AN ASSOCIATIVITY THEOREM FOR ALTERNATIVE RINGS

by

GYÖRGY GRÄTZER and E. TAMÁS SCHMIDT

It is well-known that (with the aid of the usual method due to HILBERT) every projective plane satisfying the special Desargues' theorem may be co-ordinated with an alternative division ring (of characteristic not 2). So it occured in the geometry the necessity of the investigation of a non-associative ring.[1]

The ring R is called alternative if for all $\alpha, \beta \in R$

$$(\alpha\alpha)\beta = \alpha(\alpha\beta) \quad \text{and} \quad (\beta\alpha)\alpha = \beta(\alpha\alpha).$$

The reason why the name alternative has been given is that the associator function

$$(\alpha, \beta, \gamma) = (\alpha\beta)\,\gamma - \alpha\,(\beta\gamma)$$

is an alternative function of its variables,[2] that is,

$$(\alpha, \beta, \gamma) = -(\alpha, \gamma, \beta) = -(\beta, \alpha, \gamma) = -(\gamma, \beta, \alpha).$$

The co-ordinate ring R is associative if and only if Desargues' theorem holds upon the projective plane. Therefore the consideration of associative subrings of alternative rings is of importance. At first E. ARTIN proved a theorem of this kind: in an alternative ring any subring generated by two elements is associative. Later on M. ZORN [5] generalized Artin's theorem. Further generalization was achieved by R. MOUFANG [2] for some special type of alternative rings, and this result was extended to the general case by M. F. SMILEY [4]. All the proofs need many computations. R. H. BRUCK and E. KLEINFELD [1] established a sufficient condition for the associativity of a subring which is a far-reaching generalization of the earlier results. The proof of this theorem — though it is not simple — is essentially simpler than that of the earlier ones. The proof of the theorem of BRUCK and KLEINFELD is simplified in nuances in the book [3] of G. PICKERT.

Our aim is to get a necessary and sufficient condition (the former ones were only sufficient) which contains all the conditions so far mentioned as special cases. The proof of this theorem is comparatively simple and requires

[1] Non-associative ring means a not necessarily associative ring. From now on we use the term "ring" for "non-associative ring".

[2] In rings of characteristic not 2 this property used to serve as a definition of alternative rings.

almost no computation. From this theorem we shall derive a further theorem of BRUCK and KLEINFELD [1] (which was not to be deduced from their associativity condition as a consequence of its being only sufficient).

If A, B and C are subsets of the alternative ring R, (A, B, C) denotes the set of all associators (α, β, γ) with $\alpha \in A$, $\beta \in B$, $\gamma \in C$. Let $D = \bigvee_{i=1}^{n} A_i$, $n \geqq 2$, the union of the subsets A_i of R. Let $A_{ij} (i \neq j$, i and $j \leqq n, n > 2)$ consist of products of at most $n-2$ factors taken from different A_k-s, $k \neq i$, $k \neq j$. If $n = 2$ or $i = j$ we put $A_{ij} = 0$. Finally we define $\overline{D}$ as the subring of R generated by D. Now, the general associativity condition is as follows:

Theorem. *Let R be an alternative ring and $D = \bigvee_{i=1}^{n} A_i$ $(n \geqq 2)$ a subset of R. D generates an associative subring $\overline{D}$ of R if and only if the following two conditions are fulfilled:*

$$(1) \qquad (A_i, A_i, \overline{D}) = 0,$$

$$(2) \qquad (A_i, A_j, A_{ij}) = 0. \qquad (i \text{ and } j = 1,2,\ldots, n)$$

The **proof** is based on the following observation:

(*) If $(\alpha_1, \beta, \gamma) = (\alpha_2, \beta, \gamma) = (\alpha_1, \alpha_2, \beta) = 0$

then

$$(\alpha_1\alpha_2, \beta, \gamma) = (\alpha_1, \beta, \alpha_2\gamma) = -(\alpha_1, \beta, \gamma\alpha_2).$$

