ON A THEOREM OF PAUL LÉVY

G. SZEKERES[1]

1. Let $f(x)$ be a continuous strictly increasing function such that

$$f(x) > x \quad \text{for } x > a. \tag{1}$$

A family of fractional iterates of $f(x)$ is obtained by considering Abel's equation

$$A(f(x)) = A(x) + 1, \qquad x > a. \tag{2}$$

If $A(x)$ is a continuous and strictly increasing solution of this functional equation and $A_{-1}(y)$ is the inverse of $A(x)$ (so that $A_{-1}(A(x)) = x$ for $x > a$), then

$$f_\sigma(x) = A_{-1}(A(x) + \sigma), \qquad -\infty < \sigma < \infty \tag{3}$$

defines a family of functions with the property that

$$f_\sigma(f_\tau(x)) = f_{\sigma+\tau}(x), \qquad f_1(x) = f(x). \tag{4}$$

In particular $f_0(x) = x$ and $f_{-1}(x)$ is the inverse of $f(x)$. The interpretation of (3) and (4) is that they hold for sufficiently large x; for instance $f_\sigma(x)$ in (3) is defined for $x > a$ if $\sigma \geqq 0$ and for $x > A_{-1}(A(a) - \sigma)$ if $\sigma < 0$.

For $\sigma = n$, $n = 1, 2, \ldots$, $f_n(x)$ is the n-th natural iterate of $f(x)$,

$$f_{n+1}(x) = f(f_n(x)), \qquad n = 1, 2, \ldots,$$

hence independent of $A(x)$. For non-integer values of σ, $f_\sigma(x)$ is not determined uniquely but depends on the particular solution of the functional equation (1). To enforce uniqueness we need more information about the expected behaviour of the iterates.

Suppose that

$$f(x) = x + \omega(x) \tag{5}$$

where $\omega(x)$ is differentiable and $\omega'(x) \to 0$ as $x \to \infty$. By induction one easily verifies that

$$f_n(x) = x + \omega_n(x), \qquad n = 1, 2, 3, \ldots \tag{6}$$

---

[1] University of Adelaide, Adelaide, South Australia.

where $\omega_n'(x) \to 0$ as $x \to \infty$ and

(7) $$\lim_{x\to\infty} \frac{\omega_n(x)}{\omega(x)} = n\,.$$

To prove (7) for $n+1$, note that

$$\omega_{n+1}(x) = f(x+\omega_n(x)) - x = \omega_n(x) + \omega\big(x+\omega_n(x)\big) =$$
$$= \omega_n(x) + \omega(x) + \omega_n(x)\,\omega'\big(x+\theta\,\omega_n(x)\big), \qquad 0<\theta<1,$$

and this is asymptotically equal to $(n+1)\,\omega(x)$ if (7) is true for n, since $\omega'\big(x+\theta\omega_n(x)\big) \to 0$ as $n\to\infty$. Note that

(8) $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\omega_n(x) = \infty$$

for every $x > a$, by (1). Similarly it can be shown that

$$f_{-n}(x) = x + \omega_{-n}(x), \qquad n = 1, 2, \dots$$

where

$$\lim_{x\to\infty}\frac{\omega_{-n}(x)}{\omega(x)} = -n\,.$$

It is therefore quite natural to ask whether there exists a family of iterates

(9) $$f_\sigma(x) = x + \omega_\sigma(x)$$

such that

(10) $$\lim_{x\to\infty}\frac{\omega_\sigma(x)}{\omega(x)} = \sigma$$

for every real σ.

An affirmative answer was given by PAUL LÉVY in 1928;[2] he showed that if $\omega'(x)$ is of bounded variation then such a family does in fact exist and is uniquely determined by $f(x)$. This is briefly LÉVY's argument:

Suppose first that there exists a family of iterates (9) with the asymptotic property (10). Let $y = f_\sigma(x)$ and write $x_n = f_n(x)$, $y_n = f_n(y)$ so that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ by (8). We have $f_n(y) = f_n\big(f_\sigma(x)\big) = f_\sigma\big(f_n(x)\big)$, i.e.

$$y_n = x_n + \omega_\sigma(x_n)$$

so that

$$\lim_{n\to\infty}\frac{y_n - x_n}{\omega(x_n)} = \lim_{n\to\infty}\frac{y_n - x_n}{x_{n+1} - x_n} = \sigma$$

by (10). In other words, the index of iteration σ is determined in a perfectly unique manner from the formula

(11) $$\sigma = \lim_{n\to\infty}\frac{y_n - x_n}{x_{n+1} - x_n}$$

for any pair of values $x > a$, $y > a$.

---

[2] Ann. Mat. Pura Appl. (4) **5** (1928), p. 282.

