# ON CYCLIC EQUATIONS

by

J. ACZÉL, M. GHERMANESCU and M. HOSSZÚ

In this paper we consider the functional equation

$$(1)\qquad F(x_1, x_2, \ldots, x_{p-1}, x_p) + F(x_2, x_3, \ldots, x_p, x_{p+1}) + \ldots + F(x_{n-p+1}, x_{n-p+2}, \ldots, x_{n-1}, x_n) + F(x_{n-p+2}, x_{n-p+3}, \ldots, x_n, x_1) + \ldots + F(x_n, x_1, \ldots, x_{p-2}, x_{p-1}) = 0,$$

where $p \leqq n$ are two arbitrary positive integers, $x_i \in S$ are independent variables and the values of the function $F$ lie in a module, i.e. in an additive abelian group $M$. We do not impose any restriction upon the set $S$ and *we suppose only that $M$ be a module in which for every positive integer $m \leqq n$ the equation $mX = A$ has a unique solution $X = A/m$.*

In certain particular cases (1) was solved by elementary methods ([4], [1], [2]). Here we give equally elementary methods for solving (1) in every possible case.

1. Let first be $p = n$.

$$(2)\qquad F(x_1, x_2, \ldots, x_{n-1}, x_n) + F(x_2, x_3, \ldots, x_n, x_1) + \ldots + F(x_{n-1}, x_n, x_1, \ldots, x_{n-3}, x_{n-2}) + F(x_n, x_1, \ldots, x_{n-2}, x_{n-1}) = 0.$$

Then

$$F(x_1, x_2, \ldots, x_{n-1}, x_n) = -F(x_2, x_3, \ldots, x_n, x_1) - F(x_3, x_4, \ldots, x_1, x_2) - \ldots - F(x_n, x_1, \ldots, x_{n-2}, x_{n-1})$$

and

$$\begin{aligned} nF(x_1, x_2, \ldots, x_{n-1}, x_n) &= F(x_1, x_2, \ldots, x_{n-1}, x_n) - F(x_2, x_3, \ldots, x_n, x_1) + \\ &+ F(x_1, x_2, \ldots, x_{n-1}, x_n) - F(x_3, x_4, \ldots, x_1, x_2) + \ldots + \\ &+ F(x_1, x_2, \ldots, x_{n-1}, x_n) - F(x_n, x_1, \ldots, x_{n-2}, x_{n-1}) = \\ &= F(x_1, x_2, \ldots, x_{n-1}, x_n) - F(x_2, x_3, \ldots, x_n, x_1) + \\ &+ F(x_1, x_2, \ldots, x_{n-1}, x_n) - F(x_2, x_3, \ldots, x_n, x_1) + \\ &+ F(x_2, x_3, \ldots, x_n, x_1) - F(x_3, x_4, \ldots, x_1, x_2) + \\ &+ F(x_1, x_2, \ldots, x_{n-1}, x_n) - F(x_2, x_3, \ldots, x_n, x_1) + \\ &+ F(x_2, x_3, \ldots, x_n, x_1) - F(x_3, x_4, \ldots, x_1, x_2) + \end{aligned}$$

$$+ F(x_3, x_4, \ldots, x_1, x_2) - F(x_4, x_5, \ldots, x_2, x_3) + \ldots +$$
$$+ F(x_1, x_2, \ldots, x_{n-1}, x_n) - F(x_2, x_3, \ldots, x_n, x_1) +$$
$$+ F(x_2, x_3, \ldots, x_n, x_1) - F(x_3, x_4, \ldots, x_1, x_2) + \ldots +$$
$$+ F(x_{n-1}, x_n, \ldots, x_{n-3}, x_{n-2}) - F(x_n, x_1, \ldots, x_{n-2}, x_{n-1}).$$

By denoting the sum of members with positive sign by $n\, G\, (x_1, x_2, \ldots, x_n)$ (from $nG = A$ $G$ can be uniquely determined) we have

$$F\,(x_1, x_2, \ldots, x_{n-1}, x_n) = G\,(x_1, x_2, \ldots, x_{n-1}, x_n) - \\ - G\,(x_2, x_3, \ldots, x_n, x_1). \tag{3}$$

