ON THE PROPAGATION OF GRAVITATIONAL WAVES

by

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Dedicated to Professor K. Novobátzky, for his 75^{th} birthday.

§ 1. Introduction

The study of gravitational waves in general relativity suffers from an inherent difficulty which arises from the principle of equivalence and the corresponding lack of definitiveness of coordinate systems. If other aspects of the surrounding Universe are disregarded, it is quite possible to produce gravitational waves (even under perfectly flat conditions) by introducing suitable spacetime coordinates, and the objectivity of wave phenomena depends partly on the objectivity with which one can select a reliable coordinate system. The difficulty affects both the dynamical and kinematical aspects of wave propagation. Since field energy can only be calculated from the well known pseudotensor, it is impossible to make precise statements concerning the amount of energy transferred by gravitational radiation unless the coordinate system is specified; and the same is true for statements on the propagation of gravitational signals.

For the case of weak fields, EINSTEIN himself proposed a solution in the early days of relativity. EINSTEIN showed that if there exists a coordinate system in which deviations from the Minkowskian values are everywhere small then one can select by means of suitable auxiliary conditions a class of reasonably good almost Minkowskian coordinate systems in which gravitational signals propagate with the velocity of light. The idea of auxiliary conditions has been exploited with great success in recent times, particularly by V. FOCK and his school. FOCK calls a coordinate system *harmonic*[2] if it satisfies

$$(1) \qquad \sum_{\mu=0}^{3} \frac{\delta}{\delta x_\mu} \left((-g)^{1/2} g^{\mu\nu}\right) = 0, \; g = \det g_{\mu\nu}, \quad (\nu = 0, 1, 2, 3);$$

it is usually possible to satisfy these conditions, and the coordinate system so obtained has some rather attractive properties, particularly where wave propagation is concerned. One of these properties is that under ordinary circumstances and suitable boundary conditions at infinity, harmonic coordinates are uniquely determined, apart from the arbitrariness of a Lorentz transformation. Hence the suggestion, put forward by FOCK, that harmonic

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[2] For a concise account see V. Fock, Rev. Mod. Phys. **29** (1957), 325—333.

coordinates should have the same status in general relativity theory as inertial frames have in special relativity.

It is not altogether easy to assess the correct significance and position of the harmonic condition (1) within the framework of general relativity. Superficially, it resembles the famous Lorentz condition on the vector potential in electrodynamics which is essentially a mathematical device to obtain solutions of the field equations in a convenient form. But whereas the Lorentz condition has no influence on the (physically observable) electromagnetic field forces, the harmonic condition is to characterize a class of "natural" frames of reference in which for instance gravitational energy changes can conveniently be represented. But when a physicist chooses a space-time coordinate system to describe gravitational effects (e. g. in the solar system), he will mainly be guided by visual considerations such as observations of the position of distant stars or measurements of DOPPLER shift in the radiation from these distant sources. Now there is no a priori reason why this "visual" frame should satisfy the harmonic condition, and in the present paper we shall discuss a situation where the two coordinate systems (namely the visual and the harmonic) are definitely not identical.

In this connection it is interesting to note that the Schwartzschild frame, which is most commonly used to describe the centrosymmetrical statical field because of its formal simplicity, does not satisfy the harmonic condition. Of course the centrosymmetric harmonic frame with the line element

$$\left(1-\frac{\mu}{r}\right)\left(1+\frac{\mu}{r}\right)^{-1}\left(dt^2-\frac{\mu^2}{r^2}\,dr^2\right)-\left(1+\frac{\mu}{r}\right)^2\sum_{k=1}^{3}dx_k^2$$

is physically just as acceptable and leads to the same observational effects as the Schwartzschild line element

$$\left(1-\frac{2\mu}{r}\right)dt^2-\frac{2\mu}{r-2\mu}\,dr^2-\sum_{k=1}^{3}dx_k^2 .$$

But beyond this assertion the harmonic system does not offer any particular advantages.

This example shows at any rate that under suitable circumstances symmetry conditions can be quite an effective substitute for auxiliary conditions. The intrinsic strength of symmetry conditions is that if they are applicable to all, they are comparatively immune to objections; if a configuration admits certain geometrical symmetries, it seems to be sound philosophy to use a coordinate system which exhibits these symmetries.

In the present note we shall use this principle to examine the existence and propagation of pure gravitational waves, in the special case of a rotating ellipsoidal body. The inherent symmetries of this model enable us to guess the general form of an adequate line element with reasonable certainty, and it will be possible to obtain a reliable picture of the generated waves without imposing the harmonic condition. As expected, we shall find spherical (or almost spherical) gravitational potential waves spreading out with the velocity of light, but the form of the waves and in particular the law of decrease of their amplitude, will depend quite essentially on whether the visual or the harmonic frame is used.

