COMPLETELY CONTINUOUS OPERATORS WITH UNIFORMLY BOUNDED ITERATES

by

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I have proved in 1945 the following theorem [1]. Let $\mathbf{T}$ be an invertible operator[1] in (real or complex) euclidean space of a finite or infinite number of dimensions, such that all the iterates of $\mathbf{T}$ and of $\mathbf{T}^{-1}$ have a common bound, i. e.

$$\|\mathbf{T}^n\| \leqq M \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then $\mathbf{T}$ is similar to a unitary operator. More precisely, there exists a self-adjoint invertible operator $\mathbf{Q}$ such that

$$\mathbf{QTQ}^{-1}$$

is *unitary*.

If $\mathbf{T}$ is not invertible, but all its direct iterates $\mathbf{T}, \mathbf{T}^2, \mathbf{T}^3, \ldots$ have a common bound, then, obviously, $\mathbf{T}$ cannot be similar to any unitary operator. All that one may surmise in this case is that $\mathbf{T}$ is similar to a *contraction*, i. e. to an operator $\mathbf{T}'$ with $\|\mathbf{T}'\| \leqq 1$.

In this note we shall show that this is in fact true at least if the space is complex and $\mathbf{T}$ is completely continuous.[2] Since in finite dimensional euclidean space every operator is completely continuous, this settles our problem for these spaces in the positive sense. For non completely continuous operators in infinite dimensional euclidean space, however, the problem remains as yet open.

Thus, we shall prove the following

Theorem. *If* $\mathbf{T}$ *is a completely continuous operator on a complex euclidean space* $\mathcal{R}$ *of a finite or infinite number of dimensions, such that*

$$\|\mathbf{T}^n\| \leqq M \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

then $\mathbf{T}$ *is similar to a contraction; more precisely, there exists an invertible self-*

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[1] By an "operator" we shall mean always a bounded linear operator, which is everywhere defined in the space; an operator will be said "invertible" if it has an everywhere defined, bounded inverse. For any operator $\mathbf{T}$ the norm $\|\mathbf{T}\|$ is defined as usual by the least upper bound of the length of the vector $\mathbf{Tx}$ when the vector $\mathbf{x}$ varies on the unit sphere.

[2] I. e. if $\mathbf{T}$ transforms any bounded set of vectors into a compact set; see [2] No. 85.

adjoint operator $\mathbf{Q}$ *on* $\mathcal{R}$ *such that the operator* $\mathbf{T}' = \mathbf{QTQ}^{-1}$ *is a contraction, i. e.*

$$\|\mathbf{T}'\| \leq 1 .$$

Proof. We make use of the „spectral radius theorem" (due to I. GELFAND in its general form for normed rings). For operators (on an arbitrary Banach space) it states that

$$\lim_{k\to\infty} \|\mathbf{T}^k\|^{\frac{1}{k}} = r(\mathbf{T})$$

where $r(\mathbf{T})$ is the „spectral radius" of the operator $\mathbf{T}$, i. e. the radius of the smallest closed disk in the plane of complex numbers, with centre at the point 0, which contains all the points of the spectrum of $\mathbf{T}$ (see [2] No. 149).

Thus the condition (1) implies that $r(\mathbf{T}) \leq 1$. Since $\mathbf{T}$ is completely continuous, its spectrum consists of a denumerable set of complex numbers having no accumulation point different from 0. Let $\lambda_1, \ldots, \lambda_n$ be those points of the spectrum of $\mathbf{T}$ (if any) which lie on the unit circle. The other points of the spectrum form a set σ_0 lying in some concentric circle of radius *smaller* than 1. To the decomposition

$$\sigma = \sigma_0 \cup \{\lambda_1\} \cup \ldots \cup \{\lambda_n\}$$

of the spectrum σ of $\mathbf{T}$ into isolated parts, there corresponds a decomposition

$$\mathbf{I} = \mathbf{P}_0 + \mathbf{P}_1 + \ldots + \mathbf{P}_n ,$$

of the identity operator $\mathbf{I}$ into a sum of idempotent operators with

$$\mathbf{P}_i \mathbf{P}_k = \mathbf{O} \qquad (i \neq k);$$

this decomposition is characterized by the property that $\mathbf{T}$ is permutable with all $\mathbf{P}_i$'s, and when considered as an operator on the subspace

$$\mathcal{R}_i = \mathbf{P}_i \mathcal{R} ,$$

$\mathbf{T}$ has the spectrum σ_0 if $i = 0$ and the one-point spectrum $\{\lambda_i\}$ if $i \neq 0$ (see [2] No. 148) ; moreover, as another consequence of complete continuity, the subspaces $\mathcal{R}_i$ $(i \neq 0)$ are all finite dimensional.