On the other hand every function of the form (3) satisfies the equation (2) and thus we have (cf. [2]) the

**Theorem 1.** (3) *is the most general solution of the functional equation* (2).

We remark that the same consideration shows also that

$$F(x) = G(x) - G(Cx)$$

is the general solution of the functional equation

$$F(x) + F(Cx) + \ldots + F(C^{n-1}\, x) = 0$$

where $x$ is element of an arbitrary set, $C$ is a cyclic operator with period $n$ ($C^n x = x$) defined on this set and the values of the function $F$ lie in a module in which the equation $nX = A$ has a unique solution (cf. [3]).

**2.** Now we take $n \geqq 2p - 1$ and write (1) in more detail

$$F_p(x_1, x_2, \ldots, x_{p-1}, x_p) + F_p(x_2, x_3, \ldots, x_p, x_{p+1}) + \ldots + \tag{4}$$
$$+ F_p(x_p, x_{p+1}, \ldots x_{2p-2}, x_{2p-1}) + F_p(x_{p+1}, x_{p+2}, \ldots, x_{2p-1}, x_{2p}) + \ldots +$$
$$+ F_p(x_{n-p+1}, x_{n-p+2}, \ldots, x_{n-1}, x_n) + F_p(x_{n-p+2}, x_{n-p+3}, \ldots, x_n, x_1) + \ldots +$$
$$+ F_p(x_n, x_1, \ldots, x_{p-2}, x_{p-1}) = 0.$$

We keep the variables $x_{p+1}, x_{p+2}, \ldots, x_n$ constants and carry all members of (4) except $F_p(x_1, x_2, \ldots, x_{p-1}, x_p)$ on the right hand side. Thus we have there $p - 1$ members depending only upon $x_2, x_3, \ldots, x_{p-1}, x_p$ and $n - p$ members depending only upon $x_1, x_2, \ldots, x_{p-2}, x_{p-1}$:

$$F_p(x_1, x_2, \ldots, x_{p-1}, x_p) = H(x_2, x_3, \ldots, x_{p-1}, x_p) + \\ + G(x_1, x_2, \ldots, x_{p-2}, x_{p-1}). \tag{5}$$

Putting this back into (4) we have

$$G(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) + H(x_2, x_3, \ldots, x_{p-1}, x_p) +$$
$$+ G(x_2, x_3, \ldots, x_{p-1}, x_p) + H(x_3, x_4, \ldots, x_p, x_{p+1}) + \ldots +$$
$$+ G(x_n, x_1, \ldots, x_{p-3}, x_{p-2}) + H(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) = 0$$

or by denoting

$$F_{p-1}\,(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) = G(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) + \\ + H(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) \tag{6}$$

we arrive at

$$F_{p-1}(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) + F_{p-1}(x_2, x_3, \ldots, x_{p-1}, x_p) + \ldots + \\ + F_{p-1}(x_n, x_1, \ldots, x_{p-3}, x_{p-2}) = 0.$$

We observe that this is an equation of the form (4) with $p-1$ instead of $p$. For $p = 1$ this equation is

$$F_1(x_1) + F_1(x_2) + \ldots + F_1(x_n) = 0$$

this involves

$$F_1(x) = 0 \tag{7}$$

(by keeping $x_2, \ldots, x_n$ constant $F_1(x) = C$ and from $nC = 0$, by our supposition on the unique solution of $nX = A$ in $M$, $C = 0$ follows). We prove that

$$F_k(x_1, x_2, \ldots, x_{k-1}, x_k) = G_k(x_1, x_2, \ldots, x_{k-2}, x_{k-1}) - \\ - G_k(x_2, x_3, \ldots, x_{k-1}, x_k). \tag{8}$$