§ 2. The rotating ellipsoid

We consider an ellipsoidal body of gravitational radius μ and principal semi-axes $a(1+\varepsilon_1)$, $a(1+\varepsilon_2)$, $a(1+\varepsilon_3)$, $\varepsilon_1 \neq \varepsilon_2$,

(2) $$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0\,,$$

which rotates with constant angular velocity ω about the axis of $a(1+\varepsilon_3)$. The velocity of light at infinity is taken to be 1. In order to avoid the complexities of an exact solution, we assume that the eccentricities ε_p are small and neglect all quantities in which ε_p^2 or $\omega^2\mu^2\varepsilon_p$ appears. These approximations are quite adequate for the purpose of finding out the essential characteristics of the generated waves. Occasionnally we shall refer to "small" values of r; by this we mean that ω_r is small. Otherwise no restriction will be placed on the angular velocity itself.

With the above mentioned approximation the Newtonian potential outside the body at rest is

(3) $$V = \mu/r + \frac{3}{2} M_r^{-3} \sum_{p=1}^{3} \varepsilon_p H_p^2\,.$$
$$H_p = \sum_{k=1}^{3} A_{pk}\,\xi_k, \quad \xi_k = x_k/r\,,$$

where M is the moment of inertia of the ellipsoid about its axis of rotation and

$$\mathbf{A}_p = (A_{p1}, A_{p2}, A_{p3}), \quad p = 1, 2, 3$$

are mutually perpendicular unit vectors in the directions of the (body-fixed) principal axes. Suffices such as m, n, p will always appear in the lower position and run from 1 to 3; the time suffix 0 will usually be written out separately. The summation convention will only be used in conjunction with ε_p (or ε_p^* defined below) and then in a rather unconventional manner: the index is required to appear three times in the term. Thus $\varepsilon_p H_p A_{pm}$ denotes $\sum_{p=1}^{3} \varepsilon_p H_p A_{pm}$ and $\varepsilon_p H_p^2$ denotes $\sum_{p=1}^{3} \varepsilon_p H_p H_p$. On the other hand $\varepsilon_m H_m$, $\varepsilon_m^* \xi_m$ are not to be summed for m. Note that

(4) $$\sum_{m=1}^{3} \varepsilon_p A_{pm} A_{pm} = \sum_{p=1}^{3} \varepsilon_p = 0$$

because of the orthogonality of the $\mathbf{A}_p$ and the normalization (2).

If the x_3-axis is placed in the direction of the (space and body-fixed) rotation axis and the time origin is suitably chosen we can set

(5) $$\mathbf{A}_1 = (\cos\omega t, \sin\omega t, 0), \quad \mathbf{A}_2 = (-\sin\omega t, \cos\omega t, 0),$$
$$\mathbf{A}_3 = (0, 0, 1)$$

hence

$$H_1^2 = \frac{1}{2}(\xi_1^2 + \xi_2^2) + \frac{1}{2}(\xi_1^2 - \xi_2^2)\cos 2\,\omega t + \xi_1\,\xi_2 \sin 2\,\omega t$$

$$H_2^2 = \frac{1}{2}(\xi_1^2 + \xi_2^2) - \frac{1}{2}(\xi_1^2 - \xi_2^2)\cos 2\,\omega t - \xi_1\,\xi_2 \sin 2\,\omega t$$

$$H_3^2 = \xi_3^2\,,$$

and

(6) $$\varepsilon_p\,H_p^2 = \varepsilon_p^*\,\xi_p^2 + \varepsilon\cos^2\Theta\cos 2(\varphi - \omega t)$$

where

(7) $$\varepsilon_1^* = \varepsilon_2^* = \frac{1}{2}(\varepsilon_1 + \varepsilon_2),\ \ \varepsilon_3^* = \varepsilon_3,\ \ \varepsilon = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)$$

and

(8) $$\xi_1 = \cos\varphi\cos\Theta,\ \ \xi_2 = \sin\varphi\cos\Theta,\ \ \xi_3 = \sin\Theta.$$

Note furthermore that

(9a) $$\varepsilon_p\,H_p^2 = \varepsilon_p^*\,\xi_p^2 - \frac{1}{4\,\omega^2}\,\varepsilon_p(H_p^2)^{\cdot\cdot}$$

(9b) $$\varepsilon_p\,H_p\,A_{pm} = \varepsilon_m^*\,\xi_m - \frac{1}{4\,\omega^2}\,\varepsilon_p(H_p A_{pm})^{\cdot\cdot}$$

