EVALUATION OF INTEGRALS BY MONTE CARLO METHODS BASED ON THE ONE-DIMENSIONAL RANDOM SPACE FILLING

by

GEORGE BÁNKÖVI

§ 1. Introduction

The fundamental problem. In paper [1] A. RÉNYI solved the following problem. Let us place at random on the interval $(0, x)$ successively unit intervals, the left endpoints uniformly distributed on the interval $(0, x - 1)$, but discarded if the new interval intersects with one of the previous intervals. The procedure comes to an end when no more *"free interval"* longer than unity remains. $M(x)$ denotes the expectation of the number of unit intervals thus placed. $M(x)$ satisfies the functional equation

$$M(x+1) = \frac{2}{x} \int_0^x M(t)\, dt + 1 \qquad (x > 0) \tag{1}$$

and the initial condition

$$M(x) = 0 \qquad \text{for} \qquad x \leqq 1. \tag{2}$$

It is proved that

$$\lim_{x \to +\infty} \frac{M(x)}{x} = \int_0^\infty \exp\left(-2 \int_0^z \frac{1 - e^{-u}}{u}\, du\right) dz. \tag{3}$$

The fundamental idea of this paper. The model described above is a very special case of the random space filling. There are various ways of generalization by modification of the placing procedure. This paper deals with generalizations in direction of Monte Carlo methods resulting in procedures for obtaining the approximative value of integrals of type

$$I\{g\} = \int_0^\infty \exp\left(\int_0^z g(u)\, du\right) dz. \tag{4}$$

Before a specified discussion we wish to mention in the following section two general Monte Carlo methods applied for the evaluation of integrals (see [2], [3], [4]).

Two general methods. a) A general Monte Carlo method applied for evaluating is as follows. We wish to evaluate the integral

$$I = \int_{D_n} h(x)\, dx \tag{5}$$

where D_n is a domain in n-space and x is a vector. Let ξ be a random vector variable with density function $f(x)$,

$$\int_{D_n} f(x)\,dx = 1\,, \tag{6}$$

($f(x) \geqq 0$ ($x \in D_n$), and equality can hold only for $h(x) = 0$); let $x_1, x_2, \ldots, x_N$ be independent observations concerning the values of ξ. Then the random variable

$$S = \frac{1}{N} \sum_{j=1}^{N} \frac{h(x_j)}{f(x_j)} \tag{7}$$

is an unbiassed estimator of I, i. e.

$$\mathbf{E}(S) = \int_{D_n} \frac{h(x)}{f(x)} f(x)\,dx = I\,, \tag{8}$$

and

$$\mathbf{D}^2(S) = \frac{1}{N} \int_{D_n} \left(\frac{h(x)}{f(x)} - I\right)^2 f(x)\,dx\,. \tag{9}$$

b) An other method is the following: It can be assumed without loss of generality that $0 \leqq h(x) \leqq R$. Let us define two random variables ξ and η uniformly distributed on D_n and $(0, R)$, respectively. Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_N, \eta_N)$ be independent pairs of observations concerning the values (ξ, η) and s_j a set of random variables defined by

$$s_j = \begin{cases} 1, & \text{if} \quad h(\xi_j) \leqq \eta_j \\ 0, & \text{if} \quad h(\xi_j) > \eta_j \end{cases} \qquad (j = 1, 2, \ldots, N)\,. \tag{10}$$

Then the random variable

$$S = \frac{R \operatorname{mes} D_n}{N} \sum_{j=1}^{N} s_j \tag{11}$$

is an unbiassed estimator of I and

$$\mathbf{D}^2(S) = \frac{I(R \operatorname{mes} D_n - I)}{N}\,. \tag{12}$$

Several Monte Carlo techniques applied for evaluating integrals are simple modifications of these methods.

Natural models. The great advantage of the methods mentioned above is their general applicability, i. e. theoretically almost every integral can be evaluated by these methods. Considering the goodness of these methods their great generality is not always favourable; it seems that the probabilistic models are related to the *operation of integration* but not the *nature of the particular integrals considered.* We shall call such models *artificial models;* while models having a close connection with the nature of the problem will be called *natural models.* The field of applicability of the natural models is in general smaller than that of the artificial ones, but the natural models (if their application is possible) come up to the non-probabilistic approximative methods in such cases too, where the application of artificial models is obviously unfavourable. In this paper a class of natural models is connected to a class G of functions; by realization of these models estimates for the integrals $I\{g\}$ are obtained. Integrals of this type occur e. g. in the solution of Riccati differential equations (see [5]).

