ON SOME RANDOM SPACE FILLING PROBLEMS

by

ILONA PALÁSTI

§ 1. Introduction

A. RÉNYI solved in his paper [1] the one-dimensional case of a problem raised by L. SCHMETTERER, i. e. he considered the following problem. Let us place at random with uniform distribution disjoint unit intervals on the interval $(0, x)$ till this is possible, namely until the distances of all neighbouring unit intervals will be smaller than one. RÉNYI determined the mean value $M(x)$ of the number of unit intervals and showed that

$$\lim_{x \to \infty} \frac{M(x)}{x} = c .$$

He solved the problem proving that $M(x)$ satisfies the following difference-differential equation

$$(x-1)\, M'(x) + M(x) = 2\, M(x-1) + 1 \qquad \text{for} \quad x \geqq 1$$

with the initial condition $M(x) = 0$, for $0 \leqq x \leqq 1$, further by showing that for $M(x)$ the asymptotical formula

$$M(x) = cx - (1-c) + O\left(\frac{1}{x^n}\right) \tag{1}$$

holds, where n is an arbitrarily large number and

$$c = \int_0^\infty e^{-2\int_0^t \frac{1-e^{-u}}{u}\,du}\, dt . \tag{2}$$

The numerical value of c was given in [1] to three decimals as 0,748.

This paper deals with the corresponding problem in the two- and three-dimensional case.

We consider in § 2 the following problem. Let us place at random (with uniform distribution) disjoint domains which are congruent and parallel, with a given domain D of unit area (this means that any domain can be carried over into D by a shift transformation $x' = x - a$, $y' = y - b$), into a rectangle with sides $x \gg 1$, $y \gg 1$. The process is repeated until this is possible. The question is, what is the mean value of the number of domains placed in such a way, i. e. how large part of the rectangle will be filled in average with these domains.

Let us denote by $M(x, y)$ the mean value of the number of domains placed in the above mentioned way into a rectangle T_{xy} with vertices (0,0), $(x, 0)$, (x, y), $(0, y)$ (where $x, y \gg 1$). By starting from a hypothesis which seems very plausible (to the proof of which we intend to return in an other paper) we shall show that the limit

$$\lim_{\substack{x \to \infty \\ y \to \infty}} \frac{M(x, y)}{xy} = \alpha(D) \tag{3}$$

exists.

Our proof depends on a generalization of a theorem of D. H. HYERS [2]. In the case when D is the unit square S_1 a heuristic reasoning has led us to the conjecture that the corresponding constant $\alpha(S_1)$ is equal to $c^2 \sim 0,56$ where c is the constant (2) obtained by RÉNYI in the linear case.

Concerning this conjecture experiments were carried out by the Monte Carlo method, in order to determine an approximative value of $\frac{M(x, y)}{x\, y}$. The results obtained which are reproduced in § 3 strongly support our conjecture.

Finally in § 4 we give some remarks on the three-dimensional case.

§ 2. On the existence of the density of space filling

We shall start from the following hypothesis:

Hypothesis A. Let $M(x, y)$ denote the mean value of the number of domains congruent and parallel with the convex domain D of unit area placed at random as described in the introduction on the rectangle T_{xy} the sides of which are equal to $x > 0$ and $y > 0$. Then there exists a constant $A > 0$ depending on D, such that for any $x_1 > 0$, $x_2 > 0$ and $y > 0$ we have

$$| M(x_1 + x_2, y) - M(x_1, y) - M(x_2, y) | \leqq Ay \tag{4a}$$

resp. for any $x > 0$, $y_1 > 0$, $y_2 > 0$

$$| M(x, y_1 + y_2) - M(x, y_1) - M(x, y_2) | \leqq Ax. \tag{4b}$$

Remark. It can be seen from (1) that the analogue of the above hypothesis, i. e. the inequality

$$| M(x + y) - M(x) - M(y) | \leqq A_1 \tag{5}$$

is valid. In fact, (5) is true with $A_1 = 1$, but we do not need this result here.

D. H. HYERS deals with functional inequalities similar to (5) in his paper [2]. He called the transformation $f(x)$ of the Banach space E into the Banach space E' a δ-linear transformation if it satisfies the following inequality.

$$|| f(x + y) - f(x) - f(y) || < \delta \tag{6}$$

for any $x \in E$, $y \in E$. He showed that for any δ-linear transformation $f(x)$ of E into E' there can be found one and only one linear transformation $l(x)$ which satisfies

$$|| f(x) - l(x) || < \delta . \tag{7}$$

E and E' are both the set of real numbers, and as in our case there are functions and thus clearly $M(x) < x$, $l(x)$ can not be else than a linear function. Hence it follows from (5) and (7) that there exists a constant c such that

$$| M(x) - cx | \leqq 1, \tag{8}$$

which implies

$$\lim_{x \to \infty} \frac{M(x)}{x} = c. \tag{9}$$

Let us consider now the *two-dimensional case.* As we can not make use of the above mentioned theorem of HYERS in its original form, we prove a lemma which is similar to the theorem of HYERS and which is suitable for our purposes. The idea of the proof is similar, but deviates somewhat from that of the theorem of HYERS.