(9c) $$\varepsilon_p\,A_{pm}\,A_{pn} = \frac{1}{2}(\varepsilon_m^* + \varepsilon_n^*)\,\delta_{mn} - \frac{1}{4\,\omega^2}\,\varepsilon_p(A_{pm}\,A_{pn})^{\cdot\cdot}$$

where $(\,\cdot\,) = \partial/\partial t$. Hence

(10a) $$\varepsilon_p(H_p^2)^{\cdot\cdot\cdot} = -\,4\,\omega^2\,\varepsilon_p(H_p^2)^{\cdot}$$

and similar expressions for $\varepsilon_p(H_p\,A_{pm})^{\cdot\cdot\cdot}$, $\varepsilon_p(A_{pm}\,A_{pn})^{\cdot\cdot\cdot}$.

To obtain the most general form of a coordinate system with symmetries appropriate to the problem, we employ the following device: We set tentatively

$$g_{00} = 1 - 2\,V,\ \ g_{0n} = 0,\ \ g_{mn} = -\,\delta_{mn}$$

where V is given by (3), and calculate the RICCI tensor. In order to be able to satisfy EINSTEIN's equations, we must clearly allow the metric tensor to contain all types of terms which appear in the RICCI tensor. This consideration finally leads to the following assumption:

(11a) $$g_{00} = 1 - 2\,\mu r^{-1} - 3\,Mr^{-3}\,\varepsilon_p^*\,\xi_p^* - \omega^2\,V_1\,\varepsilon_p(H_p^2)^{\cdot} - \omega V_2\,\varepsilon_p(H_p^2)^{\cdot\cdot},$$

(11b) $$g_{0n} = \omega^2\,B_1\,\varepsilon_p(H_p\,A_{pn})^{\cdot} + \omega\,B_2\,\varepsilon_p(H_p\,A_{pn})^{\cdot\cdot} + \omega^2\,C_1\,\varepsilon_p(H_p^2)^{\cdot}\,\xi_n + \\ + \omega\,C_2\,\varepsilon_p(H_p^2)^{\cdot\cdot}\,\xi_n,$$

$$
\begin{aligned}
-g_{mn} = {} & \delta_{mn} + 2\mu r^{-1}\xi_m\xi_n + D_0 r^{-3}\varepsilon_p^*\xi_p^2\delta_{mn} + E_0 r^{-3}\varepsilon_p^*\xi_p^2\xi_m\xi_n + \\
& + F_0 r^{-3}(\varepsilon_m^* + \varepsilon_n^*)\xi_m\xi_n + G_0 r^{-3}(\varepsilon_m^* + \varepsilon_n^*)\delta_{mn} + \\
& + \omega^2 D_1\varepsilon_p(H_p^2)^{\cdot}\delta_{mn} + \omega D_2\varepsilon_p(H_p^2)^{\cdot\cdot}\delta_{mn} + \\
& + \omega^2 E_1\varepsilon_p(H_p^2)^{\cdot}\xi_m\xi_n + \omega E_2\varepsilon_p(H_p^2)^{\cdot\cdot}\xi_m\xi_n + \\
& + \omega^2 F_1\varepsilon_p[(H_p A_{pm})^{\cdot}\xi_n + (H_p A_{pn})^{\cdot}\xi_m] + \\
& + \omega F_2\varepsilon_p[(H_p A_{pm})^{\cdot\cdot}\xi_n + (H_p A_{pn})^{\cdot\cdot}\xi_m] + \\
& + \omega^2 G_1\varepsilon_p(A_{pm}A_{pn})^{\cdot} + \omega G_2\varepsilon_p(A_{pm}A_{pn})^{\cdot\cdot},
\end{aligned} \tag{11c}
$$

where $D_0, \ldots, G_0$ are constants and $B_1, B_2, \ldots, V_2$ are functions of ωr. Hence $\partial B_1/\partial r = \omega B_1'$ etc.

Strictly speaking one ought to add a term of the form $2Mr^{-2}\Omega_m$ to the expression for g_{0n} where $\Omega_1 = -\varepsilon_2\omega$, $\Omega_2 = \varepsilon_1\omega$, $\Omega_3 = 0$ to take into account the motion of the rotating matter relatively to the coordinate frame[3] but it can be omitted as it has no relevance to our problem. We may imagine the field created by pulsation rather than actual rotation in which case the term does not appear at all. Also terms containing $(\mu/r)^2$ have been omitted as they have no effect on further calculations.