By (7) this holds for $k = 1$ and if it is true for $k = p-1$ it holds for $k = p$ too. In fact, if

$$F_{p-1}(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) = G_{p-1}(x_1, x_2, \ldots, x_{p-3}, x_{p-2}) - \\ - G_{p-1}(x_2, x_3, \ldots, x_{p-2}, x_{p-1}),$$

then by (6) and denoting

$$G_p(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) = G(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) + G_{p-1}(x_2, x_3, \ldots, x_{p-2}, x_{p-1})$$

we have

$$H(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) = - G(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) + \\ + G_{p-1}(x_1, x_2, \ldots, x_{p-3}, x_{p-2}) - G_{p-1}(x_2, x_3, \ldots, x_{p-2}, x_{p-3}) = \\ = G_{p-1}(x_1, x_2, \ldots, x_{p-3}, x_{p-2}) - G_p(x_1, x_2, \ldots, x_{p-2}, x_{p-1})$$

and by (5)

$$F_p(x_1, x_2, \ldots, x_{p-1}, x_p) = G_p(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) - \\ - G_p(x_2, x_3, \ldots, x_{p-1}, x_p) \tag{9}$$

i.e. (8) remains valid for $k = p$, qu. e. d. Thus we have proved that every solution of (4) is of the form (9).

On the other hand every function of the form (9) satisfies the equation (4). Thus we have the

**Theorem 2.** *(9) is the most general solution of the functional equation* (4) ($n \geq 2p - 1$).

**3.** Now we consider the cases $p < n < 2p - 1$ and prove the following

**Theorem 3.** *The most general solution of the functional equation* (1) *is for* $p < n < 2p - 1$

$$(10)\quad F(x_1, x_2, \ldots, x_{p-1}, x_p) = G_0(x_1, x_2, \ldots, x_{p-2}, x_{p-1}) - G_0(x_2, x_3, \ldots, x_{p-1}, x_p) + \sum_{k=1}^{[(2p-n)/2]} [G_k(x_1, x_2, \ldots, x_k, x_{n-p+k+1}, \ldots, x_{p-1}, x_p) - G_k(x_{p-k+1}, x_{p-k+2}, \ldots, x_p, x_1, \ldots, x_{2p-n-k})] .$$

For sake of better understanding we effectuate the proof in the case $n = 4$, $p = 3$ but in a manner valid also in the general case. Thus we prove that

$$(11)\quad F(x_1, x_2, x_3) = G_0(x_1, x_2) - G_0(x_2, x_3) + G_1(x_1, x_3) - G_1(x_3, x_1)$$

is the most general solution of the functional equation

$$(12)\quad F(x_1, x_2, x_3) + F(x_2, x_3, x_4) + F(x_3, x_4, x_1) + F(x_4, x_1, x_2) = 0 .$$

In fact, put $x_4 = c$ in (12):

$$(13)\quad F(x_1, x_2, x_3) + F(x_2, x_3, c) + F(x_3, c, x_1) + F(c, x_1, x_2) = 0 .$$

We see that here already all members except $F(x_1, x_2, x_3)$ depend only from two ($< p$) variables. By adding members of the form of the right-hand side of (11) (we will call expressions of this form *G-expressions)* we can change these latter members into such ones, where the $x_i$-s if not kept constant stand just on the $i$-th places in the function $F$:

$$F(x_1, x_2, x_3) + F(x_1, x_2, c) + F(x_1, c, x_3) + F(c, x_2, x_3) + F(x_2, x_3, c) + F(x_3, c, x_1) + F(c, x_1, x_2) - F(x_1, x_2, c) - F(x_1, c, x_3) - F(c, x_2, x_3) = 0$$

i.e.

$$(14)\quad F(x_1, x_2, x_3) = G_0^1(x_1, x_2) - G_0^1(x_2, x_3) + G_1^1(x_1, x_3) - G_1^1(x_3, x_1) - [F(x_1, x_2, c) + F(x_1, c, x_3) + F(c, x_2, x_3)] ,$$

where

$$(15)\quad G_0^1(x_1, x_2) = F(x_1, x_2, c) - F(c, x_1, x_2) ,\quad G_1^1(x_1, x_3) = F(x_1, c, x_3) .$$

