A NOTE ON A PROBLEM OF RÉNYI

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A. Rényi[1] recently propounded the following problem. An $n \times n$ matrix $\mathbf{X} = \{x_{ij}\}$, whose elements are all real and nonnegative, has to satisfy the conditions $\left\{\sum_j x_{ij} = g_i\right\}$ and $\left\{\sum_i x_{ij} = h_j\right\}$, and also has to minimize the linear form $\sum_{ij} c_{ij} x_{ij}$. Under these conditions, a certain number of the elements of $\mathbf{X}$ will be zero. If more than one matrix satisfies the conditions, we select one with as many zero elements as possible. If now we consider every possible set of values for $\{g_i\}$, $\{h_j\}$ and $\{c_{ij}\}$, the resulting matrices $\mathbf{X}$ will have various numbers of zero elements ; let $f(n)$ be the smallest such number for a given value of n. It is the form of $f(n)$ which is required.

It is easily shown that $f(n) \leq (n-1)^2$. Consider for instance the situation where the linear form to be minimized consists of the sum of all the elements of $\mathbf{X}$ except for the first row and the first column, and where $g_1 = h_1 = n$, and $g_i = h_j = 1$ for i and $j \neq 1$. In fact, as will be shown below, $f(n) = (n-1)^2$.

Treat the elements of $\mathbf{X}$ as coordinates of a point in euclidean space of n^2 dimensions. The fixing of g_i and h_j impose $2n - 1$ independent conditions, which restrict the point to a linear subspace of $(n-1)^2$ dimensions. Now the conditions $\{x_{ij} \geqq 0\}$ have the added effect of restricting the point to a region $\mathcal{R}$ of this subspace, where $\mathcal{R}$ is convex and is bounded by surfaces of the form $x_{ij} = 0$. This being so, it is clear that the minimum over $\mathcal{R}$ of the form $\sum c_{ij} x_{ij}$ is attained either at a vertex of $\mathcal{R}$ or over a set which contains vertices of $\mathcal{R}$. But every vertex of $\mathcal{R}$ must lie on at least $(n-1)^2$ of the bounding surfaces of $\mathcal{R}$, and thus at least this number of the elements of $\mathbf{X}$ are zero at a vertex. Therefore $f(n) \geq (n-1)^2$. Combining this with the above, we have $f(n) = (n-1)^2$.

Similarly it may be proved that the corresponding number for $n \times m$ matrices is $(n-1)(m-1)$, and similar results hold for arrays of more than 2 dimensions.

From the above it is possible to deduce the result, proved previously by other means by Birkhoff [1] and by Hammersley and Mauldon [2], that the set of $n \times n$ doubly-stochastic matrices is the convex hull of the set

[1] In a lecture at the Colloquium on Monte-Carlo methods, held in Balatonvilágos, 23rd September, 1958. A summary of this lecture is given in "Matematikai Lapok" **9** (1958) p. 353—354.

of $n \times n$ permutation matrices. **X** is doubly stochastic if $g_i = h_j = 1$. But if **X** is a vertex of $\mathcal{R}$ it has at most $n^2 - (n-1)^2 = 2n - 1$ non-zero elements. Therefore at least one row of **X** has only 1 non-zero element, which must have the value 1, and which must therefore also be the only non-zero element in its column. Delete this row and column and proceed by induction.

This note is the outcome of a private communication from DR. A. RÉNYI, and of discussion with Mr. J. M. HAMMERSLEY, to both of whom I am grateful, as I am also to the University of Illinois who are at present supporting me.

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REFERENCES

[1] BIRKHOFF, G. : *Rev. Univ. Tucuman* A, 5 (1946), 147—151.
[2] HAMMERSLEY J. M. and MAULDON J. G. : *Proc Camb. Phil. Soc.* 52 (1956) 461—481

RÉNYI ALFRÉD EGY PROBLÉMÁJÁRÓL

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Kivonat

A dolgozat egy RÉNYI ALFRÉD által felvetett problémát old meg, amenynyiben bebizonyítja a következő tételt. Vizsgáljuk azokat az $\mathbf{X} = \{x_{ij}\}$ $n \times n$-es nemnegatív elemű mátrixokat, amelyek elemei eleget tesznek a

$$\sum_j x_{ij} = g_i \quad (i = 1, 2, \ldots, n), \qquad \sum_i x_{ij} = h_j \quad (j = 1, 2, \ldots n)$$

feltételeknek ; e mátrixok közül tekintsük azokat, amelyek a $\sum_i \sum_j c_{ij} x_{ij}$ lineáris alak értékét minimalizálják. Itt g_i és h_j tetszőleges előírt pozitív számok, a c_{ij} számok pedig tetszőleges valós számok. Az ilyen matrixok között van olyan, amelyben a zérusok száma $\geqq (n-1)^2$, és ez az eredmény nem javítható.

ОБ ОДНОЙ ПРОБЛЕМЕ A. RÉNYI

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Резюме

Работа решает одну проблему A. RÉNYI, доказывая следующую теорему. Рассмотрим матрицы $\mathbf{X} = \{x_{ij}\}$ с $n \times n$ неотрицательными элементами, удовлетворяющими условиям

$$\sum_j x_{ij} = g_i \quad (i = 1, 2, \ldots, n), \qquad \sum_i x_{ij} = h_j \quad (j = 1, 2, \ldots, n);$$

среди этих матриц будем изучать те, которые минимизируют значение линейной формы $\sum_i \sum_j c_{ij} x_j$. Здесь g_i и h_j любые фиксированные положительные числа, а c_{ij} любые вещественные числа. Среди этих матриц есть такая, в которой число нулей $\geqq (n-1)^2$, и этот результат не может быть улучшен.