The amount of arbitrariness of the coordinate system is expressed by the transformation

$$
t = \bar{t} + \omega^2 T_1\varepsilon_p(H_p^2)^{\cdot} + \omega T_2\varepsilon_p(H_p^2)^{\cdot\cdot}, \tag{12a}
$$

$$
\begin{aligned}
x_m = {} & \bar{x}_m + \omega^2 M_1\varepsilon_p(H_p A_{pm})^{\cdot} + \omega M_2\varepsilon_p(H_p A_{pm})^{\cdot\cdot} + \\
& + \omega^2 N_1\varepsilon_p(H_p^2)^{\cdot}\xi_m + \omega N_2\varepsilon_p(H_p^2)^{\cdot\cdot}\xi_m
\end{aligned} \tag{12b}
$$

where $T_1, \ldots, N_2$ are functions of $\omega\bar{r}$, of the same general order of smallness as the coefficients of the metric tensor in (11). In the new coordinates

$$
\bar{g}_{00} = g_{00} - 8\omega^3 T_2\varepsilon_p(H_p^2)^{\cdot} + 2\omega^2 T_1\varepsilon_p(H_p^2)^{\cdot\cdot}, \tag{13a}
$$

$$
\begin{aligned}
\bar{g}_{0n} = {} & g_{0n} + \omega^3\left(T_1' - \frac{2}{\omega\bar{r}}T_1 + 4N_2\right)\varepsilon_p(H_p^2)^{\cdot}\xi_n + \\
& + \omega^2\left(T_2' - \frac{2}{\omega\bar{r}}T_2 - N_1\right)\varepsilon_p(H_p^2)^{\cdot\cdot}\xi_n + \\
& + \omega^3\left(\frac{2}{\omega\bar{r}}T_1 + 4M_2\right)\varepsilon_p(H_p A_{pn})^{\cdot} + \\
& + \omega^2\left(\frac{2}{\omega\bar{r}}T_2 - \omega^2 M_1\right)\varepsilon_p(H_p A_{pn})^{\cdot\cdot}
\end{aligned} \tag{13b}
$$

by (10), where g_{00}, g_{0n} are the same expressions in the new coordinates as they were in the old ones; a similar formula can be obtained for $\bar{g}_{mn}$. In the calcula-

[3] J. Lense and H. Thirring, Phys. Z. **19** (1918), 156—163.

tion of (13) we have made no distinction between differentiation with respect to t and $\bar{t}$; this is permissible in the present approximation.

By a suitable determination of M_i, N_i, T_i, $i = 1, 2$, we can evidently achieve that g_{00} be time-independent and $g_{0n} = 0$. However, the time-coordinate corresponding to the first assumption is not likely to be useful to an actual observer, and we prefer not to specify the coordinate system at this stage but try to satisfy the Einstein equations

$$R_{\mu\nu} = 0$$

with the general line element (11).

The computation of the Ricci tensor is straightforward but rather tedious, and details will be suppressed. One gets a system of 14 equations for the unknown functions $B_i, \ldots, V_i$, which turn out to be compatible and which can be reduced to the following system:

$$P_i'' + \frac{2}{x} P_i' - \frac{6}{x^2} P_i + 4 P_i = 0 \tag{14}$$

$$H_i = - x P_i' - 2 P_i + x K_i' \tag{15}$$

$$L_i = \frac{1}{2} x (P_i' + H_i') - (P_i + H_i) \tag{16}$$

$$V_i = 2 x^2 K_i + \frac{1}{2} x P_i' + 2(1 - x^2) P \tag{17}$$

$(i = 1, 2)$, where

$$x = \omega r \tag{18}$$

and the quantities P_i, H_i, K_i, L_i are given by

$$P_i = D_i - \frac{1}{2} F_i - \frac{1}{2} \overline{B_i'} + \frac{1}{2x} \overline{B_i} + \frac{1}{x} \overline{C_i}, \tag{19}$$

$$H_i = F_i + \overline{B_i'} - \frac{1}{x} \overline{B_i} + \frac{2}{x} \overline{C_i}, \tag{20}$$

$$K_i = G_i + \frac{2}{x} \overline{B_i}, \tag{21}$$

$$L_i = E_i + 2 \overline{C_i'} - \frac{6}{x} \overline{C_i} \tag{22}$$

with

$$\overline{B_1} = B_2, \quad \overline{B_2} = -\frac{1}{4} B_1, \quad \overline{C_1} = C_2, \quad \overline{C_2} = -\frac{1}{4} C_1. \tag{23}$$

Equation (14) shows that $x^{1/2} P_1$ and $x^{1/2} P_2$ are Bessel functions of, $2x$ of order $\pm \frac{5}{2}$; therefore

$$P_i = \alpha_i \left[\left(\frac{1}{x} - \frac{3}{4x^3} \right) \sin(2x + \beta_i) + \frac{3}{2x^2} \cos(2x + \beta_i) \right], \qquad i = 1, 2, \tag{24}$$

where α_i, β_i are integration constants. P_i is the only combination of the field quantities which can be determined from the equations independently of the arbitrarniness of the coordinate system. The others depend on three pairs of arbitrary functions, namely B_i, C_i and G_i, corresponding to the free choice of M_i, N_i and T_i in (12). We can dispense with the first two by setting

$$B_i = C_i = 0; \tag{25}$$

it gives $g_{0n} = 0$, that is a stationary coordinate system.[4] Thus we are left with a pair of free functions G_1 and G_2; the only restriction on these functions is that they should vanish sufficiently rapidly at infinity.

