussed from the point of view of oscillation and comparison of the solutions. In the present paper we deal with solutions of the equation

$$\frac{d}{dx}\big(P(x)\,v'\big) + Q(x)\,f(v,v') = 0\ . \tag{1}$$

Let the following conditions be assumed throughout this paper:

1. $P(x) > 0$ and $P(x) \in C_0$, $Q(x) \in C_0$ in $[a, b]$,

2. $f(v, w)$ is defined for arbitrary v, w and $f(v, w) \in \mathrm{Lip}\,(1)$ in any bounded domain,

3. $f(\lambda v, \lambda w) = \lambda f(v, w)$, $f(0, w) \equiv 0$ and $\operatorname{sgn} f(v, w) = \operatorname{sgn} v$.

Let us take two linearly independent solutions (v_1, v_2) of (1). They have no zero in common. (See [4], § 1.). We conclude herefrom that

$$\Delta(x) = v_2' v_1 - v_2 v_1' \neq 0 \quad \text{in} \quad [a, b]\ .$$

Namely regard a point c with $a \leqq c \leqq b$ where $v_1(c) \neq 0$, $v_2(c) \neq 0$ but $\Delta(c) = 0$, i. e.

$$\frac{v_1'(c)}{v_1(c)} = \frac{v_2'(c)}{v_2(c)}.$$

---

1) Polytechnic University of Budapest, Ist Department of Mathematics.

Hence $v_2(c) = \lambda v_1(c)$, $v_2'(c) = \lambda v_1'(c)$ $(\lambda \neq 0)$ and this involves $v_2 \equiv \lambda v_1$ in $[a, b]$. On the other hand if $v_1 = 0$, $\Delta(x) = v_1' v_2 \neq 0$; for $v_1' \neq 0$ (except $v_1 \equiv 0$) and $v_2 \neq 0$. It is a simple well known fact that v (if different from $v \equiv 0$) cannot have an infinity of zeros in $[a, b]$. For this would imply $v(c) = v'(c) = 0$ at a limiting point c of these zeros, and this contradicts $v \not\equiv 0$. Consider now the linear expression of v and v' $(v \not\equiv 0)$

$$\Phi(x) = \varphi_1 v - \varphi_2 P v' \quad . \tag{2}$$

Assuming suitable conditions $\Phi(x)$ cannot oscillate in $[a, b]$ infinitely often. This is expressed by

Theorem 1. *Let φ_1, φ_2 and $f(v, w)$ satisfy in $[a, b]$ the conditions*

1. $\varphi_1(x)$ *and* $\varphi_2(x)$ *are derivable,*
2. $\{\varphi_1, \varphi_2\} = \varphi_1' \varphi_2 - \varphi_2' \varphi_1 + \dfrac{\varphi_1^2}{P} + \dfrac{\varphi_2 Q}{P} f(\varphi_2 P, \varphi_1) \neq 0$,

then Φ cannot have an infinity of zeros in $[a, b]$.[2)]

Proof. In the opposite case there would exist again a point c in $[a, b]$ where

$$\Phi(c) = \Phi'(c) = 0 \quad .$$

With regard to (1) and (2) we have herefrom

$$\varphi_1 v - \varphi_2 P v' = 0 \tag{3}$$

$$\varphi_1' v + (\varphi_1 - \varphi_2' P) v' + \varphi_2 Q f(v, v') = 0 \quad , \tag{4}$$

whence being $v(c) \neq 0$ or $v'(c) \neq 0$

$$\frac{v'}{v} = \frac{\varphi_1}{\varphi_2 P} \quad \text{or} \quad \frac{v}{v'} = \frac{\varphi_2 P}{\varphi_1} \quad .$$

(Namely owing to the first equation if $v \neq 0$, then $\varphi_2 \neq 0$, if $v' \neq 0$, then $\varphi_1 \neq 0$.) Putting this in (4) we get

$$\frac{v}{\varphi_2} \{\varphi_1, \varphi_2\} = 0 \quad \text{or} \quad \frac{v'}{\varphi_1} \{\varphi_1, \varphi_2\} = 0 \qquad \text{at } x = c$$

what is impossible.

§ 2

The zeros of two linearly independent solutions (v_1, v_2) of (1) separate each other (see [4] § 2.). — We raise the problem of finding suitable conditions for the same behaviour of the functions

$$\begin{aligned} \Phi_1 &= \varphi_1 v_1 - \varphi_2 P v_1' \\ \Phi_2 &= \varphi_1 v_2 - \varphi_2 P v_2' \quad . \end{aligned} \tag{5}$$

---

2) Theorems 1. and 3. reduce to Bôcher's theorems provided that $f(v, v') \equiv v$.