Consider $\mathbf{T}$ as an operator on the finite dimensional space $\mathcal{R}_i$ $(i \neq 0)$; we shall show that its matrix (in an arbitrary base of $\mathcal{R}_i$) has no non-linear elementary divisors. In the contrary case there should exist non-zero vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ in $\mathcal{R}_i$, forming a „Jordan-chain" of length $r \geq 2$, i. e. such that

$$\mathbf{T}\mathbf{x}_1 = \lambda_i \mathbf{x}_1 , \quad \mathbf{T}\mathbf{x}_2 = \lambda_i \mathbf{x}_2 + \mathbf{x}_1 , \quad \mathbf{T}\mathbf{x}_3 = \lambda_i \mathbf{x}_3 + \mathbf{x}_2 , \ldots , \qquad \mathbf{T}\mathbf{x}_r = \lambda_i \mathbf{x}_r + \mathbf{x}_{r-1}.$$

From the first two of these relations it results by iteration

$$\mathbf{T}^k \mathbf{x}_2 = \lambda_i^k \mathbf{x}_2 + k\,\lambda_i^{k-1} \mathbf{x}_1 \qquad (k = 1, 2, \ldots) .$$

Now by (1)

$$\|\mathbf{T}^k \mathbf{x}_2\| \leq M \|\mathbf{x}_2\| ;$$

since $|\lambda_i| = 1$, this implies

$$\|\lambda_i \mathbf{x}_2 + k\,\mathbf{x}_1\| \leq M \|\mathbf{x}_2\| .$$

This is impossible, for $\|\lambda_i \mathbf{x}_2 + k\,\mathbf{x}_1\| \to \infty$ if $k \to \infty$. Thus there can be no non-linear elementary divisors, and since the only point of the spectrum of

$\mathbf{T}$ on $\mathcal{R}_i$ is λ_i, we conclude that for all $\mathbf{x} \in \mathcal{R}_i$, $\mathbf{T}\,\mathbf{x} = \lambda_i\,\mathbf{x}$. We have shown, in other words, that

$$\mathbf{T}\,\mathbf{P}_i = \mathbf{P}_i\,\mathbf{T} = \lambda_i\,\mathbf{P}_i \qquad (i = 1, 2, \ldots, n)\,. \tag{2}$$

Consider now the operator $\mathbf{T}_0 = \mathbf{T}\,\mathbf{P}_0 = \mathbf{P}_0\,\mathbf{T}$. On $\mathcal{R}_0$ it coincides with $\mathbf{T}$, thus it has there the spectrum σ_0, while on the complementary subspace $(\mathbf{I} - \mathbf{P}_0)\,\mathcal{R}$ it is identically 0. The spectrum of $\mathbf{T}_0$ is thus equal to σ_0 or $\sigma_0 \cup \{0\}$; in any case, its spectral radius $r(\mathbf{T}_0)$ is < 1. Let p be a number between $r(\mathbf{T}_0)$ and 1. Then we have, by the spectral radius theorem,

$$\lim_{k\to\infty} \|\mathbf{T}_0^k\|^{\frac{1}{k}} < p < 1\,.$$

Thus, for sufficiently large values of k,

$$\|\mathbf{T}_0^k\| < p^k\,,$$

and consequently

$$\|(\mathbf{T}_0^k)^*\,\mathbf{T}_0^k\| < p^{2k}\,.$$

This implies that the operator series

$$\sum_{k=1}^{\infty} (\mathbf{T}_0^k)^*\,\mathbf{T}_0^k$$

converges in the operator norm.

Consider the operator

$$\mathbf{A} = \sum_{i=0}^{n} \mathbf{P}_i^*\,\mathbf{P}_i + \sum_{k=1}^{\infty} (\mathbf{T}_0^*)^k\,\mathbf{T}_0^k\,;$$

it is selfadjoint, and we have for any $\mathbf{x} \in \mathcal{R}$

$$(\mathbf{A}\,\mathbf{x}, \mathbf{x}) = \sum_{i=0}^{n} \|\mathbf{P}_i\,\mathbf{x}\|^2 + \sum_{k=1}^{\infty} \|\mathbf{T}_0^k\,\mathbf{x}\|^2 \geq \sum_{i=0}^{n} \|\mathbf{P}_i\,\mathbf{x}\|^2 \geq \frac{1}{n+1}\left(\sum_{i=0}^{n} \|\mathbf{P}_i\,\mathbf{x}\|\right)^2 \geq$$