In order that V_i vanish at infinity, we must have by (17) and (24)

$$K_i = G_i = \alpha_i \left[\frac{1}{x} \sin(2x + \beta_i) + \frac{1}{x^2} \cos(2x + \beta_i) \right] - \\ - \frac{\gamma_i}{4x^3} \sin(2x + \beta_i) + \frac{Q_i}{x^2} \tag{26}$$

where $Q_i \to 0$ for $x \to \infty$ and γ_i is a constant. For clarity we have separated out the part with γ_i from Q_i; the latter can then be regarded as an "aperiodic" part whose appearance is due to an inappropriate choice of coordinates. By setting $Q_i = 0$ and substituting (25), (26) into (15)—(22), we get

$$V_i = \alpha_i \left[\left(\frac{3}{2x} - \frac{3}{8x^3} \right) \sin(2x + \beta_i) + \frac{3}{4x^2} \cos(2x + \beta_i) \right] - \frac{\gamma_i}{2x} \sin(2x + \beta_i) \tag{27}$$

$$D_i = \alpha_i \left[\left(\frac{1}{2x} - \frac{9}{8x^3} \right) \sin(2x + \beta_i) + \frac{5}{4x^2} \cos(2x + \beta_i) \right] + \\ + \gamma_i \left[\frac{3}{8x^3} \sin(2x + \beta_i) - \frac{1}{4x^2} \cos(2x + \beta_i) \right], \tag{28}$$

$$L_i = (2\alpha_i - \gamma_i) \left[\left(-\frac{1}{2x} + \frac{15}{8x^3} \right) \sin(2x + \beta_i) - \frac{7}{4x^2} \cos(2x + \beta_i) \right], \tag{29}$$

$$H_i = -\alpha_i \left[\left(\frac{1}{x} + \frac{3}{4x^3} \right) \sin(2x + \beta_i) + \frac{1}{2x^2} \cos(2x + \beta_i) \right] + \\ + \gamma_i \left[\frac{3}{4x^3} \sin(2x + \beta_i) - \frac{1}{2x^2} \cos(2x + \beta_i) \right]. \tag{30}$$

[4] We shall find later that the harmonic condition leads to a non-zero determination of B_i. The obvious advantage of stationary coordinates is that it allows a clear separation into "space" and "time". The potential function V_i itself is not affected by the choice of B_i.

The most interesting feature of the solution is the appearance of the constant γ_i in the expression for V_i. It means that the observed fluctuation of the gravitational potential depends significantly on the selected value of γ_i and its determination must be regarded as a physical problem rather than a matter of mathematical convenience.

Fortunately it is possible to fix the coordinate system on physical grounds, in spite of the principle of equivalence, because of the near-centrosymmetry and one-body character of the problem which allows us to localize the source of oscillations at the origin. Clearly a good coordinate system is one which does not take part in the radial oscillations of the gravitational acceleration. Now any oscillation of this kind can be detected visually, by observing variations of DOPPLER shift in a stream of light which comes from a distant source in radial direction. Therefore we must seek a coordinate frame in which such a DOPPLER shift effect is absent, and this will certainly be the case if we can choose the system so that the radial velocity of light at every spatially fixed point is constant in time. In terms of the metric tensor (11) we have the condition that

$$V_i + D_i + E_i + 2\,F_i + G_i = 0 \tag{31}$$

for all x. It is not altogether obvious that equation (31) can be satisfied at all. But if we substitute the expressions (26)—(30) into (31) we get

$$(3\,\alpha_i - \gamma_i)\left[\frac{1}{4\,x^3}\sin(2\,x + \beta_i) - \frac{1}{2\,x^2}\cos(2\,x + \beta_i)\right] = 0 \tag{32}$$

and we see that (32) holds provided that γ_i is determined from

$$\gamma_i = 3\,\alpha_i \qquad (i = 1, 2)\,. \tag{33}$$

There is another circumstance which favours this selection of γ_i: as seen from (27), it is the only value of γ_i which leads to a $1/x^2$ law of decrease of the gravitational potential (hence also of the periodic acceleration) for large x. We have therefore good reason to believe that (33) is the physically most acceptable value of γ_i.