We can give such conditions only for the special case where $f(v, w) = v^3/(v^2 + w^2)$, i. e. for the equation

$$(Pv')' + Q \frac{v^3}{v^2 + v'^2} = 0 \tag{6}$$

discussed several times in [4]. In the linear case — where $f(v, w) = v$ — the function $\Phi(x) = \varphi_1 v - \varphi_2 Pv'$ satisfies the second order linear homogeneous equation

$$P\{\varphi_1, \varphi_2\}(P\Phi')' = A\Phi + B\Phi'$$

A and B depending on φ_1, φ_2, P, Q by intricate formulae. This equation is not singular provided that $P \neq 0$, $\{\varphi_1, \varphi_2\} \neq 0$. As $\varphi_1^2 + \varphi_2^2 > 0$, the functions Φ_1, Φ_2 are linearly independent such as v_1 and v_2, consequently their zeros separate each other, thus Bôcher's theorem ([2], p. 48.) — relative to linear equations — is not very far reaching.

However, assuming $f(v, w)$ to be nonlinear, a nonlinear second order equation is obtained for Φ not belonging to types, for which it is known that zeros of their "independent" solutions separate each other; moreover we cannot establish whether Φ_1, Φ_2 are independent, for we do not know whether they have zeros in common or not.

It is in the esentially nonlinear case where the separation theorem concerning Φ_1, Φ_2 claims for a "separate" proof.

The actual proof shows just the fact that Φ_1 and Φ_2 are linearly independent.

In order that Φ_1 and Φ_2 should not have an infinite number of zeros in $[a, b]$ we assume here too that

$$\{\varphi_1, \varphi_2\} = \frac{\varphi_1^2}{P} + \varphi_1'\varphi_2 - \varphi_2'\varphi_1 + \frac{\varphi_2^4 P^2 Q}{\varphi_2^2 P^2 + \varphi_1^2} \neq 0 \quad \text{in} \quad [a, b] .$$

Let x_1, x_2 be two consecutive zeros of Φ_1 in $[a, b]$. Further — for the sake of brevity — we introduce the notations

$$A = \varphi_1^2 + P(\varphi_1'\varphi_2 - \varphi_2'\varphi_1) = \varphi_1^2\left[1 - P\left(\frac{\varphi_2}{\varphi_1}\right)'\right], \quad B = -\varphi_1\varphi_2 Q, \quad C = \varphi_2^2 QP .$$

Now we can formulate

Theorem 2. *If*

1) φ_1, φ_2 *are derivable*, $P > 0$, $P \in C_0$, $Q \in C_0$, $\{\varphi_1, \varphi_2\} \neq 0$,

2) $A > 0$, $A + C > 0$, $B^2(A + C) < 3\,AC(4\,A + C)$

then Φ_2 *has a zero (and only one) in* (x_1, x_2) *and conversely:* Φ_1 *has a unique zero between two successive roots of* Φ_2.

In particular, in order to satisfy the conditions 1) and 2) it is sufficient to assume the following hypotheses :

$$1')\ P > 0\,,\quad Q > 0\,,\quad \left(\frac{\varphi_2}{\varphi_1}\right)' < \frac{3P - Q}{3P^2}\,,\quad \varphi_1 \neq 0,\quad \varphi_1,\ \varphi_2$$

are derivable, $P \in C_0$, $Q \in C_0$.

Viz. condition 1') *involves*

$$A = \varphi_1^2\left[1 - P\left(\frac{\varphi_2}{\varphi_1}\right)'\right] > \frac{Q}{3P}\varphi_1^2 > 0\,,\quad A + C > 0\,,\quad \{\varphi_1, \varphi_2\} =$$

$$= \frac{1}{P}A + \frac{\varphi_2^4 P^2 Q}{\varphi_2^2 P^2 + \varphi_1^2} > 0$$

and — as it may be easily seen —

$$B^2 < 3AC\ .$$

But this implies

$$B^2(A + C) < 3AC(A + C) < 3AC(4A + C)\ .$$

Proof. If the theorem were false, Φ_2 would not vanish in (x_1, x_2). Φ_2 cannot vanish at any one of x_1 and x_2, for Φ_1 and Φ_2 have no zero in common at all ; otherwise we should have $v_1' v_2 - v_2' v_1 = 0$ as at this point $\varphi_1^2 + P^2\varphi_2^2 > 0$ (see (5)). Consequently the function Φ_1/Φ_2 is derivable in $[x_1, x_2]$ and vanishes at x_1 and x_2, therefore by the theorem of Rolle there is a point c with $x_1 < c < x_2$ such that

$$\frac{d}{dx}\left(\frac{\Phi_1}{\Phi_2}\right) = 0 \qquad \text{resp.} \qquad \Phi_1'\Phi_2 - \Phi_2'\Phi_1 = 0\ .$$