In applying artificial models or numerical integration as well the substitution in the integrand cannot be avoided; if the integrand is a complicated function (e. g. it consists of a large number of terms) this circumstance causes difficulties. These difficulties can be eliminated by applying natural models.

We wish to remark we regard the results of this paper only as first steps in this direction and our further investigations will aim at finding other types of random space filling models connected to certain problems of analysis (solution of differential equations, integral equations, etc.).

§ 2. A simple model

In this paragraph the procedure applied by A. Rényi is adapted to a simple modification and generalization of his model. Our model (in a special case) does not differ from the fundamental model in its limiting properties and it is convenient for theoretical and numerical computations.

Let $x_1, x_2, \ldots, x_N, \ldots$ be observations of a random variable ξ uniformly distributed on the interval $(0, x)$ and let us place in these points *weights* in accordance with the following rules:

1° In the point x_1 we put a weight $w(x)$ (where $w(x)$ statisfies conditions to be determined later); the argument denotes the length of the "free interval" containing the point x_1.

2° We establish a *"prohibitive interval"* with the endpoints $x_1 - a$, $x_1 + b$, where a and b are positive numbers.

3° We *"multiply the free intervals"* $(0, x_1 - a)$ and $(x_1 + b, x)$ by the positive constants c_1 and c_2, respectively, in the sense that all weights (determined by the values of the function $w(x)$) falling subsequently into these intervals must be multiplied by these constants.

4° The point x_2 is discarded if it falls into the prohibitive interval; otherwise it is placed.

5° If the point x_2 is placed we put in x_2 a weight the value of which depends on the length of the free interval containing x_2.

6° A second prohibitive interval is established with the endpoints $x_2 - a$, $x_2 + b$.

7° We multiply the two new free intervals by c_1 and c_2, respectively (i. e. the new intervals are multiplied by one of the values c_1^2, $c_1 c_2$, c_2^2). Evidently where a new prohibitive interval intersects with one of the endpoints of the interval $(0, x)$ or with one of the previous prohibitive intervals no new free interval comes to exist.

8° If the point x_3 falls into one of the prohibitive intervals it is discarded; otherwise it is placed, and so on.

9° The procedure comes to an end when the interval $(0, x)$ contains no more free intervals.

This method of the random space filling will be called *strategy* A.

The sum of the weights put thus on the interval $(0, x)$ is a random variable; let us denote its expectation by $m(x)$. Considering that the points are uniformly distributed on every free interval, it is easily shown that $m(x)$ satisfies the functional equation

$$(13)\qquad m(x)=\frac{c_1}{x}\int_0^x m(t-a)\,dt+\frac{c_2}{x}\int_0^x m(x-t-b)\,dt+w(x)\quad (x>0)$$

and the initial condition

$$(14)\qquad m(x)=0 \quad \text{for} \quad x\leqq 0.$$

In the following we are giving a limiting relation for the quotient $m(x)/x$. Let be in (13)

$$(15)\qquad w(x)=x^n \qquad (n\geqq 0).$$

In the following only the properties that $w(x)$ is a positive function $(0<x<+\infty)$ and that the Laplace-transform of $\frac{d}{dx}(x\,w(x))$ exists, are utilized.