It is interesting to compare (33) with the value obtained for γ_i in a harmonic coordinate system. It can be shown that the harmonic condition

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial x_\mu} g^{\mu\nu} + \sum_{\mu=0}^{3}\sum_{\lambda=0}^{3} \left\{ {\lambda \atop \mu\lambda} \right\} g^{\mu\nu} = 0 \qquad (\nu = 0, 1, 2, 3) \tag{34}$$

leads to the equations

$$\frac{1}{2}\,x^2(K_i'' - P_i'') + 3\,x(K_i' - P_i') + 2\,x^2(K_i - P_i) - P_i = 0\,, \tag{35}$$

$$x^2\,B_i'' + 2\,x\,B_i' + 4\,x^2\,B_i - 2\,B_i + 4\,C_i = 0\,, \tag{36}$$

$$x^2\,C_i'' + 2\,x\,C_i' + 4\,x^2\,C_i - 12\,C_i = 0\,. \tag{37}$$

From the last equation it follows that $x^{1/2} C_i$ is a Bessel function of order $\pm \frac{7}{2}$ in $2x$; but as C_i cannot be of order x^{-4} for small x, it must be of the form

$$C_i = \delta_i \left[\left(\frac{15}{8\,x^4} - \frac{3}{x^2} \right) \sin 2\,x + \left(- \frac{15}{4\,x^3} + \frac{1}{x} \right) \cos 2\,x \right].$$

From (36) we find that $x^{1/2} B_i = - \frac{2}{5} x^{1/2} C_i$ plus a Bessel function of order $\pm \frac{3}{2}$ which is admissible as it behaves well both for small and large x. Thus there is a considerable indeterminacy in the harmonic coordinate system due to the degenerate character of the problem of uniform rotation. The indeterminacy will be removed in section 3 where non-uniform rotation is considered. We shall find that the correct determination of B_i and C_i is such that

$$\overline{B}_i' - \frac{1}{x} \overline{B}_i + P_i = 0, \qquad C_i = 0. \tag{38}$$

Hence $B_i \neq 0$ and the harmonic system is not stationary. With the above determination, equations (19), (20) and (22) take the simple form

$$\frac{1}{2} P_i = D_i - \frac{1}{2} F_i\,, \tag{19*}$$

$$H_i = F_i - P_i\,, \tag{20*}$$

$$L_i = E_i\,. \tag{22*}$$

Finally (35) gives with (24)

$$K_i = \alpha_i \left[\left(\frac{1}{x} - \frac{1}{2\,x^3} \right) \sin (2\,x + \beta_i) + \frac{1}{x^2} \cos (2\,x + \beta_i) \right] \tag{39}$$

plus a term $x^{-5/2} J_i$ where J_i is a Bessel-function of order $\frac{5}{2}$ in $2x$: It will be found that in the correct frame this last part vanishes and K_i is given by (39). Comparing with (26) we find that $Q_i = 0$ and

$$\gamma_i = 2\,\alpha_i\,. \tag{40}$$

An attractive feature of the harmonic frame is that it leads to a particularly simple form of the solution, viz.,

$$V_i = D_i = \frac{1}{2} P_i\,, \quad E_i = F_i = 0\,, \tag{41}$$

as seen easily from (19*), (20*), (22*), (27)—(30) and (40). On the other hand it gives $\gamma_i = 2\alpha_i$ instead of (33), and in view of the previous discussion it is not very likely that an actual observer will accept the harmonic frame as a suitable coordinate system.

To represent outgoing waves only and to suit the given initial conditions for small r, we set $\beta_i = 0$, $\beta_2 = \frac{\pi}{2}$, $\alpha_1 = 2\alpha_2 = 2M$; it gives

$$
(42a) \quad \begin{aligned} g_{00} = 1 - 2\,\mu r^{-1} - 3\,\varepsilon_3 M r^{-3} \left(1 - \frac{3}{2}\cos^2\theta\right) - \\ - 3\,\varepsilon M \cos^2\theta \left[r^{-3}\cos 2\left(\omega(r-t)+\varphi\right) + 2\,\omega\, r^{-2} \sin 2\left(\omega(r-t)+\varphi\right)\right] \end{aligned}
$$

in the "visual" frame (33) and

$$
(42b) \quad \begin{aligned} g_{00} = 1 - 2\,\mu r^{-1} - 3\,\varepsilon_3 M r^{-3} \left(1 - \frac{3}{2}\cos^2\theta\right) + \\ + 3\,\varepsilon M \cos^2\theta \left[\left(\frac{4}{3}\,\omega^2 r^{-1} - r^{-3}\right)\cos 2\left(\omega(r-t)+\varphi\right) - 2\,\omega\, r^{-2} \sin 2\left(\omega(r-t)+\varphi\right)\right] \end{aligned}
$$

in the harmonic frame (40). Both represent a potential wave spreading radially outwards with phase velocity 1, i. e. with the velocity of light.