Writing this out in detail

$$\left(\varphi_1' v_1 + \varphi_2 Q\frac{v_1^3}{v_1^2 + v_1'^2} + (\varphi_1 - \varphi_2' P)\, v_1'\right)(\varphi_1 v_2 - \varphi_2 P v_2') -$$

$$- \left(\varphi_1' v_2 + \varphi_2 Q\frac{v_2^3}{v_2^2 + v_2'^2} + (\varphi_1 - \varphi_2' P)\, v_2'\right)(\varphi_1 v_1 - \varphi_2 P v_1') =$$

$$= \left[\varphi_1^2 + (\varphi_1'\varphi_2 - \varphi_2'\varphi_1)\, P - \varphi_1\varphi_2 Q v_1 v_2 \frac{v_1' v_2 + v_2' v_1}{(v_1^2 + v_1'^2)(v_2^2 + v_2'^2)} + \right.$$

$$\left. + \varphi_2^2 QP\frac{v_1^2 v_2^2 + v_1^2 v_2'^2 + v_1 v_2 v_1' v_2' + v_1'^2 v_2^2}{(v_1^2 + v_1'^2)(v_2^2 + v_2'^2)}\right] \cdot \varDelta = 0$$

where $\varDelta = v_1' v_2 - v_2' v_1$. Therefore the expression in the square bracket[3)]

[3)] Let us denote this expression by E.

has to vanish at $x = c$. We state that neither v_1 nor v_2 vanishes at c. Supposing e. g. $v_2(c) = 0$, then $v_1(c) \neq 0$ and we have at $x = c$

$$\varphi_1^2 + (\varphi_1' \varphi_2 - \varphi_2' \varphi_1) P + \varphi_2^2 QP \frac{1}{1 + z_1^2} = 0 \quad \text{where} \quad z_1 = \frac{v_1'}{v_1} .$$

By our notations this can be written as

$$A + C \frac{1}{1 + z_1^2} = 0 \qquad \text{or} \quad z_1^2 = - \frac{A + C}{A} . \tag{7}$$

In view of $(A + C)/A > 0$ we obtained a contradiction. Putting

$$\frac{v_1'}{v_1} = z_1 , \qquad \frac{v_2'}{v_2} = z_2$$

the expression

$$E = A + \frac{B(z_1 + z_2) + C(1 + z_1 z_2 + z_1^2 z_2^2)}{(1 + z_1^2)(1 + z_2^2)} \tag{8}$$

has to vanish according to our assumptions. Introducing the notations

$$z_1 + z_2 = \xi \qquad z_1 z_2 = \eta$$

we obtain the equation as follows

$$A + \frac{B \xi + C(1 + \eta + \eta^2)}{1 + \xi^2 - 2\eta + \eta^2} = 0$$

or

$$A \xi^2 + B \xi + [(A + C) \eta^2 + (C - 2A) \eta + A + C] = 0 . \tag{9}$$

Considering this as a second order equation in ξ the discriminant is

$$f(\eta) = B^2 - 4A [(A + C) \eta^2 + (C - 2A) \eta + A + C] = - 4A(A + + C) \eta^2 - 4A(C - 2A) \eta + B^2 - 4A(A + C) .$$

But this is not capable to assume nonnegative values because its discriminant

$$16A [B^2(A + C) - 3AC(4A + C)] < 0$$

and $-4A(A + C) < 0$. Therefore (9) has no real roots and so z_1, z_2 cannot be real. This contradiction proves the theorem.

