UNSOLVED PROBLEMS IN THE ENUMERATION OF GRAPHS[1]

by

FRANK HARARY[2]

§ 1. Introduction

Our object is to present several unsolved problems in the enumeration of graphs in the hope that it will serve to stimulate active interest among mathematicians. It is not likely that all of these problems will be settled in the near future, for included among their solutions there would be enough information to settle the four color conjecture either in the affirmative or the negative.

We first illustrate what is meant by a graph enumeration problem using graphs and directed graphs. We then develop the preliminary concepts concerning graphs in order to be able to state the unsolved problems concisely. Statements (without proofs) of several methods which have been used in the enumeration of graphs are given. The most important method in this area is provided by the elegant and powerful enumeration method of PÓLYA [45]. For Pólya's method or a variation thereof has been utilized in most known solutions to such problems. We compare problems involving the number of trees of various kinds with analogous problems for graphs. Lists of 27 solved problems and 27 unsolved problems are presented. The importance of the unsolved problems and the nature of their essential difficulties are indicated. The calculation of asymptotic numbers of graphs of various kinds is also mentioned. We conclude with a comprehensive bibliography of articles which either implicitly or explicitly involve the enumeration of graphs.

§ 2. Graphical Preliminaries

In this section, we develop the definitions of several basic graphical concepts. A *graph* (see KÖNIG [39] as a general reference on graph theory) consists of a finite set of *points* $a, b, c, \ldots$ together with a prescribed set of unordered pairs of distinct points. Each such pair of points a and b is a *line* $\alpha = ab$ of the graph G. We then say that points a and b are *adjacent* and that the point a and the line α are *incident* to each other. Note that by definition a graph has no lines joining a point with itself nor does it have

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[1] This article is based on a talk given in March 1959 at the Combinatorial Problems Seminar of the Logistics Project at Princeton University while the author was on leave from the University of Michigan. The final draft was completed at the Los Alamos Scientific Laboratory during the summer of 1959.

[2] Ann Arbor, USA.

two distinct lines joining the same pair of points. If the definition of a graph is generalized to permit more than one line joining the same pair of points, the result is called a *multigraph*, following the terminology in BERGE [1]. Two or more lines joining the same pair of distinct points are called *multiple lines*. If we further allow the presence of *loops*, i.e., lines joining a point with itself, as well as multiple lines, then we have a *general graph*.

Two graphs are *isomorphic* if there exists a one-to-one correspondence between their sets of points which preserves adjacency. In Figure 1 we show all the graphs (up to isomorphism) of four points.

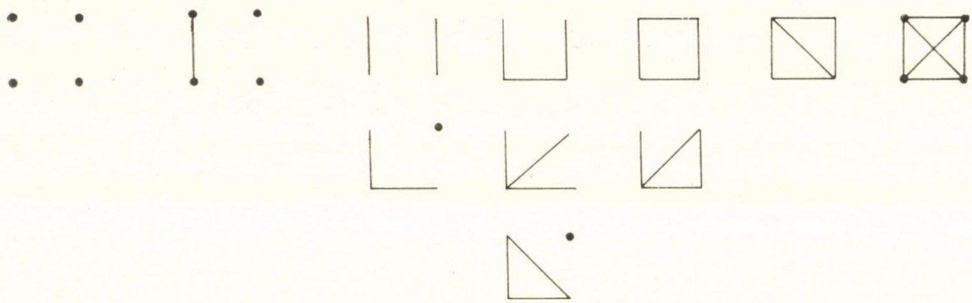


Figure 1. The graphs of four points.

Let g_{pq} be the number of graphs with p points and q lines. Let

$$g_p(x) = g_{p0} + g_{p1}\,x + g_{p2}\,x^2 + \ldots$$

be the counting series for the graphs of p points; thus the highest power of x is $p\,(p-1)/2$.

A *directed graph* (or more briefly a *digraph*) consists of a finite set of points together with a prescribed collection of *ordered pairs* of distinct points. Each such ordered pair (a, b) of points is called a *directed line* (or more briefly a *line* where the meaning is clear by context), and is denoted by $\overrightarrow{ab}$. The definition of isomorphism for digraphs is analogous to that for graphs. In Figure 2, we show all the digraphs of three points.

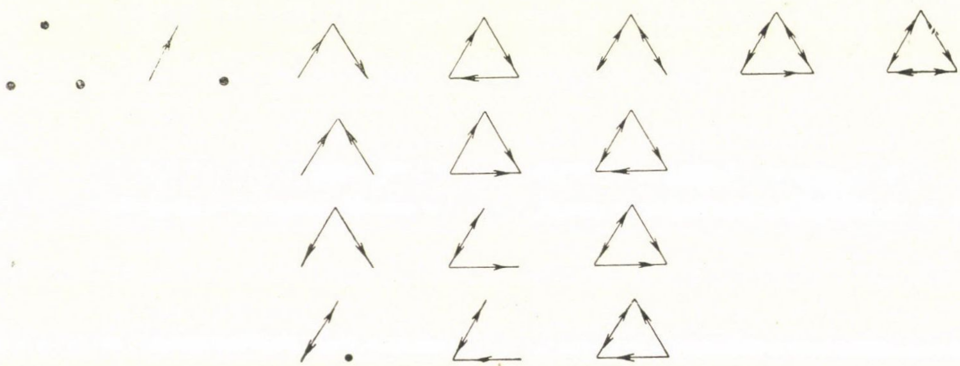


Figure 2. The digraphs of three points.

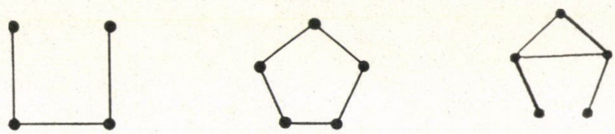
Let $\bar{g}_{pq}$ be the number of digraphs with p points and q (directed) lines. To enumerate the digraphs of p points means to find the expression for the counting series

$$\bar{g}_p(x) = \bar{g}_{p0} + \bar{g}_{p1}\,x + \bar{g}_{p2}\,x^2 + \dots$$

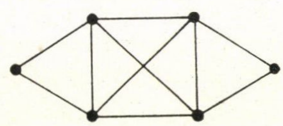
in which the highest power of x is $p(p-1)$. From Figures 1 and 2 we see that the counting series for the graphs of four points and the digraphs of three points are respectively:

$$g_4(x) = 1 + x + 2\,x^2 + 3\,x^3 + 2\,x^4 + x^5 + x^6\,,$$

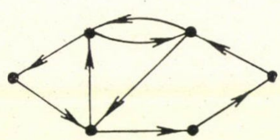
$$\bar{g}_3(x) = 1 + x + 4\,x^2 + 4\,x^3 + 4\,x^4 + x^5 + x^6\,.$$



a. Three self-complementary graphs.

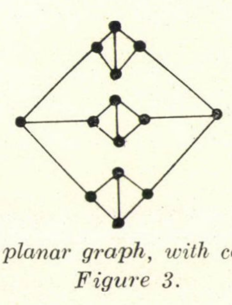


b. An Euler graph, with lower girth 3 and upper girth 6.

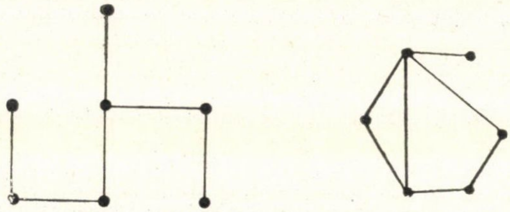


c. An Euler digraph.

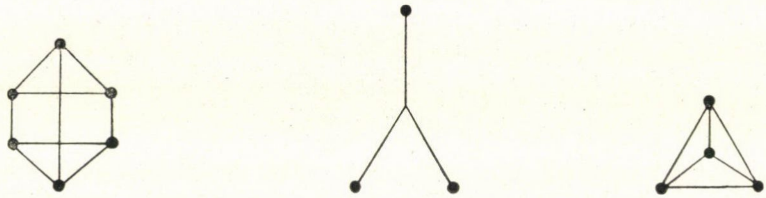
d. Four digraphs with various kinds of connectedness.



e. A cubic planar graph, with connectivity 2.
Figure 3.

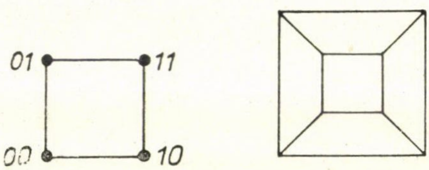


f. Identity graphs.



point symmetric *line symmetric* *symmetric*

g. Symmetric graphs.



a 2-cube *a 3-cube*

h. Two cubes.

Figure 3.

Let G be a graph with p points and q lines. The *complement* G' of G contains the same set of points as G and two points are adjacent in G' if and only if they are not adjacent in G. A graph is *self-complementary* if it is isomorphic to its complement. (See Figure 3a.)

Two lines of a graph are *adjacent* if they contain a common point. A *path* is a collection of successively adjacent lines of the form a_1a_2, a_2a_3, $\ldots$, $a_{n-1}a_n$ and the n distinct points a_i. The *length* of a path is the number of lines in it. A graph is *connected* if there is a path between any two points. The *diameter* of a connected graph is the maximum distance between any two points, where their *distance* is the length of a shortest path between them.