Multiplying (13) by x and taking derivatives we obtain

$$(16)\qquad m(x)+xm'(x)=c_1\,m(x-a)+c_2\,m(x-b)+(n+1)x^n.$$

Multiplying (16) by e^{-sx} and integrating with respect to x from 0 to ∞:

$$(17)\qquad \int_0^\infty m(x)\,e^{-sx}\,dx+\int_0^\infty x\,m'(x)\,e^{-sx}\,dx= =c_1\int_0^\infty m(x-a)\,e^{-sx}\,dx+c_2\int_0^\infty m(x-b)\,e^{-sx}\,dx+\frac{\Gamma(n+2)}{s^{n+1}}.$$

Introducing the Laplace transform

$$(18)\qquad \varphi(s)=\int_0^\infty m(x)\,e^{-sx}\,dx \qquad (\mathrm{Re}\,s>0)$$

equation (17) obtains the form

$$(19)\qquad \varphi'(s)+\varphi(s)\frac{c_1e^{-as}+c_2e^{-bs}}{s}+\frac{\Gamma(n+2)}{s^{n+2}}=0.$$

In our further considerations the following three lemmas are applied:

Lemma 1.

$$(20)\qquad \lim_{s\to+\infty}\varphi(s)=0,$$

where $\varphi(s)$ *is defined by* (18).

Proof. It can be seen by simple considerations that there exists a constant K depending on a and b only so that the number of points resulting from strategy A is not greater then $Kx+1$; thus

$$(21)\qquad m(x)\ll L(x)\,(K\,x+1)\,x^n$$

where

$$L(x) = \left(\max(c_1, c_2, 1)\right)^{Kx+1}. \tag{22}$$

From (21) considering (18)

$$\varphi(s) \leqslant \int_0^\infty L(x)(Kx+1)x^n e^{-sx}\,dx, \tag{23}$$

and from this fact the statement of the lemma follows.

Lemma 2. (See [6] Theorem 108 and [1].) *If* $\alpha(x)$ *is a monotonically increasing function* $(0 < x < +\infty)$, $\beta > 0$ *and*

$$\lim_{s\to+0} s^\beta \int_0^\infty e^{-sx}\,d\,\alpha(x) = C, \tag{24}$$

then

$$\lim_{x\to+\infty} \frac{\alpha(x)}{x^\beta} = \frac{C}{\Gamma(\beta+1)}. \tag{25}$$

Lemma 3. *The function* $m(x)$ *satisfying* (13) *and* (14) *is monotonically increasing* $(0 < x < +\infty)$ *for the weight function* (15).

Proof. Let us arrange the numbers $a, 2a, 3a, \ldots, b, 2b, 3b, \ldots$ according to their order of magnitude and denote the elements of the ordered set (discarding the duplicates) by $0 < a_1^* < a_2^* < a_3^* < \ldots$. It follows from (13)—(16)

$$xm'(x) = c_1\left(m(x-a) - \frac{1}{x}\int_0^{x-a} m(t)\,dt\right) + c_2\left(m(x-b) - \frac{1}{x}\int_0^{x-b} m(t)\,dt\right) + nx^n. \tag{26}$$

From (14) and (26) it follows that

$$m'(x) \geqq 0 \quad \text{for} \quad 0 < x \leqq a_1^*. \tag{27}$$

Equality holds only for $n = 0$.) Assuming that

$$m'(x) \geqq 0 \quad \text{for} \quad 0 < x \leqq a_k^* \qquad (k = 1, 2, \ldots), \tag{28}$$

(and so $m(x)$ is monotonically increasing in this interval), it is shown that

$$m'(x) > 0 \quad \text{for} \quad a_k^* < x \leqq a_{k+1}^*. \tag{29}$$

Namely if $f(x)$ is a monotonically increasing function in the interval $(0, x_0)$ then for all values of $0 < x \leqq x_0$

$$f(x) - \frac{1}{x}\int_0^x f(t)\,dt \geqq 0, \tag{30}$$

and as

$$m(x-\alpha) - \frac{1}{x}\int_0^{x-\alpha} m(t)\,dt > m(x-\alpha) - \frac{1}{x-\alpha}\int_0^{x-\alpha} m(t)\,dt \tag{31}$$

$$(\alpha = a \quad \text{or} \quad \alpha = b, \quad \alpha < x \leqq a_k^* + \alpha),$$

considering (26) the statement of the lemma holds. By an analogous induction the monotony of the function $m(x)$ can be proved in case of more complicated models as well.