§ 3

Taking a unique not identically vanishing solution v of (1) we shall investigate the zeros of the two expressions

$$\begin{aligned} \Phi &= \varphi_1 v - \varphi_2 P v' \\ \Psi &= \psi_1 v - \psi_2 P v' . \end{aligned} \tag{10}$$

We ask whether one can obtain similar results as we got in the previous §? The answer turned out to be affirmative (as for linear equations). This is the content of

Theorem 3. *If* $\varphi_1, \varphi_2, \psi_1$ and ψ_2 *are derivable, and* $\{\varphi_1, \varphi_2\} \neq 0, \{\psi_1 \psi_2\} \neq 0$, $D = \varphi_1 \psi_2 - \psi_1 \varphi_2 \neq 0$ *in* $[a, b]$, *then the zeros of* Φ *separate those of* Ψ *and conversely.*

This theorem is meaningless unless one at least of Φ and Ψ has several zeros in $[a, b]$.

Proof. A differential equation will be deduced satisfied by Φ and Ψ (resp. by Φ/Ψ). — We start from the identity

$$\Phi(\psi_1 v - \psi_2 P v') - \Psi(\varphi_1 v - \varphi_2 P v') = 0$$

what can be brought to the form

$$(\psi_1 \Phi - \varphi_1 \Psi)\, v - (\psi_2 \Phi - \varphi_2 \Psi) P v' = 0 \ . \tag{11}$$

With regard to equation (1) we get by derivation herefrom

$$\begin{aligned} (\psi_1 \Phi' - \varphi_1 \Psi' + \psi_1' \Phi - \varphi_1' \Psi)\, v + [\psi_1 \Phi - \varphi_1 \Psi - (\psi_2' \Phi - \varphi_2' \Psi + \\ + \psi_2 \Phi' - \varphi_2 \Psi') P]\, v' + (\psi_2 \Phi - \varphi_2 \Psi)\, Q f(v, v') = 0 \ . \end{aligned} \tag{12}$$

From (11) identically

$$\frac{v'}{v} = \frac{1}{P} \frac{\psi_1 \Phi - \varphi_1 \Psi}{\psi_2 \Phi - \varphi_2 \Psi} \ . \tag{13}$$

Putting this into (12)

$$\begin{aligned} (\psi_1 \Phi' - \varphi_1 \Psi' + \psi_1' \Phi - \varphi_1' \Psi) + [\psi_1 \Phi - \varphi_1 \Psi - (\psi_2' \Phi - \varphi_2' \Psi + \\ + \psi_2 \Phi' - \varphi_2 \Psi') P] \frac{1}{P} \frac{\psi_1 \Phi - \varphi_1 \Psi}{\psi_2 \Phi - \varphi_2 \Psi} + (\psi_2 \Phi - \varphi_2 \Psi)\, Q f\left(1, \frac{1}{P} \frac{\psi_1 \Phi - \varphi_1 \Psi}{\psi_2 \Phi - \varphi_2 \Psi}\right) = 0 . \end{aligned}$$

As $D = \varphi_1 \psi_2 - \varphi_2 \psi_1 \neq 0$, we can express Φ' from this equation at a point where $\Phi = 0$ and Ψ' at a point where $\Psi = 0$, obtaining

$$\Phi' = - \frac{\{\varphi_1, \varphi_2\}}{D} \Psi$$

$$\Psi' = \frac{\{\psi_1, \psi_2\}}{D} \Phi \ ,$$

where $\{\varphi_1, \varphi_2\}$ and $\{\psi_1, \psi_2\}$ are the functions defined in § 1. The functions Φ and Ψ — as (10) shows — cannot vanish simultaneously. The remainder of the proof proceeds perfectly on the lines followed by Bôcher (see [1], p. 430) and we omit its mere reproduction. The rest of Bôcher's results in [1] may also be extended to equation (1). We mention only this : If we assume furthermore that $\{\varphi_1, \varphi_2\}$ and $\{\psi_1, \psi_2\}$ have opposite signs in $[a, b]$, then one at most of Φ and Ψ can vanish and moreover only once.

Application to equation (6). Take $\varphi_1 = \varphi$, $\varphi_2 = 1$, $\psi_1 = 1, \psi_2 = 0$, then $\Phi = \varphi v - Pv'$, $\Psi = v$ and

$$\{\varphi_1, \varphi_2\} = \varphi' + \frac{\varphi^2}{P} + \frac{QP^2}{\varphi^2 + P^2}, \qquad \{\psi_1, \psi_2\} = \frac{1}{P} > 0 \ .$$

Therefore if we choose φ so that

$$\varphi' + \frac{\varphi^2}{P} + \frac{QP^2}{\varphi^2 + P^2} < 0 \qquad \text{in } [a, b] \ , \tag{14}$$

then only one of v and $\varphi v - Pv'$ can vanish in $[a, b]$ and at most once, i. e. equation (6) is non-oscillatory (no solution of (6) can have several zeros in $[a, b]$). Specializing φ in (14) we obtain various sufficient conditions for the non-oscillatory character of (6). — E. g. $\varphi \equiv 0$ results in $Q < 0$. In this case v and v' together can have at most one zero in $[a, b]$.