$$\geq \frac{1}{n+1}\left\|\sum_{i=0}^{n} \mathbf{P}_i\,\mathbf{x}\right\|^2 = \frac{1}{n+1}\|\mathbf{x}\|^2\,,$$

thus

$$\mathbf{A} \geq \frac{1}{n+1}\,\mathbf{I}\,. \tag{3}$$

Let $\mathbf{Q}$ be the positive selfadjoint square-root of $\mathbf{A}$; from (3) it follows that $\mathbf{Q}$ has the positive lower bound $(n+1)^{-1/2}$. This assures that $\mathbf{Q}$ is invertible and $\|\mathbf{Q}^{-1}\| \leq (n+1)^{1/2}$.

Now we shall calculate $\mathbf{T}^*\,\mathbf{A}\,\mathbf{T}$. For $k \geq 1$ we have

$$\mathbf{T}_0^k\,\mathbf{T} = \mathbf{T}_0^{k-1}\,\mathbf{T}\,\mathbf{P}_0\,\mathbf{T} = \mathbf{T}_0^{k-1}\,\mathbf{T}\,\mathbf{P}_0^2\,\mathbf{T} = \mathbf{T}_0^{k-1}\,\mathbf{T}\,\mathbf{P}_0\,\mathbf{T}\,\mathbf{P}_0 = \mathbf{T}_0^{k+1}$$

and

$$\mathbf{T}^*(\mathbf{T}_0^k)^*\,\mathbf{T}_0^k\,\mathbf{T} = (\mathbf{T}_0^k\,\mathbf{T})^*\,(\mathbf{T}_0^k\,\mathbf{T}) = (\mathbf{T}_0^{k+1})^*\,\mathbf{T}_0^{k+1}\,,$$

thus

$$T^* \left(\sum_{k=1}^{\infty} (T_0^k)^* T_0^k \right) T = \sum_{k=2}^{\infty} (T_0^k)^* T_0^k .$$

Further we have

$$T^* P_0^* P_0 T = T_0^* T_0 ,$$

and for $i \neq 0$, using (2),

$$T^* P_i^* P_i T = (\lambda_i P_i)^* (\lambda_i P_i) = \bar{\lambda}_i \lambda_i P_i^* P_i = P_i^* P_i .$$

Summing these results we get

$$T^* A T = A - P_0^* P_0 .$$

From this relation it follows for any $y \in \mathcal{R}$

$$\|QTy\|^2 = (QTy, QTy) = (T^* Q^2 Ty, y) = (T^* ATy, y) =$$
$$= (Ay, y) - (P_0^* P_0 y, y) = (Qy, Qy) - (P_0 y, P_0 y) \leq \|Qy\|^2 .$$

For any $x \in \mathcal{R}$ put $y = Q^{-1} x$, then we get from this result that

$$\|QTQ^{-1} x\| \leq \|x\| ;$$

thus QTQ^{-1} is a contraction.

This concludes the proof of the theorem.

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REFERENCES

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TELJESEN FOLYTONOS OPERÁTOROKRÓL, AMELYEK ITERÁLTJAI EGYENLETESEN KORLÁTOSAK

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Kivonat

A következő tételt bizonyítjuk be: Legyen T a véges vagy végtelen dimenziós komplex euklidesi tér teljesen folytonos operátora, amelyre $\|T^n\| \leq M$ $(n = 1, 2, \ldots)$, ahol M állandó. Ekkor létezik egy olyan korlátos, és korlátos inverzű, önadjugált Q operátor, amelyre a $T' = QTQ^{-1}$ operátor *kontrakció*, azaz amelyre $\|T'\| \leq 1$.

О ВПОЛНЕ НЕПРЕРЫВНЫХ ОПЕРАТОРАХ С РАВНОМЕРНО ОГРАНИЧЕННЫМИ ИТЕРИРОВАННЫМИ

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Резюме

Доказывается следующая теорема: Пусть **T** вполне непрерывный оператор в комплексном эвклидовом пространстве конечной или бесконечной размерности, такой, что $\|\mathbf{T}^n\| \leqq M$ $(n = 1, 2, \ldots)$, где M постоянное. Тогда существует самосопряженный оператор **Q**, ограниченный и с ограниченным обратным $\mathbf{Q}^{-1}$, такой, что оператор $\mathbf{T}' = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^{-1}$ является сжатием, т. е. $\|\mathbf{T}'\| \leqq 1$.