A *trajectory* is a sequence of successively adjacent distinct lines in which the points need not be distinct. A *line sequence* is a sequence of successively adjacent lines in which neither the points nor the lines need be distinct. A trajectory, or line sequence is *open* if its first and last points are distinct; otherwise it is *closed*. An *Euler line* of a connected graph G is a closed trajectory which contains all the lines of G. An *Euler graph* is one which contains an Euler line. (See Figure 3b.)

A *directed path from a_1 to a_n* in a digraph is similarly given by a sequence of lines $\overrightarrow{a_1a_2}, \ldots, \overrightarrow{a_{n-1}a_n}$ on n distinct points. Then, as before, an *Euler digraph D* is one which contains a closed directed trajectory containing all the lines of D. (See Figure 3c.)

If digraph D has a path from a to b, we say that b is *accessible* from a. A *point basis* of a digraph is a minimal collection of points from which all other points are accessible. A *singleton point basis* consists of exactly one point. A digraph D is *strongly connected* or *strong* if each point is accessible from every other point. D is *unilaterally connected* or *unilateral* if for any two points, at least one is accessible from the other. D is *weakly connected* or *weak* if for any partition of its set of points into two nonempty subsets, there exists a line between a point of one subset and a point of the other. Finally, D is *disconnected* if it is not even weak. (See Figure 3d.)

A *cut point* of a connected graph is one whose removal results in a disconnected graph. A *block* of a graph is a maximal-connected subgraph containing no cut points of itself.

The *degree* of a point of a graph is the number of lines to which it is incident. A *regular graph* is one in which every point has the same degree; a *cubic graph* is a regular graph of degree 3. A graph is *homeomorphically irreducible* if it has no points of degree 2. (See Figure 3e.)

A *cycle* of a graph consists of a path $a_1a_2 \ldots a_n$ together with the line a_1a_n. A *complete cycle* is one which passes through all the points of the graph; in the graphical literature a complete cycle is often called a *hamilton line*, and a graph is *hamiltonian* if it contains a complete cycle. The *length* of a cycle is the number of lines in it. The *lower girth* of graph G is the length of any smallest cycle; the *upper girth* is the length of a longest cycle. (See Figure 3b.) A *tree* is a connected graph with no cycles. (See the first graph of Figure 3f.)

The *index* of a connected graph is the smallest number of lines whose removal results in a tree. The *connectivity* of a graph is the smallest number of points whose removal results in a disconnected graph. (See Figure 3e.)

An *automorphism* of a graph is an isomorphism with itself. The *group of a graph* is the collection of all its automorphisms. An *identity graph* is one in which the only automorphism is the identity mapping on the set of points. (See Figure 3f.) Two points of a graph are *similar* if there is an automorphism which maps one into the other; similarity of two lines is analogous. A graph is *point-symmetric* if all its points are similar, it is *line-symmetric* if all its lines are similar, and it is *symmetric* if it is both point-symmetric and line-symmetric. (See Figure 3g.)

A graph is *k-colored* if each point is assigned one of k colors in such a way that no two points of the same color are adjacent, and all k colors are used. A graph is *k-chromatic* or has *chromatic number k* if it can be k-colored but not $(k - 1)$-colored. A *labeled graph* is one in which each point is distinguished from every other point.

The *partition* of a graph of p points and q lines is the expression for $2q$ as the sum of the degrees of the points. The *partition* of a digraph is the vector sum of the ordered pair at each point which gives the number of directed lines to and from that point.

A *planar graph* is one which can be drawn in the plane in such a way that none of its lines intersect each other.

A *subgraph* of a graph G is a subset of its points and lines which forms a graph. A *spanning subgraph* of G has the same point set as G.

We conclude this section with the definitions of some miscellaneous concepts. An *n-cube* is a graph with 2^n points each of which is a binary number with n digits, in which two points are adjacent whenever they differ in exactly one digit. (See Figure 3h.) A *boolean function* of two variables x and y is a finite combination of sums, products, and complements, of expressions in x and y. For each assignment of the values 0 and 1 to the variables x and y, a given boolean function has the value 0 or 1.

An *(abstract) simplicial complex* consists of a set P of *points* and a collection S of subsets of P called *simplexes,* which satisfy the following two conditions:

1. Every point is a simplex.
2. Every nonempty subset of a simplex is a simplex.

A *Latin square* of order n is a square matrix of order n in which every row and every column is a permutation of the integers $1, 2, \ldots, n$.

A *finite automaton* or a *sequential machine* with two inputs 0, 1 and a finite number of states may be defined as follows. There is a directed graph whose points are called states in which one point is distinguished or *rooted* and called the *initial state.* Each point has exactly two lines from it, one line labeled 0 and the other labeled 1. These two labels on lines of the digraph are called *inputs* and serve to determine the next state of the machine when the given state and the input are known. We note that directed lines from a point to itself (loops) are permitted here as well as two directed lines both from one point to another. Also, it is stipulated that every state is accessible from the initial state. (See Figure 9 below.) An *automaton with outputs* 0 and 1 is defined by providing a table of outputs which associate one of the output symbols 0 or 1 given the present state and the input.

List I

UNSOLVED PROBLEMS IN THE ENUMERATION OF GRAPHS

I. Digraphs	1. Strong 2. Unilateral 3. Singleton point basis
II. Partitions	4. Graphs with given partition 5. Homeomorphically irreducible graphs 6. Regular graphs 7. Euler graphs
III. Planarity	8. Planar graphs 9. k-chromatic and k-colored graphs 10. Planar graphs with additional properties
IV. Connectivity	11. Graphs of given girth and diameter 12. Graphs of given index and connectivity 13. Blocks

V. Ising	14. 2-dimensional Ising problem and non-nearest neighbors
	15. 3-dimensional Ising problem
	16. Paving problem
	17. Cell-growth problem
VI. Switching	18. Types of complete cycle in an n-cube
	19. Finite automata
	20. Indecomposable two-terminal networks
VII. Topological	21. Self-complementary graphs
	22. Simplicial complexes
VIII. Combinatorial	23. Latin spuares
	24. Line graphs
IX. Groups	25. Symmetric graphs
	26. Identity graphs
	27. Graphs with given group

§ 3. Statements of the Unsolved Problems

We regard a solution of each of the unsolved problems of List I as a generating function in closed form for the number of graphs of each given kind with a given number p of points and a given number q of lines (or directed lines for digraphs). These problems are divided into nine categories which combine related problems.

I. Problems involving digraphs

1. Strong digraphs

We see from Figure 2 that the counting series for the strong digraphs of three points is

$$x^3 + 2x^4 + x^5 + x^6 .$$

2. Unilateral digraphs

Again, we see from Figure 2 that the counting series for unilateral digraphs of three points is

$$x^2 + 4x^3 + 4x^4 + x^5 + x^6 .$$

3. Digraphs with a singleton point basis

Figure 2 shows that the counting series for these digraphs with three points is

$$2x^2 + 4x^3 + 4x^4 + x^5 + x^6 .$$

It is easy to show that every unilateral digraph has a singleton point basis.

II. Problems involving partition

4. Graphs with a given partition

From Figure 1, we see that each graph of four points has a different partition. For example, the graph consisting of a single cycle of length 4 has partition $2 + 2 + 2 + 2$ and is the only graph with this partition. However, starting with graphs of five points, there exist partitions which belong to more than one graph. An example is given by the two graphs shown in Figure 4, each of which has the partition $1 + 1 + 2 + 2 + 2$.

Figure 4. Two graphs with the same partition.

5. Homeomorphically irreducible graphs

Inspection of Figure 1 shows that the counting series for homeomorphically irreducible graphs of four points is

$$1 + x + x^2 + x^3 + x^6$$

while that for connected homeomorphically irreducible graphs of four points is $x^3 + x^6$.

6. Regular graphs

This is an interesting special case of graphs with a given partition.

Every regular graph of degree one has an even number $2n$ of points which are joined by n lines to form n connected components. Every regular graph of degree 2 has a cycle for each of its components. The first interesting case of regular graphs is given by cubic graphs. The only cubic graph of four points is the complete graph shown in Figure 1; hence the counting series for cubic graphs of four points is simply given by x^3.

7. Euler graphs and Euler digraphs

Euler himself showed that a graph has a closed trajectory containing all the lines if and only if it is connected and every point is *even* (of even degree). Hence Euler graphs are subsumed in the category of graphs with a given partition. Namely, they are those graphs whose partitions have no odd parts.

III. Problems involving planarity and colorability

8. Planar graphs

KURATOWSKI has shown that a graph is planar if and only if it contains no subgraph homeomorphic to either of the two "skew graphs" K_5 or K_{33} shown in Figure 5.

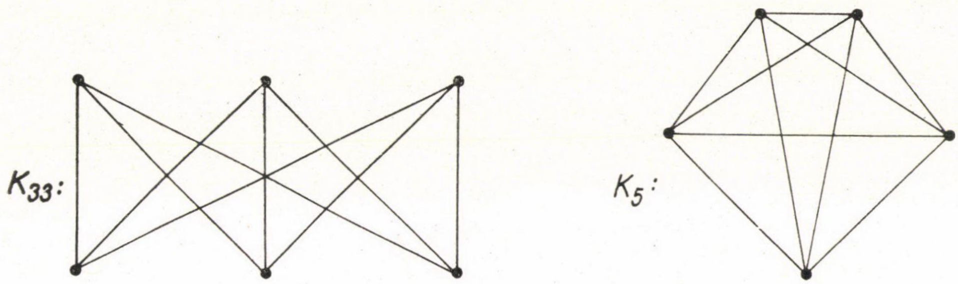


Figure 5. The two skew graphs.