Solving the equation (19) under the initial condition (20) we obtain

$$\varphi(s) = \frac{\Gamma(n+2)}{s^{n+2}} \int_s^\infty \exp\left(\int_s^z \frac{c_1 e^{-au} + c_2 e^{-bu} - (n+2)}{u}\, du\right) dz. \tag{32}$$

Applying Lemma 2 (and Lemma 3) with $\alpha(x) = m(x)$, $\beta = n+1$, and considering that

$$\int_0^\infty e^{-sx}\, d\, m(x) = s\,\varphi(s)\,, \tag{33}$$

we obtain

$$\lim_{x\to+\infty} \frac{m(x)}{x^{n+1}} = \int_0^\infty \exp\left(\int_0^z \frac{c_1 e^{-au} + c_2 e^{-bu} - (n+2)}{u}\, du\right) dz. \tag{34}$$

Obviously only the case $c_1 + c_2 = n + 2$ deserves attention as otherwise the value of the integral in (34) equals either 0 or $+\infty$. This model is advisable only for integer values of n, as otherwise a modification given in § 3 is more suitable for computations.

Connection between the functions $M(x)$ and $m(x)$. Let us consider the special case of

$$a = b = c_1 = c_2 = 1 \qquad \text{and} \qquad n = 0; \tag{35}$$

thus the value of $m(x)$ is equal to the expectation of the number of points placed and we denote it by $m_0(x)$. Substituting in (34) we obtain the expression

$$\lim_{x\to+\infty} \frac{m_0(x)}{x} = \int_0^\infty \exp\left(-2\int_0^z \frac{1-e^{-u}}{u}\, du\right) dz. \tag{36}$$

The function $M(x)/x$ in [1] converges to the same limit.

This fact and the circumstance that RÉNYI's placing procedure is similar to strategy A in the case of (35) (namely the restriction, that the unit intervals must not intersect with one another, means in our terminology for the left endpoints of the subsequent unit intervals the extension of each of the unit prohibitive intervals already placed with a further unit to the left), suggest a close connection between the functions $M(x)$ and $m_0(x)$. Indeed the following lemma is true:

Lemma 4. *The function $m_0(x)$ is identically equal to $M(x+1)$.*

Proof. It follows from the initial conditions (2) and (14) that

$$m_0(x) = M(x+1) = 0 \quad \text{for} \quad x \leqq 0 \tag{37}$$

and obviously both functions satisfy the following equations:

$$m_0(x) = M(x+1) = 1 \qquad \text{for} \qquad 0 < x \leqq 1. \tag{38}$$

Let us suppose that the statement of the lemma holds for all $x \leq x_0 - 1$. From this assumption it follows that

$$\frac{2}{x}\int_0^x M(t)\,dt = \frac{2}{x}\int_0^x m_0(t-1)\,dt \quad \text{for} \quad x \leqq x_0, \tag{39}$$

and thus considering (1), (13), (15) and (35)

$$M(x+1) = m_0(x) \qquad \text{for} \qquad x \leqq x_0. \tag{40}$$

The proof is completed by induction.

§ 3. Some further examples

In this paragraph some tricks are described by the aid of which the value of $I\{g\}$ can be evaluated for various $g \in G$. The according models are modifications of *Model* 1. The index ν of the function $m_\nu(x)$ characterizes the model in question. We remark that *per definitionem*

$$m_\nu(x) = 0 \quad \text{for} \quad x \leqq 0 \quad \text{and for all } \nu. \tag{41}$$

Analogous lemmas to Lemma 1 and Lemma 3 can be obtained by simple modifications of the proofs.

The models described below must be considered only as illustrative simple examples; further models can be obtained by the combination of these and the introduction of new tricks in the placing procedure. In practice if an integral similar to one of those occuring in the known models is given the problem arises how to find a model corresponding to this integral.