In order to reduce the members $F(x_1, x_2, c) + F(x_1, c, x_3) + F(c, x_2, x_3)$ in (14) which have not yet the form of $G$-expressions into functions of less variables (here already only one variable) we put $x_3 = c$ resp. $x_2 = c$ resp. $x_1 = c$ into (13) and get

$$F(x_1, x_2, c) + F(x_2, c, c) + F(c, c, x_1) + F(c, x_1, x_2) = 0 ,$$

$$F(x_1, c, x_3) + F(c, x_3, c) + F(x_3, c, x_1) + F(c, x_1, c) = 0 ,$$

$$F(c, x_2, x_3) + F(x_2, x_3, c) + F(x_3, c, c) + F(c, c, x_2) = 0 .$$

We apply the same way of transformations to these equations by which we got (14) from (13):

$$\begin{aligned} F(x_1, x_2, c) &+ F(x_1, c, c) + F(c, c, x_3) + F(c, x_2, x_3) + \\ &+ F(x_2, c, c) + F(c, c, x_1) + F(c, x_1, x_2) - \\ &- F(x_1, c, c) - F(c, c, x_3) - F(c, x_2, x_3) = 0, \\ F(x_1, c, x_3) &+ F(c, x_2, c) + F(x_1, c, x_3) + F(c, x_2, c) + \\ &+ F(c, x_3, c) + F(x_3, c, x_1) + F(c, x_1, c) - \\ &- F(c, x_2, c) - F(x_1, c, x_3) - F(c, x_2, c) = 0, \\ F(c, x_2, x_3) &+ F(x_1, x_2, c) + F(x_1, c, c) + F(c, c, x_3) + \\ &+ F(x_2, x_3, c) + F(x_3, c, c) + F(c, c, x_2) - \\ &- F(x_1, x_2, c) - F(x_1, c, c) - F(c, c, x_3) = 0. \end{aligned}$$

In the equation obtained by adding the last three equations there will figure besides the requested members $F(x_1, x_2, c) + F(x_1, c, x_3) + F(c, x_2, x_3)$ only $G$-expressions and functions depending on only one variable:

$$\begin{aligned} &2[F(x_1, x_2, c) + F(x_1, c, x_3) + F(c, x_2, x_3) + \\ &\quad + F(x_1, c, c) + F(c, x_2, c) + F(c, c, x_3)] = \\ &\quad = G_0^2(x_1, x_2) - G_0^2(x_2, x_3) + G_1^2(x_1, x_3) - G_1^2(x_3, x_1), \end{aligned} \tag{16}$$

where

$$\left\{ \begin{aligned} G_0^2(x_1, x_2) &= F(x_1, x_2, c) - F(c, x_1, x_2) + F(x_1, c, c) - F(c, x_1, c) + \\ &\quad + F(c, x_2, c) - F(c, c, x_2), \\ G_1^2(x_1, x_3) &= F(x_1, c, x_3) + F(x_1, c, c) + F(c, c, x_3). \end{aligned} \right. \tag{17}$$

By our supposition on the solvability of $mX = A$ $(m = 2)$ we can write over (16) as follows:

$$\begin{aligned} &F(x_1, x_2, c) + F(x_1, c, x_3) + F(c, x_2, x_3) = \\ &\quad = \frac{1}{2}[G_0^2(x_1, x_2) - G_0^2(x_2, x_3) + G_1^2(x_1, x_3) - G_1^2(x_3, x_1)] - \\ &\quad - [F(x_1, c, c) + F(c, x_2, c) + F(c, c, x_3)]. \end{aligned} \tag{18}$$

Finally we apply the same transformation process as before upon the last bracket of (18). We put into (13) $x_2 = x_3 = c$ resp. $x_1 = x_3 = c$ resp. $x_1 = x_2 = c$ and get

$$\left\{ \begin{aligned} &F(x_1, c, c) + F(c, c, c) + F(c, c, x_1) + F(c, x_1, c) = 0, \\ &F(c, x_2, c) + F(x_2, c, c) + F(c, c, c) + F(c, c, x_2) = 0, \\ &F(c, c, x_3) + F(c, x_3, c) + F(x_3, c, c) + F(c, c, c) = 0. \end{aligned} \right. \tag{19}$$