Hence it follows that every graph of four points is planar and that the counting series for the planar graphs of five points is obtained from that of all graphs of five points by subtracting x^5.

9. k-chromatic graphs and k-colored graphs

Only the number of bicolored graphs has been found in closed form, [23]. For example, the bicolored graphs with two points of each color are shown in Figure 6, in which the two points of each graph to the left are regarded as colored with the first color while the two points to the right are colored with the second color.

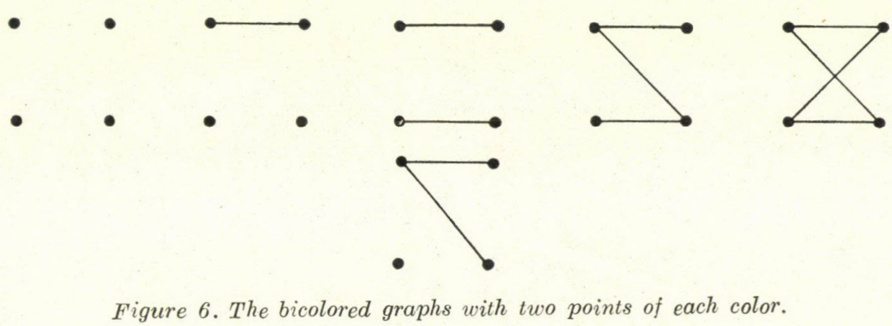


Figure 6. The bicolored graphs with two points of each color.

By a theorem of KÖNIG [39] a graph is bichromatic if and only if all its cycles are even (of even length). Thus we see from Figure 1 that the number of bichromatic graphs of four points is given by the series

$$1 + x + 2x^2 + 2x^3 + x^4 .$$

Further, it is clear that there is a one-to-one correspondence between connected bicolored graphs and connected bichromatic graphs. But there are more disconnected bicolored graphs than disconnected bichromatic graphs. Bicolored graphs are regarded as isomorphic even when their two colors are interchanged.

10. Planar graphs with additional properties

These problems include planar cubic graphs, planar Euler graphs, and planar k-chromatic graphs.

IV. Problems involving connectivity

11. Graphs with given girth and given diameter

From Figure 1, we see that there are exactly three connected graphs with lower girth 3 and the same number with upper girth 4, and that the counting series for the connected graphs of four points with diameter 2 is $x^3 + x^4 + x^5$.

12. Graphs of given index and given connectivity

Among the connected graphs of four points there are two graphs of index 1, one of index 2, and one of index 3.

A connected graph has connectivity 1 if and only if it has a cut point. Hence the counting series for connected graphs of four points of connectivity 1 is $2x^3 + x^4$. The sum of this solution and that of Problem 13 is the known number of connected graphs.

13. Blocks

In view of the definitions of a block and of the connectivity of a graph, it follows at once that blocks are connected graphs with connectivity greater than 1. The counting series for blocks of four points is (from Figure 1) $x^4 + x^5 + x^6$.

V. Ising model problems

14. The two-dimensional Ising problem

Consider a labeled graph which is an n-dimensional lattice. A subgraph of this lattice is called *admissible* if and only if every point is even. Let A_q be the number of different labeled admissible subgraphs with q lines. Find a generating function for the quantity A_q. This problem was solved for $n = 1$ by Ising himself [35], and for $n = 2$ by Onsager [43]. However, Onsager did not use combinatorial methods and his procedures have not generalized to higher dimensions. Hence even though the two-dimensional Ising problem has been solved, it is still an unsolved problem to derive a purely combinatorial solution.[3]

[3] This problem has just been solved by. S. Sherman in an article to appear in vol. I, May 1960, Journal of Mathematical Physics. Sherman's method may also solve the rest of Problem 14.

As a variation of this problem we have the case which is called in the physical literature "interaction between non-nearest neighbors". We show in Figure 7a the graph of a two-dimensional lattice and in Figure 7b the graph obtained from this lattice on joining those pairs of points which are nearest to each other without already being adjacent. We note that in physical applications, each of these graphs would usually be considered as being drawn on a torus, i.e., both pairs of opposite sides are identified.

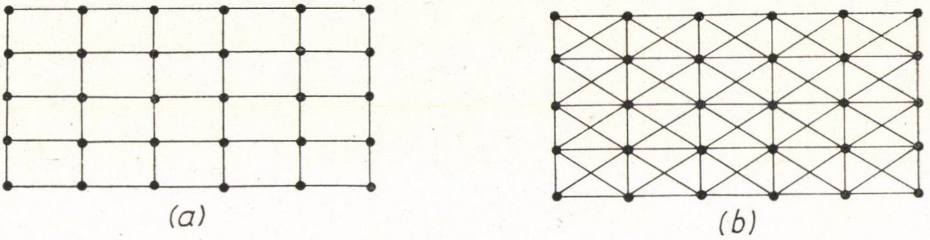


Figure 7. Graphs of a 2-dimensional lattice without and with diagonals.

15. The three-dimensional Ising problem

This problem is obtained from the preceding one on replacing the dimension $n = 2$ by $n = 3$. No real beginning has been made toward its solution. Of course the n dimensional Ising problems for $n > 3$ are also unsolved.

By the *area* of an admissible labeled subgraph of a two-dimensional lattice we mean the minimum area enclosed by disjoint cycles constituting this subgraph. Let $A_{q,r}$ be the number of admissible labeled subgraphs with q lines and area r. Find a generating function for the quantities $A_{q,r}$. In the physical literature, this is shown to be the "two-dimensional Ising problem with a magnetic field".

16. A paving problem

Let us start with a two-dimensional lattice with N squares. Consider n_1 squares and n_2 double squares (like dominoes) such that $n_1 + n_2 = N$. In how many ways can the labeled lattice be "paved" by these ?

17. The cell growth problem

Consider a one-celled animal which has a square shape and can grow in the plane by adding a cell to any of its four sides. How many connected animals A_r with area r are there up to isomorphism ? The animals are assumed to be simply connected in the sense that there are no "holes".

In Figure 8 we show all the animals with area 1, 2, 3, 4, and 5.

Thus we see that the counting series for the cell growth problem is of the form

$$A(x) = \sum_{1}^{\infty} A_r x^r = x + x^2 + 2x^3 + 5x^4 + 12x^5 + \dots$$

It is also known that $A_6 = 35$ and $A_7 = 107$. In GOLOMB [14], these animals are studied under the name of *polyminoes* since they are regarded as a generalization of dominoes. See Addendum II.

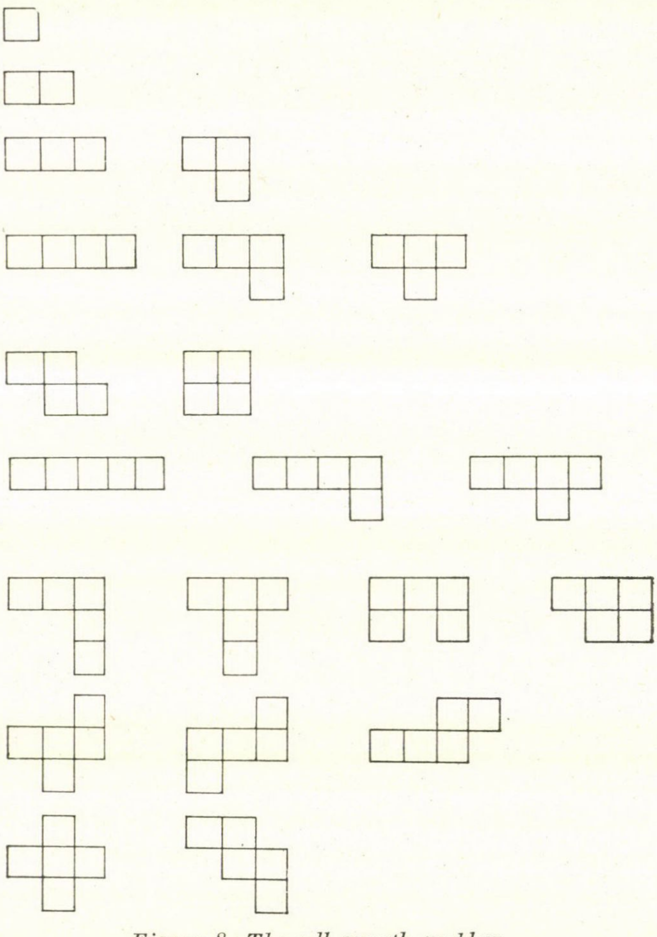


Figure 8. The cell growth problem.

VI. Switching Problems

18. The number of dissimilar complete cycles in an n-cube

In Figure 3, we see a 3-cube. It is very easy to convince oneself that there is exactly one similarity type of complete cycle in a 3-cube. It has been shown by GILBERT [13] that the counting series for this problem is of the form $x^2 + x^3 + 9x^4 + \ldots$ where the coefficient of x^n is the number of dissimilar complete cycles in an n-cube. The coefficient is not known even for x^5.

19. The number of finite automata

In Figure 9, we have the digraph representation of a finite automaton. Every point of this digraph is accessible from the point designated as the initial state.