On the other hand, it is easy to show that the limit (11) actually exists, at least for $x \leq y \leq f(x)$, provided that $\omega'(x)$ is of bounded variation. For denoting by σ_n the right hand member of (11), one finds by a simple calculation

$$\sigma_{n+1} - \sigma_n = \sigma_n \frac{\omega(x_n)}{\omega(x_{n+1})} [\omega'(\xi_n) - \omega'(\xi'_n)] \tag{12}$$

where ξ_n, ξ'_n are between x_n and x_{n+1}. But $0 \leq \sigma_n \leq 1$ since $x \leq y \leq f(x)$, $\omega(x_n)/\omega(x_{n+1}) \to 1$ since $\omega'(x_n) \to 0$, and $\sum_n | \omega'(\xi_n) - \omega'(\xi'_n)|$ converges since $\omega'(x)$ is of bounded variation. Hence $\sum_n | \sigma_{n+1} - \sigma_n |$ converges and $\lim_{n\to\infty} \sigma_n = \sigma$ exists. Note that the convergence of $\sum_n | \sigma_{n+1} - \sigma_n|$ is uniform for fixed x and $x \leq y \leq f(x)$ and in fact uniform for $a < b \leq x \leq f(b)$, $x \leq y \leq f(x)$.

Lévy's argument is incomplete in several respects.[3] First, if we write $\sigma = \lambda(x, y)$, it is necessary to show that for fixed x, $\lambda(x, y)$ is continuous and strictly increasing in y. For only then can we say with certainty that $\sigma = \lambda(x, y)$ is solvable for y and that the function $y = f_\sigma(x)$ does indeed exist (for sufficiently large x). Secondly, it is necessary to show that $f_\sigma(x)$ has the required asymptotic properties.

The purpose of this note is to establish Lévy's result in a rigorous manner. More precisely, we shall prove:

Theorem. *Suppose that $f(x) = x + \omega(x)$ where $\omega(x) > 0$, $\omega'(x)$ is of bounded variation for $x > a$, and $\omega'(x) \to 0$ as $x \to \infty$. Then (a) the limit* (11) *exists for every pair of values $x > a$, $y > a$. (b) $\sigma = \lambda(x, y)$ is continuous and strictly increasing in y. (c) $\lambda(y, x) = -\lambda(x, y)$. (d) If $y = f_\sigma(x)$ denotes the solution for y of $\sigma = \lambda(x, y)$ then the $f_\sigma(x)$ form a family of fractional iterates of $f(x)$ with the asymptotic property* (10).

A similar result holds for functions which have the form $f(x) = x - \omega(x)$ in a (right) neighbourhood of 0. If $\omega(x) > 0$, $\omega'(x)$ is of bounded variation for $0 < x < a$ and $\omega'(x) \to 0$ as $x \to 0+$, then $f(x)$ has a uniquely determined family of fractional iterates $f_\sigma(x) = x - \omega_\sigma(x)$ with

$$\lim_{x\to 0+} \frac{\omega_\sigma(x)}{\omega(x)} = \sigma$$

where σ is again given by (11). Modifications of the proof are trivial and details will be omitted.

The requirement that $\omega'(x)$ be of bounded variation is essential and relaxation of this condition seems hardly possible. If $\omega'(x)$ is of unbounded variation, the limit (11) need not exist at all, as for instance when $f(x) = x + 1 + \frac{1}{x} \sin x$, $x > 1$. In other cases the limit (11) may exist for all

[3] Lévy's chief aim was a theory of regular growth of real functions and the above theorem appeared as an auxiliary result in a largely heuristic work. From the point of view of the theory of iterations, the theorem obviously has an interest of its own.

pairs x, y, but $\sigma = \lambda(x, y)$ is a constant in an interval of y so that the equation is not solvable for y. An example of this kind is

$$f(x) = x + 1 - \frac{1}{n(n+1)} \sin^2 \frac{1}{2} \pi n(x-n), \qquad n \leqq x \leqq n + \frac{2}{n},$$

$$f(x) = x + 1, \qquad n + \frac{2}{n} \leqq x \leqq n + 1, \qquad n = 2, 3, \ldots .$$

In fact, if $x = 2$ then $x_n = n + 2$ and if $y = \frac{5}{2}$ then $y_n = n + 2 + \frac{1}{n+2}$:

Hence

$$\lim_{n\to\infty} \frac{y_n - x_n}{x_{n+1} - x_n} = \lim_{n\to\infty} \frac{1}{n+2} = 0 .$$

2. Proof of the theorem. We have already verified (a). Continuity of $\lambda(x, y)$ for $x \leqq y \leqq f(x)$ is a straightforward consequence of the uniformity of convergence of $\lim\limits_{n\to\infty} \sigma_n = \sigma$. Strict monotonity at $y = x$ follows from (12) which shows that $\sum\limits_n \frac{\sigma_{n+1} - \sigma_n}{\sigma_n}$ converges absolutely provided that $x < y \leqq f(x)$. Therefore $\sigma/\sigma_0 = \prod\limits_{n=0}^{\infty} \sigma_{n+1}/\sigma_n$ converges to a positive value and we have $\sigma > 0$.