We remark that (13) with $x_1 = x_2 = x_3 = c$ implies

$$4F(c, c, c) = 0$$

and by the uniqueness of the solution of $nX = A$ $(n = 4)$

$$F(c, c, c) = 0 .$$

Taking this into account we write the equations (19) in the form transformed with $G$-expressions and add them:

$$3\,[F(x_1, c, c) + F(c, x_2, c) + F(c, c, x_3)] = \tag{20}$$
$$= G_0^3(x_1, x_2) - G_0^3(x_2, x_3) + G_1^3(x_1, x_3) - G_1^3(x_3, x_1) ,$$

where

$$\begin{cases} G_0^3(x_1, x_2) = F(x_1, c, c) - F(c, x_1, c) + F(c, x_2, c) - F(c, c, x_2) , \\ G_1^3(x_1, x_3) = F(x_1, c, c) + F(c, c, x_3) . \end{cases} \tag{21}$$

We divide (20) by 3 ($3X = A$ has a unique solution), we put this into (18) and substitute finally the equation thus obtained into (14) to arrive at last to

$$F(x_1, x_2, x_3) = G_0(x_1, x_2) - G_0(x_2, x_3) + G_1(x_1, x_3) - G_1(x_3, x_1) \tag{11}$$

where

$$G_i = G_i^1 - G_i^2/2 + G_i^3/3 \qquad (i = 0, 1)$$

$G_i^j$ $(i = 0, 1;\ j = 1, 2, 3)$ being the functions defined in (15), (17), (21).

Thus we have proved that (12) implies (11). On the other hand one verifies immediately that every function of the form (11) satisfies the equation (12). So it is in the general case $p < n < 2p - 1$ too and thus Theorem 3 is proved.

The reader may remark that we did not use in this proof the equation (12) in its full generality only the particular case $x_4 = c$ (13) of it. This is so in the general case too: in order to get the solution (10) it is enough to suppose the validity of the functional equation (1) for one special constant value $c$ of the variables $x_{p+1}, x_{p+2}, \ldots, x_{n-1}, x_n$, while $x_1, x_2, \ldots, x_{p-1}, x_p$ vary and our result (10) shows that then (1) remains valid also for independently variable $x_1, x_2, \ldots, x_{p-1}, x_p, x_{p+1}, \ldots, x_n$.

We remark that also Theorem 2 can be proved by the same method by which we proved Theorem 3. On the other hand the methods applied to prove Theorems 1 and 2 can also be applied to prove the special cases of Theorem 3, moreover in these considerations (cf. **1**, **2**) the existence of a unqiue solution of $mX = A$ is needed only for $m = n$.

(Received October 27, 1959.)

REFERENCES

[1] GHERMANESCU, M.: „Sur quelques équations fonctionnelles linéaires." *Bulletin Soc. Math. France* **68** (1940) 109—128.
[2] GHERMANESCU, M.: „O clasa de ecuaţii funcţionale." *Pozitiva* **1** (1940) 121—125.
[3] GHERMANESCU, M.: „Ecuaţii funcţionale cu argument funcţional $n$-periodic." *Acad. R. P. Romine Buletin şti. mat. fiz.* **9** (1957) 43—78.
[4] SINZOW, D. M.: „Über eine Funktionalgleichung." *Archiv der Math. und Physik* **6** (1903) 216—217.

# О ЦИКЛИЧЕСКИХ УРАВНЕНИЯХ

J. ACZÉL, M. GHERMANESCU и M. HOSSZÚ

## Резюме

Работа занимается решением функционального уравнения (1), где переменные являются элементами любого множества, а значения функции элементы такого модуля, где уравнения вида $mX = A$ при любых натуральных $m$, не превосходящих $n$, имеют единственное решение $X$. Наиболее общее решение (без всяких условий регулярности) имеет вид (3), (9) или (10), в зависимости от того, будет ли $p = n$, $n \geqq 2p - 1$ или $p < n < 2p - 1$. В этих трех случаях фигурируют и разные доказательства, хотя некоторые из них могут быть применены и в других случаях.