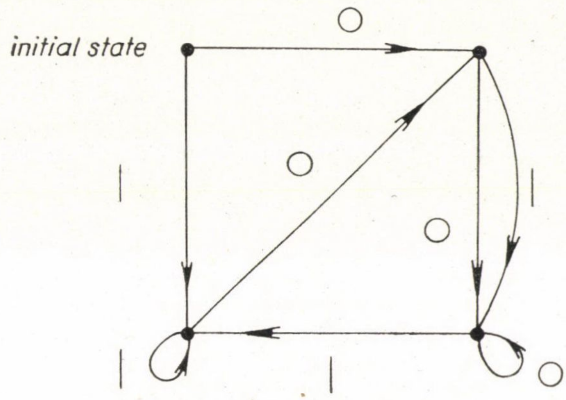


Figure 9. A finite automaton.

20. Indecomposable two-terminal networks

A *two-terminal network* is a connected multigraph in which two points are marked u and v and are called the first terminal and the second terminal. The *product* or *series connection* $N = N_1N_2$ of two 2-terminal networks N_1 and N_2 is the network obtained on identifying the points v_1 and u_2. The *sum* or *parallel connection* $N = N_1+N_2$ is obtained on identifying u_1 with u_2 and also v_1 with v_2. These two operations on networks are illustrated in Figure 10.

A two-terminal network is *series-parallel* if it may be constructed from a finite succession of series and parallel connections starting with the network having exactly two adjacent points u and v. It is well-known [50] that a two-terminal network is series-parallel if and only if it is *unidirectional*, i.e., no two paths from u to v contain any two points a and b in opposite orders.

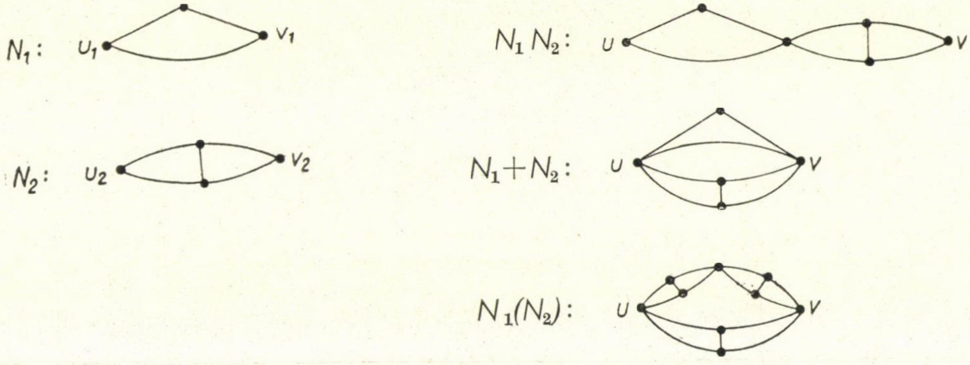


Figure 10. The product, sum, and composition of two-terminal networks.

The *composition* $N = N_1(N_2)$, where N_1 is series-parallel is obtained on replacing each line of N_1, using unidirectionality, by the network N_2. In Figure 10, the composition of two networks, the first of which is series-parallel, is also illustrated.

A network N is *indecomposable* if it is not possible to write it in the form $N = N_1(N_2)$. VETUCHNOVSKY [57] has obtained upper and lower bounds for the number of indecomposable two-terminal series-parallel networks with a given number of points. The exact number is not known, and constitutes the present problem.

VII. Topological Problems

21. Self-complementary graphs

It is easy to show that any self-complementary graph has its number of points of the form $p = 4n$ or $p = 4n + 1$. In Figure 3, we have the self-complementary graphs of four and five points. The next self-complementary graphs will therefore have eight and nine points. The counting series for self-complementary graphs is therefore of the form

$$x^4 + 2x^5 + s_8 x^8 + s_9 x^9 + s_{12} x^{12} + s_{13} x^{13} + \dots$$

R. READ finds that $s_8 = 10$ and that these graphs are all planar. Clearly every self-complementary graph on 13 or more points is nonplanar.

22. Simplicial complexes

How many isomorphism types of simplicial complexes are there with a given number of simplex of each dimension? We illustrate by applying Figure 1 to write down the counting series for the simplicial complexes with four points, and a given number of 1-simplexes (lines) and 2-simplexes. Letting x and y be the variables standing for the 1-simplexes and 2-simplexes respectively, we find that this series is of the form

$$1 + x + 2x^2 + 3x^3 + x^3 y + x^4 + x^4 y + x^5 + x^5 y + x^5 y^2 + x^6 + \\ + x^6 y^2 + x^6 y^3 + x^6 y^4 .$$

VIII. Combinatorial Problems

23. Latin squares

Let L_n be the number of Latin squares in which the first the first row and the first column are in the standard order 1, 2, ..., n. Then the counting series for Latin squares is known to be (cf. RIORDAN [52])

$$x^2 + x^3 + 4x^4 + 56x^5 + 9408x^6 + 16942080x^7 + \dots$$

The result for $n > 7$ is not known.

Every Latin square may be regarded as a bicolored graph with the same number of points of each color in which the lines are also colored. Let K_{nn} be the graph whose points are

$$a_1, \dots, a_n, b_1, \dots, b_n$$

and whose lines are all n^2 lines of the form $a_i b_j$. The points of the first color correspond to the rows of a Latin square while the points of the second color designate its columns. Each of the lines of K_{nn} is colored with exactly one of n colors in such a way that at each point there is exactly one line of each color. The matrix interpretation of such a graph is that the color of the line joining points a_i and b_j is the element in the (i, j) place of the matrix.

24. Line graphs

The *line graph of a given graph G* is that graph $L(G)$ whose points correspond to the lines of G and in which two points are adjacent whenever the corresponding lines of G are adjacent. A criterion for a graph to be the line graph of some graph is known, KRAUSZ [40]. We call such a graph a *line graph.* The present problem is to find the number of line graphs with a given number of points and lines.

IX. Problems involving groups

25. Symmetric graphs

In Figure 3, we have diagrams of graphs which are point-symmetric but not line-symmetric, line-symmetric but not point-symmetric, and symmetric. The problem is to enumerate each of these three kinds of graphs with a given number of points and lines.

26. Identity graphs

The smallest identity graph which is a tree and the smallest one not a tree are shown in Figure 3.

27. Graphs with a given group

The group of a graph is by definition a permutation group acting on the set of points. It is known, FRUCHT [11], that every finite group is abstractly isomorphic to the group of some graph. But it is not known in general whether a given permutation group is a graph group. The general problem, which includes this question, is to find the number of (nonisomorphic) graphs with a given (permutation) group. The *line group* of a graph is the permutation group acting on the set of all lines of the graph consistent with the group of the graphs. As variations and extensions of the above problem, we may ask for the number of graphs with a given line group and also for the number of graphs whose group and line group are a given ordered pair of permutation groups.

§ 4. Various Graph Counting Methods

In this section we shall discuss six methods which have been used in the enumeration of various kinds of graphs. By far the most important of these has been PÓLYA's powerful and elegant enumeration theorem [45]. After a statement of PÓLYA's Theorem, we present a special case which has been derived independently by DAVIS [6] and SLEPIAN [55]. We then discuss a recent interesting theorem of READ [49], which is based on the same kind of group theoretic approach as the theorem of PÓLYA. The dissimilarity characteristic theorems of OTTER [44] for trees and NORMAN [42] for graphs in terms of its blocks are then reviewed. After some comments on the enumeration of labeled graphs, we conclude this section with a discussion of the considerations involved in finding asymptotic numbers for graphs.

a. Pólya's Theorem

We shall state PÓLYA's Theorem in the form which is useful in deriving the counting polynomials for various kinds of graphs. The desired form is a specialization of PÓLYA's statement to one variable.

Let *figure* be an undefined term. To each figure there is assigned a non-negative integer called its *content*. Let a_k denote the number of different figures of content k. Then the *figure counting series* $a(x)$ is defined by

$$a(x) = \sum_{k=0}^{\infty} a_k x^k . \tag{1}$$

Let Y be a permutation group of degree s and order h. A *configuration* of length s is a sequence of s figures. The *content of a configuration* is the sum of the contents of its figures. Two configurations are *Y-equivalent* if there is a permutation of Y sending one into the other. Let F_k denote the number of Y-inequivalent configurations of content k. The configuration counting series $F(x)$ is defined by

$$F(x) = \sum_{k=0}^{\infty} F_k x^k . \tag{2}$$

We shall call Y the *configuration group*.

The object of PÓLYA's Theorem is to express $F(x)$ in terms of $a(x)$ and Y. This is accomplished using the cycle index of Y, defined as follows. Let $h(j)$ denote the number of elements of Y of type $(j) = (j_1, j_2, \ldots, j_x)$, i.e., having j_k cycles of length k, for $k = 1, 2, \ldots, s$, so that

$$j_1 + 2j_2 + \ldots + sj_s = s . \tag{3}$$

Let $y_1, y_2, \ldots, y_s$ be s indeterminates. Then $Z(Y)$, the *cycle index* of Y, is defined as

$$Z(Y) = \frac{1}{h} \sum_{(j)} h(j)\, y_1^{j_1} y_2^{j_2} \ldots y_s^{j_s} , \tag{4}$$

where the sum is taken over all partitions (j) of s satisfying (3). For any function $f(x)$, let $Z(Y, f(x))$ denote the function obtained from $Z(Y)$ by replacing each indeterminate y_k by $f(x^k)$. Using these definitions, we are able to give a concise statement of:

Pólya's Theorem. *The configuration series is obtained by substituting the figure counting series into the cycle index of the configuration group.* Symbolically,

$$F(x) = Z(Y, a(x)) . \tag{5}$$

This theorem reduces the problem of finding the configuration counting series to the determination of the figure counting series and the cycle index of the configuration group. See Addendum I.