For the proof of (*) we define — following [1] — a function as follows

$$f(w, x, y, z) = (wx, y, z) - (x, y, z)\, w - x\,(w, y, z).$$

It is easy to verify that this functions is multi-linear and skew-symmetric in its variables (i.e. it changes sign when two of its neighbouring variables are interchanged). Using these facts and the hypotheses we get

$$(\alpha_1\alpha_2, \beta, \gamma) = \alpha_2(\alpha_1, \beta, \gamma) + (\alpha_2, \beta, \gamma)\,\alpha_1 + f(\alpha_1, \alpha_2, \beta, \gamma) = f(\alpha_1, \alpha_2, \beta, \gamma).$$

On the other hand, in a similar way

$$(\alpha_1, \beta, \alpha_2\gamma) = f(\alpha_2, \gamma, \beta, \alpha_1).$$

Since $f(w, x, y, y)$ is skew-symmetric, we have $f(\alpha_1, \alpha_2, \beta, \gamma) = -f(\gamma, \alpha_2, \beta, \alpha_1) = f(\alpha_2, \gamma, \beta, \alpha_1)$. Comparing the three equalities we get the first part of (*). Similarly we get the second part of (*).

Now, we turn to the proof of our theorem. The necessity of the conditions (1) and (2) is obvious, for the assiciativity of $\overline{D}$ means — according to the definition of the associator — that $(\overline{D}, \overline{D}, \overline{D}) = 0$ and (1) and (2) are trivial consequences.

On the other hand, let us suppose the validity of (1) and (2); then we prove the associativity of $\overline{D}$. Let D^k denote the set of all products of k factors from D. Obviously, $\overline{D}$ is the module generated by all the D^k $(k = 1,2,\ldots)$. Since the associator function satisfies

$$(A_1 + A_2, B, C) = (A_1, B, C) + (A_2, B, C)$$

·t is enough to prove that the associator function vanishes on the elements of $\bigvee_{k=1}^{\infty} D^k$. Let u, v, w be positive integers; we use an induction on $u+v+w$ to verify $(D^u, D^v, D^w) = 0$. If $u + v + w = 3$ (which is the smaller possible value of $u + v + w$) we have to prove $(D, D, D) = 0$. This is trivial because every element of (D, D, D) is in the left side of (1) or (2), and so it is equal to 0. Now, we suppose the assertion is proved for $u + v + w < N$; let $u + v + w = N$ and $\alpha \in D^u$, $\beta \in D^v$, $\gamma \in D^w$. The elements α, β and γ are products of N factors altogether. Suppose, two of these factors are from the same A_i, say from A_1. Then we apply (*) repeatedly. From the induction hypothesis it follows that the assumptions of (*) are fulfilled. We decompose always that element[3] (or those elements) which contains a factor from A_1; and that factor is put in the first, later in the second argument which contains a factor from A_1. All other factors are gathered up in the third argument. Proceeding in this way we get finally a position, where in the first and in the second argument of the associator there will be elements of A_1 and so by (1) in this case the associator is 0.

Now let us suppose that all the N factors of α, β and γ are from different A_i-s. Then applying repeatedly (*) again, we let in the first and in the second arguments only single factors and all the others are gathered up in the third argument. Then by the hypothesis if in the first argument the factor is from A_i, whilst that in the second from A_j, then in the third argument the element belongs to A_{ij}, and so the associator is 0 according to (2). This completes the proof of the theorem.

We mention that the theorem might be generalized to the case when the number of the A_i-s is infinite. Naturally, then there is no bound on the number of factors of the elements of A_{ij}. Otherwise the theorem remains true, without any alteration.

Next, we turn our attention to special cases of this theorem.

Corollary 1. (Theorem of BRUCK and KLEINFELD.) *Let A_1, A_2, A_3 be subsets of an alternative ring R such that $(A_1, A_1, R) = (A_2, A_2, R) = (A_3, A_3, R) = (A_1, A_2, A_3) = 0$. Then the subset $D = A_1 \cup A_2 \cup A_3$ is contained in an associative subring of R.*

In case $A_1 = A_2$ this assertion was first proved by M. ZORN.

Indeed in case $n = 3$ A_{12} means A_3, thus the hypotheses of our theorem are fulfilled and we get the theorem of BRUCK and KLEINFELD.