A simple modification of Model 1. The model described in § 2 is applicable by a simple modification to evaluate $I\{g_1^*\}$, where

$$g_1^*(u) = \frac{\sum_{j=1}^{r} c_j^* e^{-a_j u} - (n+2)}{u}, \quad a_j, c_j^* > 0 \ (j=1,2,\ldots,r) \text{ and } \sum_{j=1}^{r} c_j^* = n+2. \tag{42}$$

In this case (if we denote by t the place of the first point) we consider the intervals $(0, t-a_1)$, $(0, t-a_2)$, $\ldots$, $(0, t-a_r)$ as free intervals, multiplied by c_1^*, c_2^*, $\ldots$, c_r^* respectively (instead of considering the intervals $(0, t-a)$, $(t+b, x)$ multiplied by c_1 and c_2, resp.). Denoting by $m_1^*(x)$ the expectation of the sum of weights placed in this way we obtain the equation

$$m_1^*(x) = \frac{1}{x}\sum_{j=1}^{r} c_j^* \int_0^x m_1^*(t - a_j^*)\,dt + x^n \qquad (x > 0). \tag{43}$$

Making use of the method described in § 2 we obtain from (43)

$$\lim_{x\to+\infty} \frac{m_1^*(x)}{x^{n+1}} = I\{g_1^*\}. \tag{44}$$

We wish to remark that applying the weight-function

$$w(x) = x^{n-a} \qquad (0 \leq a \leq n)\,, \tag{45}$$

the result is modified in the form

$$\lim_{x \to +\infty} \frac{m_1^*(x)}{x^{n+1}} = \int_0^\infty z^a \exp\left(\int_0^z g_1^*(u)\,du\right) dz\,. \tag{46}$$

Mixed strategy of placing. If in the function

$$g_1(u) = \frac{c_1 e^{-au} + c_2 e^{-bu} - y}{u} \qquad (c_1 + c_2 = y) \tag{47}$$

$1 < y < 2$ holds the evaluation of $I\{g_1\}$ by the aid of strategy A meets theoretical and practical difficulties as the weight-function x^n $(-1 < n < 0)$ has a singularity in 0.

The problem can be simply solved however by introducing a new strategy of placing called strategy B.

Strategy B consists of the following rules:

1° In a "free interval" $(0, x)$ a point t is placed at random. Thus the interval $(0, x)$ is divided into two parts.

2° The intervals $(0, t)$ and (t, x) are multiplied by the positive constant c_0 (in the sense mentioned in § 2).

Evidently strategy B cannot be applied alone but the mixed strategy

$$\mathsf{C} = p\,\mathsf{A} + q\,\mathsf{B} \tag{48}$$

(i. e. every move of strategy C represents either a move according to strategy A or the application of strategy B with probabilities p and $q = 1 - p$, resp.) is suitable for our purpose.

In this manner we obtain the equation (modification of (13))

$$m_1(x) = \frac{2c_0 q}{x} \int_0^x m_1(t)\,dt + \frac{c_1 p}{x} \int_0^x m_1(t-a)\,dt + \frac{c_2 p}{x} \int_0^x m_1(t-b)\,dt + px^n (x > 0)\,, \tag{49}$$

where

$$n \geq 0, \qquad n + 2 = c_1 p + c_2 p + 2c_0 q \tag{50}$$

and $m_1(x)$ denotes the expectation of the sum of weights which can be placed on the interval $(0, x)$ according to strategy C. Using the method described in § 2 we obtain from (49)

$$\lim_{x \to +\infty} \frac{m_1(x)}{x^{n+1}} = p\,I\{g_1\} = p \int_0^\infty \exp\left(-y \int_0^z \frac{1 - y_1 e^{-au} - y_2 e^{-bu}}{u}\,du\right) dz\,, \tag{51}$$

where

$$y_1 = \frac{c_1 p}{y}\,, \quad y_2 = \frac{c_2 p}{y}\,, \quad y = (c_1 + c_2)\,p = n + 2 - 2c_0 q\,. \tag{52}$$

We wish to remark that the value of $I\{g_1\}$ is equal to $+\infty$ if $y \leqq 1$. This fact can be shown integrating by parts.

In the special case (35) a modification of (36) follows from (51):

$$\lim_{x\to+\infty} \frac{m_1(x)}{x} = p \int_0^\infty \exp\left(-2p \int_0^z \frac{1-e^{-u}}{u}\right) dz. \tag{53}$$

This method is of advantage for not integer values of y since in this case n can be chosen as an integer. Theoretically the placing procedure may consists of an infinite number of moves; but this event has probability zero. In practice the procedure always comes to an end as instead of intervals a finite number of points is considered.