We mention that the cycle index of the symmetric group S_n of degree n is

$$Z(S_n) = \frac{1}{n!} \sum_{(j)} \frac{n!}{1^{j_1} j_1! \ldots n^{j_n} j_n!} y_1^{j_1} \ldots y_n^{j_n},$$

where the sum is taken over all partitions (j) of n satisfying (3) with $s = n$.

b. A special case of Pólya's Theorem

The following special case of Pólya's Theorem has been independently discovered by Davis and Slepian. In addition, the result is also known to have been found independently by Gleason (unpublished).

Very simply stated, this special case is obtained from Pólya's Theorem, equation (5), by substituting $x = 1$. Formally, this gives $F(1) = Z(Y, a(1))$. But from equation (2), $F(1) = \sum F_k$ and from (1), $a(1) = \sum a_k$. But $F(1)$ is the total number of (inequivalent) configurations without regard to content, and similarly $a(1)$ is the total number of figures without regard to content. Hence the substitution of $x = 1$ in (5) results in the following formula for the total number of configurations in terms of total number of figures and the configuration group. Using the notation of [18], let $B = F(1)$ and $b = a(1)$. Then (5) becomes

$$B = \frac{1}{h} \sum_{(j)} h(j)\, b^{\Sigma j_k}. \tag{6}$$

Thus B is obtained at once from the cycle index of the configuration group.

c. A generalization of Pólya's Theorem

In a recent article, de Bruijn [2] has developed an interesting generalization of Pólya's Theorem. He first restates the method of Pólya in more abstract and less geometric language as follows: Let D be the domain and R the range of a collection of functions $f, f_1, f_2, \ldots$. The elements of the range correspond to figures while the range itself stands for the figure collection in Pólya's terminology. The elements of the domain correspond to the "places" at which the figures are to be located. Then each function mapping the domain into the range becomes a configuration. Let A be a permutation group which acts on D. Then in Pólya's method, two functions (configurations) f_1 and f_2 are *equivalent* if there is a permutation α in A such that for all $x \in D$,

$$f_1(x) = f_2(\alpha x).$$

de Bruijn considers the more general situation in which there is also a permutation group B acting on R. He then defines two functions as *equivalent* if there exist permutations $\alpha \in A$ and $\beta \in B$ such that for all $x \in D$,

$$f_1(x) = \beta f_2(\alpha x).$$

Rather than state the main formula of [2] in all its generality, we state the following special case, in which $R = D$ and $B = A$.

The number of classes of functions of a finite set D into itself, with respect to group A acting on D, is given by the formula

$$Z\left(A; \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots\right)[(1-y_1)^{-1}(1-2y_2)^{-1}(1-3y_3)^{-1}\ldots] \tag{7}$$

where these partial derivatives are evaluated at $y_1 = y_2 = \ldots = 0$.

The most general result of the article [2] gives a formula for the number of inequivalent functions from D into R where A acts on D and B acts on R. It is easy to see that this kind of combination of the two permutation groups A and B is closely related to the operation which we [23, 27] have called "exponentiation" of permutation groups.

The *exponentiation* B^A of two permutation groups A and B which act respectively on sets D and R is as follows. Let A and B have degree d and r and order m and n. Then B^A acts on R^D, the set of all functions from D into R, so that the degree of B^A is r^d. The function $f \in R^D$ is mapped into the following function f' by the permutation $\gamma \in B^A$ determined by any permutation $\alpha \in A$ and any d permutations (repetitions permitted) $\beta_1, \beta_2, \ldots, \beta_d \in B$, where $D = \{x_1, x_2, \ldots, x_d\}$:

$$f'(x_i) = \beta_i f(\alpha\, x_i)\,. \tag{8}$$

Thus the order of B^A is mn^d.

Hence we see that DE BRUIJN's generalization of PÓLYA's Theorem may be regarded as an application of PÓLYA's Theorem to a new kind of permutation group, the *diagonal of the exponentiation*, whose definition is obtained from (8) on taking all the β_i as the same permutation of B. This concept will be developed in detail elsewhere.

We note that the cycle index of $S_n^{S_2}$ has been used in [23] to count bicolored graphs with the same number of points of each color, and using different terminology the cycle index of $S_2^{S_n}$ was found by SLEPIAN [55] to enumerate the types of boolean functions of n variables. A general formula for $Z(B^A)$ has not been found.

d. Read's Theorem

The results of READ have just appeared [49]. The main result is his "Superposition Theorem". By the superposition of two graphs on the same set of points is meant the graph obtained by forming the union of their sets of lines, including multiplicity. For example, we show in Figure 11 the graph obtained by the superposition of three graphs on the same collection of six points. It must be noted that the lines of G_i have color i and these colors are preserved in G. When these three graphs are placed differently on the same set of points, the resulting superposed graph need not be isomorphic with the graph G of Figure 11. The question is then: Given three graphs G_1, G_2, G_3, how many distinct superposed graphs can be formed by them? It turns out that this number depends only on the automorphism groups Y_1, Y_2, Y_3 of the three graphs and is given by an expression which we may denote $N(Y_1, Y_2, Y_3)$. In order to state the superposition theorem, let h_i be the order of the group Y_i and let $h_i(j)$ be the number of permutations in the group Y_i of type (j) as defined by equation (3). This gives enough notation to write the cycle index as in (4) of each of these three groups Y_i.

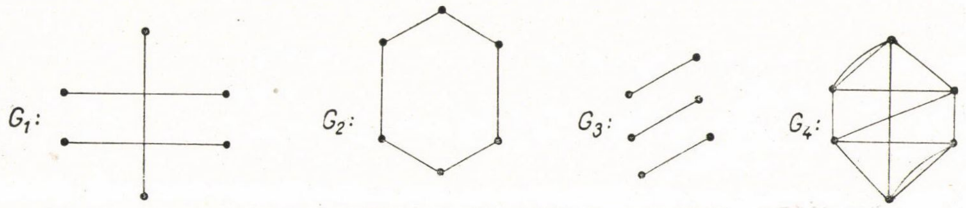


Figure 11. The superposed graph G of the graphs G_1, G_2, G_3.

Superposition theorem.

$$(9)\qquad N(Y_1, Y_2, Y_3) = \frac{1}{h_1 h_2 h_3} \sum_{(j)} h_1(j)\, h_2(j)\, h_3(j)\, (1^{j_1} 2^{j_2} \dots s^{j_s} j_1!\, j_2! \dots j_s!)^2 .$$

The theorem holds for the superpositions of any number n of graphs; the exponent 2 in the right-hand member of (9) being replaced by $n-1$. This theorem is an important and interesting contribution by READ to the art of counting. See Addendum I.

e. Otter's Theorem and Norman's Theorem

OTTER's dissimilarity characteristic theorem (10) for trees [44] was used as an essential lemma in his elegant enumeration of trees in terms of rooted trees. A generalization (11) of this theorem by NORMAN [42] enabled him to solve the more general enumeration theorem of finding the number of graphs with given blocks. Derivations from equation (11) of (10) and other formulas are given in [30].

Let T be any tree, and p and q be the number of *dissimilar* points and lines of T. An *exceptional line* of T is one whose two points are similar; let q_e be the number of exceptional lines of T. Thus $q_e = 0$ or 1.

Dissimilarity-characteristic theorem for trees.

$$(10)\qquad p - (q - q_e) = 1 .$$

Let G be a connected graph with n blocks. Let p be the total number of dissimilar points in G and p_k the number of dissimilar points in the k'th dissimilar block of G.

Dissimilarity characteristic theorem for graphs.

$$(11)\qquad \sum_{k=1}^{n} (p_k - 1) = p - 1 .$$

The application of both equations (10) and (11) to graph counting problems is made by summing each of these equations over the collection of all graphs to be enumerated. The term 1 when summed over all graphs obviously gives the total number of graphs while the term p becomes the number of rooted graphs under consideration. Clever combinatorial devices then serve to yield formulas for the summation of the remaining terms in these formulas.

f. Labeled graphs

The enumeration of labeled graphs of any given kind is *always* easier than that of unlabeled graphs. We shall mention later some comparison between the enumeration of unlabeled and labeled graphs of various kinds. The essential difference is as follows. Regardless of what configuration group is required in the process of enumerating the ordinary graphs of a given kind, this group is replaced by the identity group of the same degree for the labeled case. Since the cycle index of the identity group of degree n is f_1^n, it follows from PÓLYA's Theorem that relatively straightforward combinatorial procedures serve for the enumeration of labeled graphs; see for example FORD and UHLENBECK [9, I] and GILBERT [12].

g. Asymptotic problems

The asymptotic number of trees was first studied by PÓLYA [45]. Further contributions were made by OTTER [44]. In a more recent study, FORD and UHLENBECK [9, IV] have made a systematic investigation of the number of asymptotic graphs with various properties. We have developed in an expository note [25] asymptotic formulas for certain kinds of binary relations based on the corresponding kinds of graph. READ has also studied asymptotic problems in connection with the results obtained by his superposition theorem.