Corollary 2. *Let D be a subset of an alternative ring R and a_ν $(\nu \in \Omega)$ the elements of D. If $D_{\mu\omega}$ $(\mu, w \in \Omega)$ denotes the set of the products of the a_ν-s with a_μ and a_ω excluded, then D generates an associative subring in R if and only if $(a_\mu, a_\omega, D_{\mu\omega}) = 0$ for all $\mu, w \in \Omega$, $\mu \neq \omega$.*

A special case of Corollary 2 is

Corollary 3. (Theorem of SMILEY.) *The elements α, β, γ of the alternative ring R generate an associative subring if and only if $(\alpha, \beta, \gamma) = 0$.*

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[3] An element of R may be considered as a product of one factor; multiplying a product of k by one of l factors, we get a product of $k + l$ factors. So a product of n factors always might be decomposed to two elements, both of them having less than n factors.

By the aid of our theorem we can easily prove also the following assertion of BRUCK and KLEINFELD [1]:

Corollary 4. *Let A be an associative subring of the alternative ring R, and B a subset of R subject to $(A, A, B) = (B, B, R) = 0$. Then $D = A \cup B$ generates an associative subring of R.*

Proof. In case $n = 2$ condition (2) is trivially satisfied, further $(B, B, R) = 0$ is now supposed, hence in order to prove the validity of the conditions of the theorem it is enough to see that $(A, A, \bar{D}) = 0$. Let $\gamma \in \bigvee_{k=1}^{\infty} D^k$ and denote by n the least integer such that γ is the product of n factors taken from A and B. We choose $a_1, a_2 \in A$ and by an induction on n we prove $(\gamma, a_1, a_2) = 0$ which will complete the proof. In case $n = 1$, γ is from A or from B and so $(\gamma, a_1, a_2) = 0$ follows from the associativity of A, resp. from $(A, A, B) = 0$. If the assertion is true for all $k < n$, then we again apply (*): decomposing γ we can modify the associator (γ, a_1, a_2) in such a way, that all the factors of γ which are taken from B are gathered up in the third argument, while the others are collected in the first argument. Since the associator (without altering its value) will be an element of (A, A, B), it is equal to 0.

We note — following [1] — that Corollary 4 immediately implies the following result:

All the maximal associative subrings of the alternative ring R are maximal associative subsets, that is, they can not be enlarged so that all the associators should remain 0.

Indeed, in case the maximal associative subring A can be enlarged with the element β so that the associators remain 0, then $(A, A, \beta) = 0$ and obviously $(\beta, \beta, R) = 0$, so, by Corollary 4, A and β generate an associative subring, in contradiction to the maximality of A.

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EGY ASSZOCIATIVITÁSI TÉTEL ALTERNATÍV GYŰRŰKRE

GRÄTZER GY. és SCHMIDT E. T.

Kivonat

Legyen R egy alternatív gyűrű. Ha α, β, γ elemei R-nek, akkor (α, β, γ) jelöli az $(\alpha\beta)\gamma - \alpha(\beta\gamma)$ elemet. Az (α, β, γ) függvényt asszociátor-függvénynek nevezzük, s R alternativitása azt jelenti, hogy az asszociátor-függvény változóinak alternatív függvénye. Ha A, B, C részhalmazai R-nek, akkor (A, B, C)

jelölje az összes olyan (α, β, γ) asszociátor halmazát, amelyre $\alpha \in A$, $\beta \in B$, $\gamma \in C$. Legyenek az A_i-k az R gyűrűk részhalmazai, s $D = \bigvee_{i=1}^{n} A_i$ $(n \geqq 2)$. Definiáljuk az A_{ij} részhalmazt, hogy $i \neq j$, $n > 2$ esetén álljon mindazon legfeljebb $n-2$ tényezős szorzatokból, amelyeknek minden tényezőjét különböző A_i-kból vettük, s $i = j$ vagy $n = 2$ esetén legyen $A_{ij} = 0$. Végezetül legyen $\bar{D}$ az R gyűrűnek a D részhalmaz által generált részgyűrűje. A dolgozat célja, hogy bebizonyítsa a következő állítást:

Tétel. *Legyen R egy alternatív gyűrű, A_i részhalmazai R-nek, s $D = \bigvee_{i=1}^{n} A_i$ $(n \geqq 2)$. D akkor és csak akkor generálja R-nek asszociatív részgyűrűjét, ha a következő két feltétel teljesül:*

$$(1) \qquad (A_i, A_i, \bar{D}) = 0, \qquad (i, j = 1, 2, \ldots, n)$$

$$(2) \qquad (A_i, A_j, A_{ij}) = 0.$$

A tétel bizonyítása a következő lemmából egyszerűen adódik:

Tegyük fel, hogy az R alternatív gyűrű $\alpha_1, \alpha_2, \beta, \gamma$ elemeire

$$(\alpha_1, \beta, \gamma) = (\alpha_2, \beta, \gamma) = (\alpha_1, \alpha_2, \beta) = 0,$$

ekkor

$$(\alpha_1\alpha_2, \beta, \gamma) = (\alpha_1, \beta, \alpha_2\gamma) = -(\alpha_1, \beta, \gamma\alpha_2).$$

A bizonyítás egyszerű teljes indukciós meggondolással, minden számolás nélkül adódik.

A tételből az $n = 3$ esetben nyerhető Bruck és Kleinfeld [1] asszociativitás-feltétele, s tovább specializálva Smiley és Zorn tételei. Szintén egyszerűen nyerhető Bruck és Kleinfeld [1] egy további tétele is. A dolgozatban szereplő tétel bizonyítása egyszerűbb, mint az eddigi speciális eseteké.

ОДНА ТЕОРЕМА ОБ АССОЦИАТИВНОСТИ ДЛЯ АЛЬТЕРНАТИВНЫХ КОЛЕЦ

GY. GRÄTZER и E. T. SCHMIDT

Резюме

Пусть R есть альтернативное кольцо. Если α, β, γ элементы R, то (α, β, γ) обозначает элемент $(\alpha\beta)\gamma - \alpha(\beta\gamma)$. Функция (α, β, γ) называется ассоциатор-функцией, и альтернативность R означает альтернативность этой функции. Если A, B, C подмножества R, то пусть (A, B, C) обозначает множество всех ассоцаторов (α, β, γ), для которых $\alpha \in A, \beta \in B, \gamma \in C$. Пусть A_i суть подмножества R и $D = \bigvee_{i=1}^{n} A_i$ $(n \geqq 2)$. Определим подмножество A_{ij}, чтобы при $i \neq j$, $n > 2$ оно состояло из всех произведений $n-2$ сомножителей, каждый из которых принадлежит различным A_i, а в случае $i = j$ или $n = 2$ $A_{ij} = 0$. Наконец, пусть $\bar{D}$ будет подкольцо кольца R, генерированное множеством D. Цель работы доказать следующее утверждение:

Теорема. *Пусть R альтернативное кольцо, A_i его подмножества, $D = \bigvee_{i=1}^{n} A_i$ $(n \geqq 2)$. D в том и только в том случае генерирует ассоциативное подмножество R, если выполняются следующие два условия:*

$$(1) \qquad (A_i, A_i, \overline{D}) = 0\,, \qquad (i, j = 1, 2, \ldots, n)$$

$$(2) \qquad (A_i, A_j, A_{ij}) = 0\,.$$

Доказательство теоремы легко получается из следующей леммы:

Предположим, что для элементов $\alpha_1, \alpha_2, \beta, \gamma$ альтернативного кольца R

$$(\alpha_1, \beta, \gamma) = (\alpha_2, \beta, \gamma) = (\alpha_1, \alpha_2, \beta) = 0,$$

тогда

$$(\alpha_1\alpha_2, \beta, \gamma) = (\alpha_1, \beta, \alpha_2\gamma) = -(\alpha_1, \beta, \gamma\alpha_2).$$

Доказательство получается простым соображением с помощью математической индукции без всякого счёта.

Из теоремы в случае $n = 3$ может было получено условие ассоциативности Bruck-а и Kleinfeld-а [1] и дальнейшей специализацией теорема Smiley и Zorn-а. Легко получить и ещё одну теорему Bruck-а и Kleinfeld-а [1]. Доказательство теоремы проще, чем доказательство этих специальных случаев.