Case of small intervals. The value of $m_1(x)$ can be exactly computed for $0 < x \leq \min(a, b)$ i. e. in this case by derivation of (49) and considering (41) the linear differential equation

$$m_1'(x) + \frac{1-2c_0 q}{x} m_1(x) - p(n+1)x^{n-1} = 0 \tag{54}$$

is obtained.

Solving (54) under condition (41)

$$m_1(x) = \frac{p(n+1)}{n-2c_0 q+1} x^n \qquad \big(0 < x \leq \min(a, b)\big) \tag{55}$$

follows. Making use of this result the experimentation can be simplified (see § 4).

Model 2. In this section a method is described for evaluating $I\{g_2\}$,

$$g_2(u) = \frac{\sum_{j=0}^{r} d_j u^{r-j} e^{-a_j u} - (n+2)}{u^{r+1}} \tag{56}$$

where

(57) r a positive integer, $n \geqq r$, $a_j, d_j > 0\,(j = 0, 1, \ldots, r)$, $\lim_{u\to 0} g_2(u)$ finite.

For achieving our purpose we must introduce a new trick in the placing procedure. If the first point falls in t we consider the intervals $(0, t-a_0)$, $(0, t-a_1)$, $(0, t-a_2)$, $\ldots$, $(0, t-a_r)$ and multiply them by d_0, $d_1(x-t)$, $d_2(x-t)^2/2!$, $\ldots$, $d_r(x-t)^r/r!$. In the following a branching procedure is carried out so that instead of the interval $(0, x)$ each of the intervals $(0, t-a_j)$ $(j = 0, 1, \ldots, r)$ is considered.

Applying the weight-function

$$w(x) = x^n \qquad (n \geqq r) \tag{58}$$

and denoting by $m_2(x)$ the expectation of the sum of weights thus placed in the interval $(0, x)$ we obtain

$$m_2(x) = \frac{1}{x} \sum_{j=0}^{r} \frac{d_j}{j!} \int_0^x (x-t)^j m_2(t-a_j)\, dt + x^n \qquad (x > 0). \tag{59}$$

Deriving r times and making use of the method described in § 2 we obtain

$$\lim_{x \to +\infty} \frac{m_2(x)}{x^{n+1}} = I\{g_2\}. \tag{60}$$

Model 3. Let be

$$g_3(u) = \frac{(n+2)\, k_0\, e^{-du} + k_1\, e^{-au} + k_2\, e^{-bu} - (n+2)}{u(1 - k_0\, e^{-du})} \tag{61}$$

where

$$n \geqq 0, \quad a, b, d, k_0, k_1, k_2 > 0, \quad k_0 < 1, \quad \lim_{u \to 0} g_3(u) \text{ finite.} \tag{62}$$

For evaluating $I\{g_3\}$ a new trick must be introduced. Let us define a strategy of placing D as follows:

1° The free interval $(0, x)$ is divided by the point $x - d$ into two parts. The interval $(x - d, x)$ is discarded.

2° In the point $x - d$ a weight x^n is put.

3° The interval $(0, x - d)$ is multiplied by the positive constant c_0 (in the sense mentioned in § 2).

The expectation of the sum of weights (denoted by $m_3(x)$) placed on the interval $(0, x)$ by applying the mixed strategy

$$\mathsf{E} = p\,\mathsf{A} + q\,\mathsf{D} \tag{63}$$

satisfies the equation

$$m_3(x) = c_0\, q m_3(x-d) + \frac{c_1 p}{x} \int_0^x m_3(t-a)\, dt + \frac{c_2 p}{x} \int_0^x m_3(t-b)\, dt + x^n \qquad (x > 0). \tag{64}$$

From (64) in the manner mentioned before (choosing c_0, c_1, c_2, p in such a way that $c_0\, q = k_0$, $c_1\, p = k_1$, $c_2\, p = k_2$) the relation

$$\lim_{x \to +\infty} \frac{m_3(x)}{x^{n+1}} = \frac{1}{1 - k_0} I\{g_3\} \tag{65}$$

is obtained.

Model 4. In this section a trick is shown by the aid of which the integral

$$I(v) = \int_0^\infty \exp\Big(-vz + \int_0^z g(u)\, du\Big)\, dz \qquad (v > 0) \tag{66}$$

can be evaluated for $g \in G$.