§ 5. Tree counting problems

There have been two recent papers which combine the methods of PÓLYA and OTTER to enumerate various species of trees. RIORDAN [51] obtained formulas for the number of labeled colored and chromatic trees where these three adjectives are applied in all possible arrangements to the set of points and the set of lines of a tree. In essentially a sequel to RIORDAN's article, HARARY and PRINS [31] have enumerated the following kinds of trees:

1. Trees with a given partition.
2. Homeomorphically irreducible trees.
3. Trees with a given diameter.
4. Identity trees.
5. Weighted trees.
6. Oriented trees.
7. Directed trees.
8. Signed trees.
9. Trees of given strength.
10. Trees of given type.

A *signed tree* is one whose lines are designated as either positive or negative. An *oriented graph* is one in which each line is assigned a unique direction. A *directed tree* is obtained from a tree when each line is assigned either one direction or both directions. A graph of *strength* n is one in which multiple lines are admitted, but not more than n lines join the same pair of points. A graph of *type* n has lines of n different colors and is obtained from a graph of strength n by assigning colors to its lines in such a way that any two distinct lines joining the same pair of points have different colors. (READ has just derived counting formulas for labeled trees of these various species.)

We now compare these known results for trees with corresponding unsolved problems mentioned above for graphs. The number of trees with a given partition has been found by the combination of PÓLYA's Theorem and OTTER's theorem as mentioned above. READ [49] has also found the number of general graphs with a given partition using his superposition theorem. But his method does not appear to be applicable to the case of graphs in which loops and multiple lines are not permitted. Thus there have been these two solutions of variations of problem 4, but the problem itself has not been solved. Homeomorphically irreducible graphs, being graphs with no points of degree 2, constitute a special case of graphs with a given partition. Hence READ's result serves to enumerate these also for general graphs. In addition, this counting result has been obtained for trees. The appropriate formulation for handling this problem by PÓLYA's Theorem has not been found. Such a formulation seems to be required for an attack on problem 5. Problem 6, the number of regular graphs of degree r, is also a special case of graphs with a given partition. Hence for general graphs only, READ's method serves to settle these problems. Read has also obtained an application of his superposition theorem to the case of regular graphs of degree r whose lines are colored with r colors in such a way that exactly one of each color is incident to each point. These *completely factored graphs* are multigraphs and have no loops.

Since EULER graphs may be characterized as connected graphs in which every point is even, problem 7 is also a special case of problem 4. Thus its solution for general graphs is derivable from READ's formula.

In order to state READ's formula for the number of general graphs with given partition, we require the concept of "Gruppenkranz" due to PÓLYA [45], which we call in [27] the *composition* $A[B]$ of permutation groups A and B. As above, let A and B have degrees d and r, orders m and n, and act on sets D and R. Then $A[B]$ acts on the cartesian product $D \times R$. Any permutation $\alpha \in A$ and any d permutations (repetitions permitted) $\beta_1, \beta_2, \ldots, \beta_d \in B$ determine the following permutation γ of $A[B]$:

$$\gamma(x_i, y_j) = (\alpha x_i, \beta_i y_j), \quad \text{for all} \quad x_i \in D, y_j \in R .$$

Hence the degree of the composition $A[B]$ is dr and the order is mn^d. It follows at once from their definitions that the exponentiation B^A and the composition $A[B]$ are abstractly isomorphic but not permutationally equivalent.

The *direct sum* $A + B$ acts on $D \cup R$ and for each $\alpha \in A$ and $\beta \in B$, a permutation $\gamma \in A + B$ is defined by:

$$\gamma(u) = \begin{cases} \alpha(u) & \text{if} \quad u \in D \\ \beta(u) & \text{if} \quad u \in R . \end{cases}$$

(This is called "direct product" by PÓLYA [45] and others.)

PÓLYA [45] has shown that the cycle index of the direct sum $A + B$ is the product of the cycle indices :

$$Z(A + B) = Z(A) Z(B) ,$$

and that the cycle index of the composition $A[B]$ is the functional composition of their cycle indices :

$$Z(A[B]) = Z(A, Z(B)).$$

where the right-hand member is obtained as in equation (5). For example,

$$Z(S_3) = \frac{1}{6}(y_1^3 + 3y_1 y_2 + 2y_3) \quad \text{and}$$

$$Z(S_2) = \frac{1}{2}(y_1^2 + y_2), \quad \text{so that}$$

$$Z(S_2[S_3]) = \frac{1}{2}\left((Z(S_3))^2 + \frac{1}{6}(y_2^3 + 3y_2 y_4 + 2y_6)\right).$$

With this notation, READ's formula for the number of general graphs with v_i points of degree i and $q = \frac{1}{2}\sum_{i=1}^{\infty} i\, v_i$ lines is

$$N\left(\sum_{i=1}^{\prime} S_{v_i}[S_i], S_q[S_2]\right), \tag{12}$$

where $\sum'$ denotes direct sum, and this number is determined in accordance with equation (9).

Although the number of trees with a given diameter[4] has been found [31] the method of solution appears to offer no clues to the corresponding problem for graphs. This is the second part of problem 11. The first part of problem 11 asks for the number of graphs with given lower girth and also for the number with given upper girth. The translation of this condition into an application of PÓLYA's Theorem is not straightforward. However, a special case of the number of graphs of given upper girth has been solved, namely, the number of Hamiltonian graphs. This problem is handled in the article [20] where the different graphs having a complete cycle of p points are regarded as supergraphs of a cycle of length p whose set of points consists of the points of the cycle.

The number of identity trees was found [31] by means of an application of another theorem of PÓLYA involving configurations in which all the figures are distinct, to the combined methods of PÓLYA and OTTER.

By an abuse of notation, let

$$Z(A - B) = Z(A) - Z(B),$$

where A and B are permutation groups of the same degree. PÓLYA [45] has derived the following very useful result. The counting series for the number of configurations of length n inequivalent with respect to S_n, in which all figures are distinct and the figure series is $f(x)$, is given by

$$Z(A_n - S_n, f(x)), \tag{13}$$

where A_n is the alternating group of degree n.

[4] See also RIORDAN, The number of trees by height and diameter, to appear in I. B. M. Journal of Research, 1960.

By another abuse of notation, let

$$Z(S_\infty) = \sum_{n=0}^{\infty} Z(S_n),$$

where $Z(S_0)$ is defined to be 1, and let

$$Z(A_\infty - S_\infty) = \sum_{n=0}^{\infty} Z(A_n - S_n).$$

There is a well-known combinatorial identity mentioned in [2, 16, 42, 45, 52]:

$$Z(S_\infty, f(x)) = \exp \sum_{r=1}^{\infty} f(x^r)/r. \tag{14}$$

This formula is useful in several counting problems, including the number of rooted trees and of connected graphs. For counting identity trees, a formula for $Z(A_\infty - S_\infty)$ is required. This is given in equation (15) below, recently communicated to us by J. RIORDAN.

It is readily verified that

$$Z(A_n; y_1, y_2, \ldots, y_n) = Z(S_n; y_1, y_2, \ldots, y_n) + Z(S_n; y_1, -y_2, y_3, -y_4, \ldots).$$

It follows at once from this and (14) that

$$Z(A_\infty - S_\infty, f(x)) = \exp \sum_{r=1}^{\infty} (-1)^{r+1} f(x^r)/r. \tag{15}$$

Again, the group theoretic formulation required to characterize configurations corresponding to identity graphs has not been found; such a discovery is required to handle problem 26. PRINS [48] has characterized all those permutation groups which are tree groups. A corresponding characterization for graph groups is still open. Such a criterion would give a partial answer to problem 27, that of finding the number of graphs with a given permutation group. In a previous article [24], we have proposed the more general problem of finding the number of graphs whose group and line group (defined in [19]) are a given ordered pair of permutation groups.

The number of trees of given strength was found by HARARY and PRINS [31] while the number of graphs with given strength is found in the article [16]. Similarly, both the number of trees and graphs of given type are found in the articles [31] and [16] respectively.

The kinds of graphs corresponding to oriented trees, directed trees, and signed trees have all been enumerated. The solutions appear in the articles [21], [16], and [15] respectively. We have here three cases where the enumeration of graphs corresponding to certain kinds of trees have been obtained. Their study does not suggest methods for proceeding from trees to the enumeation of the correspon ding graphs which have not yet been counted.

§ 6. Comparison between solved and unsolved problems

We begin with a list of graph enumeration problems which have been solved, omitting the tree solutions already mentioned in the preceding section.