In the wave zone (large r) the dominant periodic term of the potential in the two coordinate systems is

$$
(43a) \quad U_0 = 3\,\varepsilon M r^{-2} \cos^2\theta \sin 2\left(\omega(r-t)+\varphi\right)
$$

and

$$
(43b) \quad U_h = -2\,\varepsilon M\,\omega^2 r^{-1} \cos^2\theta \cos 2\left(\omega(r-t)+\varphi\right)
$$

respectively. Apart from a difference in phase lag as compared with the potential near the body, the amplitudes of the two potentials have quite different orders of magnitude,[5] and we have a striking illustration of the fact, implicitly contained in the principle of equivalence, that one cannot make objective statements on gravitational forces without considering the physical (nonmechanical) features of the environment at large. A local gravitational criterion such as the harmonic condition can hardly influence in a decisive manner the selection of the physically most acceptable coordinate system.

§ 3. Non-uniform rotation

We shall drop now the assumption of constant ω and consider a nonuniform (accelerated) rotation of the ellipsoidal body. The purpose of the discussion of this more general situation is to find out about the propagation of gravitational signals (as distinct from the phase propagation of pure waves), independently of auxiliary conditions; in point of fact we can regard the variable angular velocity as an information to be transmitted to the observer through gravitational waves.

We assume that the Newtonian potential outside and not very far from the rotating body is given by (3) and (5) where ω is now a function of t. To find

[5] Both decrease much slower than the Newtonian amplitude which is of order r^{-3}.

out the general form of a good line element corresponding to this assumption we shall make use of the results obtained in the previous section.

We first note that the solution (25)—(30) for the line element (11) with constant ω can be written in the form

$$g_{00} = 1 - 2\mu r^{-1} - \Sigma_\lambda v_\lambda r^{\lambda-3} \varepsilon_p (H_p^2)^{(\lambda)}. \tag{44a}$$

$$g_{0n} = \Sigma_\lambda b_\lambda r^{\lambda-3} \varepsilon_p (H_p A_{pn})^{(\lambda)} + \Sigma_\lambda c_\lambda r^{\lambda-3} (H_p^2)^{(\lambda)} \xi_n , \tag{44b}$$

$$\begin{aligned} - g_{mn} = {} & \delta_{mn} + 2\mu r^{-1} \xi_m \xi_n + \Sigma_\lambda d_\lambda r^{\lambda-3} \varepsilon_p (H_p^2)^{(\lambda)} \delta_{mn} + \\ & + \Sigma_\lambda e_\lambda r^{\lambda-3} \varepsilon_p (H_p^2)^{(\lambda)} \xi_m \xi_n + \\ & + \Sigma_\lambda f_\lambda r^{\lambda-3} \varepsilon_p [(H_p A_{pm})^{(\lambda)} \xi_n + (H_p A_{pn})^{(\lambda)} \xi_m] + \\ & + \Sigma_\lambda g_\lambda r^{\lambda-3} \varepsilon_p (A_{pm} A_{pn})^{(\lambda)} \end{aligned} \tag{44c}$$

where the coefficients $b_\lambda, \ldots, v_\lambda$ are constants ($= 0$ for $\lambda > 2$) and A_{pk} is given by

$$\begin{aligned} \mathbf{A}_1 &= \big(\cos(\omega t - \omega r), \sin(\omega t - \omega r), 0\big) \\ \mathbf{A}_2 &= \big(-\sin(\omega t - \omega r), \cos(\omega t - \omega r), 0\big) \\ \mathbf{A}_3 &= (0, 0, 1) \end{aligned} \tag{45}$$

instead of (5). Thus A_{pk}, H_p are regarded as functions of $t - r$ (and of x_k) and the symbols $(\ )^{(\lambda)}$ stand for derivatives with respect to $t - r$.