Let us consider as an example the model described in § 2. Strategy A is modified in the following way:

1° The constants c_1 and c_2 are replaced by $\frac{c_1 x}{x+v}$ and $\frac{c_2 x}{x+v}$, respectively, where x denotes the length of the free interval divided by the point t, and v is a fixed positive number.

2° Let be the weight-function

$$w(x) = \frac{x^{n+1}}{x+v} \qquad (n \geqq 0). \tag{67}$$

Then the expectation of the sum of weights placed on the interval $(0, x)$ and denoted by $m_4(x)$ satisfies the equation

$$m_4(x) = \frac{c_1}{x+v}\int_0^x m_4(t-a)\,dt + \frac{c_2}{x+v}\int_0^x m_4(t-b)\,dt + \frac{x^{n+1}}{x+v} \qquad (x > 0). \tag{68}$$

From (68)

$$\lim_{x\to+\infty} \frac{m_4(x)}{x^{n+1}} = I(v) \tag{69}$$

is obtained.

§ 4. Some remarks and an experiment

Some remarks concerning convergence and variance estimation. In § 3 limiting relations are obtained concerning the functions $m_\nu(x)$. These theoretical results can be applied in the following way: a positive number x is fixed. On the interval $(0, x)$ experiments are accomplished according to the prescribed strategy. The mean value of the sum of weights placed on the interval $(0, x)$ is an estimator of $m_\nu(x)$. From this mean value and the according limiting relation an estimator of $I\{g\}$ is obtained.

Applying this method the results have two kinds of errors, one of which originates from the fact that a finite x is considered (instead of $+\infty$); the other kind of error is caused by the random fluctuation of experimental results. Theoretically the experimentation on a very large interval is advantageous but practically this is uncomfortable (the repeating of the experiments on a smaller interval is more advisable).

In paper [1] results are found concerning the asymptotic behaviour of $M(x)$ and the variance of the number of unit intervals placed on the interval $(0, x)$. According to Lemma 4 these results are valid in case of our simplest example but for more complicated models the theoretical treatment seems to be a rather difficult problem. Experimental experiences give the impression that by the application of an interval of length $x = 1000$ or even $x = 100$ (when the length of the prohibitive intervals is about unit) only a negligible systematic error is caused as compared to the random fluctuations (unless the number of experiments is unreasonably large).

An estimate of the variance is obtained from the experimental data, the value of which is biased due to variance reducing techniques. If in the result an error of some percentages is admissible the application of the described Monte Carlo methods may be advantageous compared to numerical integration even in case of rather simple integrals.

Variance reducing techniques. There exist various techniques by the aid of which the amount of labour necessary to obtain a given accuracy can be reduced. We wish to mention below some of these techniques applicable in the case of random space filling.

Systematic sampling. In a mixed strategy of placing

$$\mathsf{S} = \sum_{k=1}^{r} p_k \mathsf{S}_k \quad \left(p_k > 0, \sum_{k=1}^{r} p_k = 1 \right) \tag{70}$$

the strategy of the first move is not chosen at random but determined systematically so that out of N experiments each strategy S_k $(k = 1, 2, \ldots, r)$ will occur about Np_k times.

Stratified sampling. The interval $(0, x)$ is divided into N equal parts. In the first move of j-th experiment $(j = 1, 2, \ldots, N)$ the point t is placed at random on the j-th subinterval.

Use of expected values. Let be x_0 such a value that for every $x \leqq x_0$ a simple analytical term can be given for $m_\nu(x)$ (see e. g. (55)). The placing procedure consists now of two parts:

1° Reducing the length of the free intervals until the maximal length does not exceed x_0;

2° On these small intervals instead of further experimentation the exact values of $m_\nu(x)$ are considered.

Result of an experiment. We wished to compare on an example the results obtained and the amount of labour involved in the Monte Carlo method and the numerical integration, respectively. Thus we determined the value of the integral

$$I_0 = \int_0^\infty \exp\left(-1{,}5 \int_0^z \frac{1 - e^{-u}}{u} du \right) dz \tag{71}$$

in these two different ways. The according Monte Carlo method was based on the placing procedure resulting in (53) with $p = 0{,}75$. We wish to give a short description of the experimentation.