List II

SOLVED PROBLEMS IN THE ENUMERATION OF GRAPHS

I. Graphs	1.	Graphs [16]
	2.	Rooted graphs [16]
	3.	Connected graphs [16]
	4.	Graphs of given strength [16]
	5.	Graphs of given type [16]
	6.	Signed graphs [15]
	7.	Subgraphs of a given graph [19]
	8.	Supergraphs of a given graph [20]
	9.	Bicolored graphs [23]
	10.	Graphs with given blocks [42, 9II]
II. Digraphs	11.	Directed graphs [16]
	12.	Weak digraphs [16]
	13.	Oriented graphs [21]
	14.	Tournaments [7]
	15.	Transitive digraphs [60]
	16.	Functional digraphs [26]
III. Partitions	17.	General graphs with given partition [49]
	18.	Multigraphs which are fully factored [49]
	19.	Digraphs with given double partition [37]
IV. Switching	20.	Two-terminal series-parallel networks [50, 52]
	21.	Types of Boolean function [46, 55]
	22.	Spanning trees of a given graph [38]
V. Labels	23.	Labeled graphs [12, 9I]
	24.	Labeled series-parallel networks [3]
	25.	Labeled graphs with a given partition [49]
	26.	Labeled graphs with given blocks [9III]
VI. Asymptotic	27.	The asymptotic number of graphs and labeled graphs [21, 9IV]

The number of graphs was found by taking the pairs of distinct points from among p given points as the figures, and the content of a figure as 0 or 1 corresponding to nonadjacency or adjacency of these two points. Thus the figure series is $1 + x$. The configuration group which serves to count graphs is then obtained from the symmetric group of degree p by considering as the objects to be permuted the pairs of distinct objects. The cycle index of the resulting group is then readily found and Pólya's Theorem gives the counting polynomial for the number of graphs with p points and a given number of lines. This beautiful result, which served as a stimulus for all of our subsequent work on graph enumeration, was communicated to the author in a letter by Pólya; exactly the same formula was found independently in an unpublished work of Slepian, who rediscovered Polya's enumeration method in [55].

The counting of rooted graphs is then an easy modification which results when one takes any one of the objects permuted by the symmetric group as fixed before forming its "pair group". The number of digraphs is also readily obtained from the number of graphs when one constructs the "ordered pair group" analogously to the pair group.

Connected graphs are enumerated in terms of the total number of graphs by a combinatorial method which is exactly parallel to the enumeration of rooted trees in terms of themselves, as derived by PÓLYA [45]. This result turns out to be particularly important because of its wide applicability. In general, it serves to count the number of connected graphs (or other configurations) having a given property when the total number of graphs, both connected and disconnected, is known. If desired, the formula also serves to give the total number of graphs of a given kind in trems of the number of connected such graphs. For example, an immediate application of the method gives the enumeration of weak digraphs. Problems 1 and 2, which ask for the number of strong and unilateral digraphs have not been found amenable to this approach. Problem 3, which asks for the number of digraphs with a singleton point basis can be regarded as a generalization of problem 1. For every strong digraph has a singleton point basis consisting of any one of its points. Problem 19, the number of finite automata, involves a combination of the properties that a digraph have a singleton point basis, that its lines be of type 2, and a kind of regularity condition that every point have out-degree 2. VYSSOTSKY [57] solves a special labeled case of this problem, and also asks the problem of the number of strongly connected finite automata.

The number of oriented graphs is found analogously to the number of digraphs, but involves a modification of both the configuration group and the figure counting series in order to take account of the condition that each line of an oriented graph has exactly one of two possible directions. Again, a figure is a pair of distinct points which are either non-adjacent or are joined by a line in exactly one direction. Hence the figure counting series is $1 + 2x$, where the content of a figure is the number of lines it contains.

The enumeration of signed graphs offers no difficulty whatsoever and is obtained immediately from the formula for the number of graphs by a modification of the figure counting series to $1 + x + y$, where the terms 1, x, and y indicate respectively no line, a positive line, and a negative line joining two points.

Using the line group of a graph as the configuration group and $1 + x$ as the figure counting series, one immediately obtains the number of dissimilar spanning subgraphs of a given graph. Analogous formulas for the number of dissimilar supergraphs of a given graph and in general for the number of types of graph between a given graph-subgraph pair are readily formulated, [22].

Z. SCHUR has kindly pointed out an error in Example 2 of the article [20]. He observes that the correct configuration group for Example 2 is the dihedral group of degree 4 and writes : "We then have

$$F_{Q_3,Q_8}(x) = 1 + x + 2x^2 + x^3 + x^4,$$

which amounts to deleting the middle row of graphs in Figure 2. The three graphs in this row are similar to the corresponding graphs in the upper row."

The problem of enumerating bicolored graphs has recently been handled by the construction of a new binary operation on permutation groups, called exponentiation (see § 4, part c. above). An elementary exposition of the algebraic interaction between this operation and other already known operations on permutation groups such as the direct sum and the cartesian product is given in the note, [27].

Similar although more complicated methods will probably serve to count the number of tricolored graphs; this has not been accomplished as yet. The number of k-colored graphs for $k > 3$ involves even further combinatorial complexities. This is part of problem 9. The other part of problem 9 asks for the enumeration of k-chromatic graphs. Let us consider the simplest case of such graphs, namely, bichromatic graphs. We have already mentioned that the number of connected bichromatic graphs and the number of connected bicolored graphs are equal. But this is not so for disconnected graphs. As a result of this observation, the entire content of Section 5 of [23] on the number of connected bicolored graphs is incorrect, and this section should be deleted.

Problem 8, the number of planar graphs, is entirely untouched. No one has been able to make even a successful beginning. An intuitive indication of the essential difficulty of this particular enumeration problem is that formulas for the number of both planar graphs and those planar graphs which are 4-chromatic would serve to settle the four-color conjecture one way or the other. If these two generating functions were obtained and shown to be equal, then the four-color conjecture would be proved true. On the other hand, if it turned out that there were more planar graphs than planar 4-chromatic graphs with a given number of points and lines, then the 4-color conjectures would thereby be disproved. The enumeration of planar graphs with additional properties is listed as problem 10.

By means of equation (9), NORMAN [42] has derived a formula for the number of connected graphs with given blocks. Nevertheless, although he and several others have tried very hard, no one has succeeded in deriving a formula for the number of blocks with a given number of points and lines, problem 13. The enumeration of graphs with given index and connectivity, problem 12, is conceptually similar. There is a rather complete set of theorems involving the index of a graph and its connectivity, but these have not proven helpful in finding the kind of permutation group characterizations of such graphs which would be useful in counting them.

As mentioned above, READ [49] has obtained a formula for the number of general graphs with given partition. But his method has not provided any procedure for eliminating graphs with loops and multiple lines. Thus for graphs, problem 4 remains unsolved, as well as problems 5, 6, and 7. However, READ has found a formula for the number of *labeled* graphs with a given partitition and without loops or multiple lines. He has also applied his superposition theorem to obtain the number of multiple regular graphs (without loops) which are fully factored. But again, his method does not give the corresponding number of graphs without multiple lines, a special case of problem 6.

The number of digraphs with a given double partitition was discussed in the article by KATZ and POWELL [37]. They reduced this question to a formulation by SUKHATME which gives recurrence relations for certain number theoretic functions. SUKHATME has constructed tables for these numbers which serve to give what KATZ and POWELL call the number of "locally restricted directed graphs" having up to 13 lines. However, a general group theoretic formula has never been found.

We now mention some recently found formulas for digraphs. LEO MOSER shown in [60] that the total number of transitive digraphs with p points has (regardless of the number of lines) is equal to

$$\frac{1}{p+1}\binom{2p}{p}.$$

DAVIS [7] has found a formula for the number of tournaments, i.e., complete oriented graphs. This result is also readily obtained as a special case of the formula in [21] for the number of oriented graphs. Functional digraphs are defined as the graphical representation of a binary relation in which each first element has a unique second element. The number of functional digraphs has been found [26] by means of a characterization to the effect that every weak component of a finite functional digraph contains exactly one directed cycle together with rooted trees located at each point of the cycle.[5] It follows that the configuration group for this problem is the cyclic group and that the figure counting series is the known generating function for rooted trees.

§ 7. Importance of enumeration problems for other fields

ISING [35] proposed the problem which now bears his name and solved it for the one-dimensional case only, leaving the two-dimensional case as an unsolved problem. The first solution to the two-dimensional Ising problem was obtained by ONSAGER [43]. Recently KAC and WARD [36] discovered a simpler procedure involving determinants which, however, is not logically complete.

Their purpose was to indicate the ideas involved in a purely combinatorial development of a proof of the two-dimensional Ising problem, and they provided heuristic arguments only. FEYNMAN's simplification (unpublished) of their treatment is even simpler and contains precisely the same logical gap; a combinatorial formulation of the statement whose proof would complete this development was given by M. COHEN in [28].

Among its many equivalent formulations, the ISING problem can be regarded as an enumeration problem for linear graphs. NEWELL and MONTROLL [41] give a very clear exposition of the problem. Consider as in Figure 7a a two-dimensional lattice with n points. For applications to statistical mechanics, only large values of n are interesting and usually the lattice is regarded as drawn on a torus.

Regarding Figure 7a as a labeled graph, the first part of problem 14 asks for a purely combinatorial method for counting the distinct admissible subgraphs, i.e., those in which every point is even. Such a combinatorial solution to the two-dimensional problem serve to fill in the logical gap in both the treatments of KAC and WARD, and that of FEYNMAN. It would also offer some hope for the eventual solution of problem 15, the three-dimensional ISING problem. (See the footnote of Problem 14.)

The second part of problem 14 asks for the number of admissible subgraphs in a two-dimensional lattice with diagonals, as shown in Figure 7b. This would also be of considerable interest in theoretical physics. The second part of problem 15 also asks for a variation of the two-dimensional problem,

[5] READ has obtained an elegant simplification of this formula, to appear in Math. Annalen.

namely for the number of admissible subgraphs with a given number of lines and a given area. This is referred to in the physical literature as the two-dimensional ISING problem with magnetic effect.