In this form the line element is suitable for immediate generalization to the case when ω is variable. As we expect a retarded dependence on t, we assume that ω is a function of $t - \beta r$ where β^{-1} is the velocity of propagation to be determined. Substitution in the Einstein equations gives a number of recursive relations for the coefficients $b_\lambda, \ldots, v_\lambda$, which however turn out to be incompatible with the boundary conditions $b_\lambda = 0, \ldots, v_\lambda = 0$ for $\lambda > 2$, except when $\beta = 1$. In that case the equations can be satisfied with the following system of coefficients (all others are zero)

$$\begin{aligned} & e_0 = 5(v_0 - d_0), \quad f_0 = -4v_0 + 2d_0 + 3b_1 - 2c_1 , \\ & g_0 = 2v_0 - \frac{2}{3} d_0 - 2b_1 + \frac{4}{3} c_1 , \end{aligned} \tag{46a}$$

$$\begin{aligned} & v_1 = v_0, \quad d_1 = \frac{2}{3} v_0 + \frac{1}{3} d_0 - \frac{2}{3} c_1 + 2c_2 , \\ & e_1 = \frac{7}{3}(v_0 - d_0) + \frac{8}{3}(c_1 - 3c_2), \quad g_1 = \frac{4}{3} v_0 - 2b_2 , \\ & f_1 = -\frac{8}{3} v_0 + \frac{2}{3} d_0 + b_1 - \frac{4}{3} c_1 + 2b_2 + 2c_2 , \end{aligned} \tag{46b}$$

$$(46c)\qquad v_2 = \frac{1}{3} d_0 - \frac{2}{3} c_1, \quad d_2 = \frac{1}{3} v_0, \quad e_2 = \frac{1}{3}(v_0 - d_0) + \frac{2}{3}(c_1 - 3 c_2),$$

$$f_2 = -\frac{2}{3} v_0 + b_2, \quad g_2 = \frac{2}{3} v_0$$

where

$$(47)\qquad v_0 = 3M$$

and d_0, b_1, c_1, b_2, c_2 are arbitrary constants. The fact that such a solution exists at all can be taken as sufficiently strong indication that we have found an appropriate coordinate system. Since the solution definitely requires $\beta = 1$, we conclude that variations of ω are transmitted to the observer with the velocity of light.

We can satisfy condition (31) by setting $d_0 = b_1 = c_1 = b_2 = c_2 = 0$; it gives

$$(48a)\qquad g_{00} = 1 - 2\mu r^{-1} - 3 M r^{-3} \varepsilon_p H_p^2 - 3 M r^{-2} \varepsilon_p (H_p^2)',$$

$$(48b)\qquad g_{0n} = 0,$$

and the expression (44c) for g_{mn} with

$$(49)\qquad \begin{aligned} & d_1 = 2M, \quad d_2 = M, \\ & e_0 = 15 M, \quad e_i = 7M, \quad e_2 = M, \\ & f_0 = -12 M, \quad f_1 = -8M, \quad f_2 = -2M, \\ & g_0 = 6M, \quad g_1 = 4M, \quad g_2 = 2M. \end{aligned}$$

Again we find an r^{-2} law of decrease of the time-variable part of the gravitational potential, and in fact (48a) goes into (42a) when ω is constant.

In comparison, the harmonic condition (34) yields the values $d_0 = v_0$, $b_1 = b_2 = \frac{2}{3} v_0$, $c_1 = c_2 = 0$ in (46), hence

$$(50a)\qquad g_{00} = 1 - 2\mu r^{-1} - 3 M r^{-3} \varepsilon_p H_p^2 - 3 M r^{-2} \varepsilon_p (H_p^2)' - M r^{-1} \varepsilon_p (H_p^2)'',$$

$$(50b)\qquad g_{0m} = 2 M r^{-2} \varepsilon_p (H_p A_{pm})' + 2 M r^{-1} \varepsilon_p (H_p A_{pm})'',$$

$$(50c)\qquad \begin{aligned} -g_{mn} = {} & \delta_{mn} + 2\mu r^{-1} \xi_m \xi_n + \\ & + [3 M r^{-3} \varepsilon_p H_p^2 + 3 M r^{-2} \varepsilon_p (\mathrm{H}_p^2)' + \mathrm{M} r^{-1} \varepsilon_p (H_p^2)''] \delta_{mn} + \\ & + 2 M r^{-1} \varepsilon_p (A_{pm} A_{pn})''. \end{aligned}$$

If ω is constant, the solution goes into the system (41) and leads to the determination (38) of B_i and C_i.

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О РАСПРОСТРАНЕНИИ ГРАВИТАЦИОННЫХ ВОЛН

G. SZEKERES

Резюме

Автор в настоящей работе исследует распространение гравитационных волн, порожденных вращающимся с постоянной угловой скоростью телом эллипсоидной формы в четырехмерном римановом пространстве общей теории относительности. Исследования он производит в системе координат, определенной свойствами симметрии проблемы, и в гармонических координатах, введенных Фоком. Полученные в результате волны приближенно являются сферическими, распространяющимися со скоростью света. Однако в двух системах координат вид волн и скорость уменьшения амплитуды оказываются различными. Наконец, автор производит исследование и в случае вращающегося с переменной угловой скоростью тела эллипсоидной формы.