The experiments were performed on an interval of length $x = 100$. Six-digit random numbers were considered (we made use of the table of random numbers of [7]); The points $x_1, x_2, \ldots,$ were marked out by the first four digits (from 00,00 to 99,99) while the last two digits were used to determine whether strategy A (from 00 to 74) or strategy B (from 75 to 99) should be applied.

On the small free intervals the technique of the "use of expected values" was applied. For this model

$$m_1(x) = \frac{p}{2p-1} \qquad \left(0 < x \leqq 1, \frac{1}{2} < p \leqq 1\right) \tag{72}$$

holds. This follows from (55) considering (35) and (50) or directly from the equation

$$m_1(x) = p + 2q\,m_1(x) \qquad \left(0 < x \leqq 1, \frac{1}{2} < p \leqq 1, p+q=1\right). \tag{73}$$

The mean value of the results of ten experiments gave for I_0 the approximation

$$I_0 \approx 1{,}362. \tag{74}$$

The standard deviation of the experimental results was

$$\sigma = 0{,}064 \tag{75}$$

with each obtained value in the interval $(I_0 - 2\sigma,\ I_0 + 2\sigma)$.The realization of one experiment required about thirty minutes.

By numerical integration we obtained the result

$$I_0 = 1{,}345 \tag{76}$$

(where the last digit is uncertain). The determination of this value by means of numerical computation required by far more efforts than that of the Monte Carlo method described above. The simplicity of the placing procedures renders the methods described in this paper suitable for machine work.

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REFERENCES

[1] RÉNYI A.: „Egy egydimenziós véletlen térkitöltési problémáról." *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **3** (1958) 109—127.
[2] MEYER, H. A. (ed.): *Symposium on Monte Carlo methods*. Wiley, New York, 1956.
[3] BROWN, G. W.: "Monte Carlo methods." Beckenbach, E. F.: Modern mathematics for the engineer. Mc Graw-Hill, New York, 1956, 279—303.
[4] PALÁSTI I.—RÉNYI A.: „A Monte-Carlo módszer mint minimax stratégia." *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **1** (1956) 529—545.
[5] ABDELKADER, M. A.: "Solutions by Quadrature of Riccati and Second-order Linear Differential Equations." *The American Mathematical Monthly* **66** (1959) 10 886—889.
[6] HARDY, G. H.: *Divergent series*. Oxford University Press, 1949.
[7] HALD, A.: *Statistical Tables and Formulas*. Wiley, New York, 1952.
[8] JAHNKE-EMDE: *Tafeln Höherer Funktionen*. Teubner, Leipzig, 1952.

РАСЧЕТ ИНТЕГРАЛОВ МЕТОДОМ МОНТЕ-КАРЛО, ОСНОВАННЫМ НА ОДНОМЕРНОМ СЛУЧАЙНОМ ЗАПОЛНЕНИИ ПРОСТРАНСТВА

G. BÁNKÖVI

Резюме

В работе предлагается метод приближенного вычисления интегралов вида

$$I\{g\} = \int_0^\infty \exp\left(\int_0^z g(u)\,du\right) dz.$$

Основная идея метода заключается в следующем: расположим на интервале $(0, x)$ точки и поместим в этих точках весы случайно, но соответственно некоторым правилам. При подходящем выборе этих правил, между моделью расположения и некоторой функцией g выполняется соотношение вида

$$\lim_{x \to +\infty} \frac{m(x)}{x^{n+1}} = I\{g\},$$

где $m(x)$ обозначает математическое ожидание суммы весов, помещенных на интервале $(0, x)$. Оценка величины $m(x)$ получается реализацией модели расположения.

Модель, описанная в § 2, является модификацией и обобщением модели A. RÉNYI [1]. В § 3 автор описывает несколько моделей, с помощью которых интеграл $I\{g\}$ вычисляемый при функциях g разных типов. В § 4 описывается расчет интеграла

$$I_0 = \int_0^\infty \exp\left(-1{,}5\int_0^z \frac{1-e^{-u}}{u}\,du\right) dz$$

методом Монте-Карло.