The paving problem stated as problem 16 was proposed to the author by G. E. UHLENBECK. It has applications in the statistical mechanics of liquids. Problem 17, which asks for the number of square-celled animals as pictured in Figure 8, was proposed independently both by UHLENBECK, who was interested in the number of different shapes of paving blocks for eventual application to problem 16, by an anonymus biologist who was interested in the number of graphs of this kind as the number of different shapes of animals with a given number of cells, and by GOLOMB [14] under the name of polyminoes, or generalized dominoes; he asserts that this is a well-known unsolved combinatorial problem.

Much effort has already been expended in problem 18, the number of types of Hamilton cycle in an n-cube, by those interested in applications to synthesis problems in switching networks. While several kinds of partial results have been obtained, usually by exhaustive methods, the general problems has never been appropriately conceptualized. The number of finite automata, problem 19, would be of considerable interest in switching theory, as well as problem 20, the number of indecomposable two-terminal series-parallel networks described above.

The number of symmetric graphs, problem 25, has been studied for applications to electrical network theory by FOSTER [10]. While not solving this problem, FOSTER has provided a useful collection of diagrams of symmetric graphs, and a classification method.

Problem 21, the number of self-complementary graphs, would be of interest because of the set theoretic operation of complementation, while problem 22, which asks for the number of nonisomorphic abstract simplical complexes, is of interest in combinatorial topology.

Combinatorial considerations play an important part in the statistical design of experiments. In these processes, Latin squares play an important role. A closed formula for the number of distinct Latin squares of arbitrary order n, problem 23, appears to be extremely difficult. The numbers have been found through $n = 7$ by exhaustive methods. In connection with "association schemes" of statistical block designs, CONNOR [5], HOFFMAN [33], and SHRIKHANDE [54] have made a study of the structural prpoperties of the line graph of a complete graph. The number of line graphs with a given number of points and lines is stated as problem 24. Further enumeration problems are suggested by the recent work of BOSE, PARKER, and SHRIKHANDE in which they disproved EULER's conjecture concerning the existence of orthogonal Latin squares of order $4n + 2$.

KRAUSZ [40] has obtained the following elegant characterization:

G is a line graph if and only if there exists a partition of the set of lines of G into complete subgraphs such that no point of G lies in more than 2 of these subgraphs.

A group theoretic description of line graphs for the purposes of applying PÓLYA's method does not appear straightforward.

Obviously, the line graph L_n of the complete graph K_n satisfies the three conditions:

1. Each point has degree $2(n - 2)$.

2. Any two nonadjacent points are mutually adjacent to exactly 4 points.
3. Any two adjacent points are mutually adjacent to exactly $n - 2$ points.

Between them, CONNOR, HOFFMAN, and SHRIKHANDE have shown that these necessary conditions are also sufficient except for $n = 8$. This gives another indication of the difficulty of conveniently characterizing and hence of counting line graphs. Surprisingly, HOFFMAN has just disproved the sufficiency for $n = 8$.

We have already noted that counting problems for labeled graphs are always easier than the corresponding problems for unlabeled graphs. Nevertheless, they are also interesting in their own right. CARLITZ and RIORDAN [3] have found the number of labeled two-terminal series-parallel networks. Ford and UHLENBECK [9 I, III] have found the number of labeled graphs and also have counted labeled graphs with given blocks. In addition, they have made a study of the asymptotic number of graphs with given properties, extending the work of PÓLYA and OTTER in this area. GILBERT [12] has also enumerated labeled graphs and labeled digraphs. GILBERT [13] has done the most work on a number of types of complete cycles in an n-cube. In a recent note [21], we have gathered together for a readership of logicians some of the asymptotic results for graphs which have been found. HUSIMI [34] has obtained the number of labeled graphs in which every block is complete. CAYLEY [4] has shown that the number of labeled trees with p points is p^{p-2}. This result has been rediscovered many times and is also a special case of HUSIMI's formula.

PÓLYA [47] has written a beautiful and clear exposition of "picture-writing" which gives an aid to intuition in thinking about graphical enumeration problems.

SENIOR [53] appears to have made the first exhaustive studies on graphs with a given partition. However, he was very much restrained in his outlook to the study of those kinds of partitions which have immediate application to organic chemistry.

We conclude by stating some typical asymptotic formulas for certain kinds of graphs. As above, t_n and T_n are the number of trees and rooted trees with n points while $t(x)$ and $T(x)$ are the corresponding generating functions. PÓLYA [45] has shown that $t(x)$ and $T(x)$ have the same radius of convergence $r = 0.3383219$. This number r occurs in asymptotic formulas for t_n and T_n, as does the number $b = 7.924780$ which is a constant associated with the power series $t(x)$ and its derivates. With these preliminaries, OTTER [44] derives the asymptotic formulas:

$$T_n \sim \frac{b r^{-n+3/2}}{2\sqrt{\pi}\, n^{3/2}}$$

$$t_n \sim \frac{b^3 r^{-n+9/2}}{4\sqrt{\pi}\, n^{5/2}}.$$

Using similar methods, FORD and UHLENBECK [9 III] obtain asymptotic values of h_n and H_n, the number of cacti (previously called "Husimi trees", which are connected graphs in which no line lies on more than one cycle) with n points. In these formulas the common radius of convergence $s = 0.22215$ of $h(x)$ and $H(x)$ occur, as do the numbers $C = 4.395$ and $d = 11.46$:

$$H_n \sim \frac{Cs^{-n+1/2}}{2\sqrt{\pi}\, n^{3/2}}$$

$$h_n \sim \frac{3ds^{-n+3/2}}{4\sqrt{\pi}\, n^{5/2}}.$$

PÓLYA has shown in [9 IV] and [25] that the number g_p of graphs with p points satisfies

$$g_p \sim \frac{1}{p!}\, 2^{p(p-1)/2}.$$

Furthermore, the number $g_{p,q}$ of graphs with p points and q lines is asymptotically given by

$$g_{p,q} \sim \frac{1}{p!}\binom{p(p-1)/2}{q/2},$$

where this formula is known to hold for large p and $0 \ll q \ll p(p-1)/2$, so that the majority of graphs is included.

The corresponding asymptotic formulas for labeled graphs of these various kinds are much more easily derived. We may multiply the number of unsolved problems proposed here by asking for the asymptotic number of graphs of each kind, and also for the number of labeled graphs of each kind and their asymptotic numbers.

ADDENDUM I

The following paper is extremely appropriate from a historical standpoint:

REDFIELD, J. H. „The theory of group-reduced distributions." *American Journal of Mathematics* **49** (1927) 433—455.

The reference to this paper was found in the book:
LITTLEWOOD, D. E. *The theory of group characters*, Oxford, 1940.

This remarkable paper by REDFIELD apparently anticipated most of the major developments in enumeration techniques and results for the next thirty years. For it contains:

(1) The exact formula of READ Superposition Theorem (9).
(2) Apparently the first published definition of the cycle index of a permutation group under the name of the "group-reduction function".
(3) Formulas for the cycle index of the symmetric, alternating, cyclic and dihedral groups.
(4) The cycle index of the group of symmetries of a 3-cube. He actually substitues $1 + x$ into this cycle index, thereby giving the first known example of PÓLYA's theorem. This also anticipates the enumeration of the symmetry types of boolean functions due to PÓLYA and SLEPIAN!

(5) A substitution of $1/(1-x)$ into this cycle index. This is a device used in the formula of [16] for enumerating graphs in which any number of lines are permitted to join the same two points.
(6) The number of graphs with p points and q lines for $p = 5$ and $q = 4$ as a solution of a problem involving the number of types of binary relations.

ADDENDUM II

The following remarks concern Problem 17, the cell growth problem. STEIN, WALDEN, and WILLIAMSON have programmed a computing machine to generate the isomorphism classes of animals. The results, which have been carefully checked, show that the number of animals with 7 cells is 107 rather than the number 109 which is stated by GOLOMB. Further, A_8, the number of 8 celled animals, is **363**. In addition to these, there is exactly one animal with 7 cells which is connected but not simply connected, and there are 6 such animals with 8 cells. This program is being carried out to exemplify a purely combinatorial application of a digital computing machine.

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НЕРЕШЕННЫЕ ПРОБЛЕМЫ О ПЕРЕЧИСЛЕНИИ ГРАФОВ

F. HARARY

Резюме

Цель работы указать на ряд проблем относительно перечисления графов, чтобы вызвать интерес математиков к таким проблемам. Кажется маловероятным, что в скором будущем все эти проблемы будут решены, так как среди них фигурирует и гипотеза о четырех цветах.

Сначало характеризуется значение проблемы о перечислении графов и направленных графов. Затем работа знакомит с основными понятиями, необходимых для того, чтобы можно было кратко сформулировать нерешенные проблемы. Далее приводятся (без доказательства) некоторые методы, которые, применяются в этой области, среди них наиболее важным является элегантный и эффективный метод PÓLYA [45]. Этот метод, или некоторое его видоизменение применяюся в большинстве известных решений этих проблем. Сравниваются проблемы относительно числа деревьев различных сортов с аналогичными проблемами относительно графов. Автор дает список 27 решенных и 27 нерешенных проблем. (см. стр. 86 и 68.) Он указывает на их значение и связанные с ними трудности. Упоминается также вычисление асимптотического числа графов различных сортов. В связи с темой дается обширная библиография.