§ 4

Suppose that in (6) $Q = \text{const.}$, $P = \text{const.} > 0, \dfrac{Q}{P} = \lambda$, then we have the equation

$$v'' + \lambda \frac{v^3}{v^2 + v'^2} = 0 \ . \tag{15}$$

— As known (see [4] § 3) — every solution of (15) is periodic with the period

$$P_\lambda = \int_{-\infty}^{\infty} \frac{f(u, \lambda)}{u} du \ , \qquad f(u, \lambda) = \frac{u(1 + u^2)}{u^2(1 + u^2) + \lambda} \ .$$

Replacing in (6) P by a smaller, Q by a larger function, the oscillation will be more rapid (see [4] § 2). Therefore *if*

$$\frac{\max Q}{\min P} = \lambda_1$$

(maximum and minimum are taken in $[a, b]$) *and* $P_{\lambda_1} > 2(b - a)$, *then (6) is non-oscillatory.* — Denoting $\min Q / \max P$ by λ_2 and by k a positive integer, *every solution of (6) has at least k zeros in $[a, b]$ provided that*

$$kp_{\lambda_2} \leqq 2\,(b - a) \ .$$

Remark. The second (resp. kth) order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + Q(x, \lambda)\, f\left(\frac{\partial^k u}{\partial y^k}, \frac{\partial u}{\partial x}\right) = 0$$

($k > 0$ integer and $f(\mu p, \mu q) = \mu f(p, q)$) leads by the substitution $u = e^y v(x)$ to the equation

$$v'' + Q(x, \lambda)\, f(v, v') = 0 \ .$$

Therefore the results obtained here and in [4] relative to this last equation may be applied in the qualitative investigation of certain solutions of the preceding equation.

(Received at 26 February, 1958. — In revised form at 24 March, 1958.)

REFERENCES

[1] Bôcher, M.: „On certain pairs of transcendental functions whose roots separate each other“. *Transactions of the American Mathematical Society* **2** (1901) 428—436.
[2] Bôcher, M.: *Leçons sur les méthodes de Sturm.* Gauthier—Villars, Paris, 1917.
[3] Sturm, Ch.: „Sur les équations différentielles linéaires de deuxième ordre“. *Journal de Mathématiques Pures et Appliquées* **1** (1836) 149—164.
[4] Bihari, I.: „Ausdehnung der Sturmschen Oscillations- und Vergleichssätzen auf die Lösungen gewisser nicht-linearer Differentialgleichungen zweiter Ordnung“. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* **2** (1957) 159—173.

BIZONYOS STURM-TIPUSÚ EREDMÉNYEK KITERJESZTÉSE MÁSODRENDŰ NEMLINEÁRIS DIFFERENCIÁLEGYENLETEKRE

BIHARI I.

Kivonat

A dolgozat célja Bôcher [1], [2] és Sturm [3] bizonyos lineáris területen nyert eredményeinek a kiterjesztése nemlineáris területre. Ezek az eredmények nemlineáris egyenletek lineárisan független megoldásaiból és azok deriváltjaiból képezett függvénypárok zérushelyeinek szétválasztására, az egyenlet oszcilláló vagy nem oszcilláló voltának, e zérushelyek közül egy adott véges intervallumba esők száma végességének megállapítására vonatkoznak. Hasonló eredmények szerepelnek *egy* megoldásból és deriváltjából képzett két és több lineáris kifejezésre.

РАСПРОСТРАНЕНИЕ НЕКОТОРЫХ РЕЗУЛЬТАТОВ ТИПА STURM-A НА ЛИНЕЙНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ ВТОРОГО ПОРЯДКА

I. BIHARI

Резюме

Цель работы — распространение на нелинейную случай некоторых результатов Bôcher-а [1], [2] и Sturm-а [3] полученных в линейной случае. Эти результаты относятся к отделению нулей пар функций, образованных из линейно независимых решений уравнений и их производных, к определению осциллируемости или неосциллируемости уравнения, конечности числа нулей в данном конечном отрезке. Аналогичные результаты фигурируют для двух и большего числа линейных выражений, полученных из *одного* решения и его производных.