To extend these results to other values of y, suppose that $y > x$ and let k be an integer such that $x_k < y \leqq x_{k+1}$, $x_{k+n} < y_n \leqq x_{k+n+1}$. Now

$$\begin{aligned} \omega(y_n) &= \omega(x_{k+n}) + (y_n - x_{k+n})\, \omega'(\xi_n) \\ &= \omega(x_{k+n}) + \theta_n \omega(x_{k+n})\, \omega'(\xi_n) \end{aligned}$$

where $0 < \theta_n \leqq 1$, $x_{k+n} < \xi_n < x_{k+n+1}$. Since $\omega'(\xi_n) \to 0$, we find that

$$\lim_{n\to\infty} \frac{\omega(y_n)}{\omega(x_{k+n})} = 1 .$$

But for fixed k,

$$\lim_{n\to\infty} \frac{\omega(x_{k+n})}{\omega(x_n)} = \lim_{n\to\infty} \frac{x_{k+n+1} - x_{k+n}}{\omega(x_n)}$$

$$= \lim_{n\to\infty} \frac{x_n + \omega_{k+1}(x_n) - x_n - \omega_k(x_n)}{\omega(x_n)} = 1$$

by (7). Therefore for every pair of values $x > a$, $y > a$,

$$\lim_{n\to\infty} \frac{\omega(y_n)}{\omega(x_n)} = \lim_{n\to\infty} \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = 1 . \tag{13}$$

This gives immediately

$$\lambda(x, f(y)) - \lambda(x, y) = 1, \tag{14}$$

and also the existence of any of the two limits on the left provided that the other one exists. Hence $\lambda(x, y)$ exists for every $x > a$, $y > a$. Furthermore, (13) gives

$$\lambda(x, t) = \lambda(x, y) + \lambda(y, t) \tag{15}$$

from which the assertions (a), (b) and (c) of the Theorem follow at once. To prove (d) we note that in the number triple $\{x, y, \sigma\}$ where $\sigma = \lambda(x, y)$, each pair determines uniquely the third one by (a), (b) and (c). Hence $f_\sigma(x)$ exists and they form a family of fractional iterates of $f(x)$ by (14), (15).

Finally we have to show that $f_\sigma(x)$ has the required asymptotic behaviour. We may assume that $0 < \sigma < 1$. Now $y = f_\sigma(x)$ implies $y_n = f_\sigma(x_n)$ for $n > 0$ therefore

$$\sigma = \lim_{n\to\infty} \frac{y_n - x_n}{\omega(x_n)} = \lim_{n\to\infty} \frac{f_\sigma(x_n) - x_n}{\omega(x_n)} = \lim_{n\to\infty} \frac{\omega_\sigma(x_n)}{\omega(x_n)}. \tag{16}$$

This holds uniformly for $a < b \leqq x \leqq f(b)$, $x \leqq y \leqq f(x)$, (see remarks after (12)), and the asymptotic formula (10) follows.

(Received February 20, 1960.)

ОБ ОДНОЙ ТЕОРЕМЕ P. LÉVY

G. SZEKERES

Резюме

Пусть $f(x)$ есть строго возрастающая функция, причем

$$f(x) > x, \qquad \text{если} \quad x > a. \tag{1}$$

Если $A(x)$ есть решение функционального уравнения

$$A\{f(x)\} = A(x) + 1 \qquad (x > a), \tag{2}$$

то функции

$$f_\sigma(x) = A_{-1}\{A(x) + \sigma\} \tag{3}$$

удовлетворяют соотношениям

$$f_\sigma(f_\tau(x)) = f_{\sigma+\tau}(x), \qquad f_1(x) = f(x). \tag{4}$$

Но так как решение (2) не единственно, то и функции $f_\sigma(x)$ определены однозначно лишь для целых значений σ.

Согласно одному замечанию P. LÉVY [1] f_σ станет однозначной, если потребовать, чтобы выполнялись условия

$$f(x) = x + \omega(x) \tag{5}$$

и

$$f_\sigma(x) = x + \omega_\sigma(x)\,, \tag{9}$$

где $\omega(x)$ дифференцируема, $\omega'(x)$ имеет ограниченное изменение, $\lim\limits_{x\to\infty}\omega'(x)=0$ и

$$\lim_{x\to\infty}\frac{\omega_\sigma(x)}{\omega(x)} = \sigma\,. \tag{10}$$

В работе автор доказывает существование и единственность так определенной f_σ.