Lemma. *Let $f(x, y)$ be a Borel-measurable function of two variables satisfying the following conditions:*

$$| f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y) | \leqq By, \text{ for } x_1 \geqq 0,\ x_2 \geqq 0,\ y \geqq 0 \tag{10a}$$

and

$$| f(x, y_1 + y_2) - f(x, y_1) - f(x, y_2) | \leqq Bx \text{ for } x \geqq 0,\ y_1 \geqq 0,\ y_2 \geqq 0 \tag{10b}$$

$B > 0$ *is a constant, not depending on the variables* x_i, y_i $(i = 1, 2)$, *and there exists a constant* $\varrho > 0$ *such that* $f(x, y) = 0$, *if* $0 < x < \varrho$ *or* $0 < y < \varrho$; *then*

$$| f(x, y) - \alpha\, xy | \leqq B(x + y) \tag{11a}$$

where α *is a constant and thus*

$$\lim_{\substack{x \to \infty \\ y \to \infty}} \frac{f(x, y)}{xy} = \alpha. \tag{11b}$$

Proof. It follows from (10a) and (10b) that

$$\begin{aligned} | f(x_1 + x_2, y_1 + y_2) - f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) - f(x_2, y_2) | &\leqq \\ \leqq B(x_1 + x_2 + y_1 + y_2). \end{aligned} \tag{10c}$$

At first put $x_1 = x_2 = \frac{x}{2}$, $y_1 = y_2 = \frac{y}{2}$ into (10c), then we obtain

$$\left| f(x, y) - 4 f\left(\frac{x}{2}, \frac{y}{2}\right) \right| \leqq B(x + y). \tag{12}$$

Replacing x and y by $2^k x$ and $2^k y$, resp. and dividing (12) by 4^k we obtain

$$\left| \frac{f(2^k x, 2^k y)}{4^k} - \frac{f(2^{k-1} x, 2^{k-1} y)}{4^{k-1}} \right| \leqq B\left(\frac{x + y}{2^k}\right). \tag{13}$$

Summing up (13) for $k = m + 1, \ldots, n$ we obtain

$$\left| \frac{f(2^n x, 2^n y)}{4^n} - \frac{f(2^m x, 2^m y)}{4^m} \right| \leqq B\left(\frac{x + y}{2^m}\right). \tag{14}$$

Let us consider x and y as fixed values. Thus if $m \to \infty$ then according to (14) the sequence $4^{-k} f(2^k x, 2^k y)$ satisfies the Cauchy-convergence criterion, i. e.

$$\lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{4^n} = g(x, y) \tag{15}$$

exists. Concerning $g(x, y)$ we know the following: Substituting in (10a) $2^n x_i$ resp. $2^n y$ instead of x_i, $(i = 1, 2)$ resp. y and dividing by 4^n we obtain

$$\left| \frac{f(2^n(x_1 + x_2), 2^n y)}{4^n} - \frac{f(2^n x_1, 2^n y)}{4^n} - \frac{f(2^n x_2, 2^n y)}{4^n} \right| \leq B \frac{y}{2^n} \tag{16}$$

that is passing to the limit $n \to \infty$ we obtain from (16) by using (15)

$$g(x_1 + x_2, y) = g(x_1, y) + g(x_2, y) . \tag{17}$$

Thus as $g(x, y)$ is for fixed y a Borel-measurable function of x, we obtain

$$g(x, y) = a(y)x . \tag{18a}$$

It follows by the same argument from (10b) and (15) that

$$g(x, y) = b(x)\, y , \tag{18b}$$

which is only possible if

$$g(x, y) = \alpha\, xy \tag{19}$$

where α is a constant, independent of x and y. From (15) thus it follows that

$$\alpha = \frac{g(x, y)}{xy} = \lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{2^n x\, 2^n y} . \tag{20}$$

Let us put now $m = 0$ into the inequality (14) and divide by xy. It follows that

$$\left| \frac{f(2^n x, 2^n y)}{2^n x\, 2^n y} - \frac{f(x, y)}{xy} \right| \leq B \frac{x + y}{xy} = B \left(\frac{1}{x} + \frac{1}{y} \right) . \tag{21}$$

If now $n \to \infty$, then — according to (20) —

$$\left| \alpha - \frac{f(x, y)}{xy} \right| \leq B \left(\frac{1}{x} + \frac{1}{y} \right) \tag{22}$$

and if $x \to \infty$, $y \to \infty$ then the right side of (22) tends to 0 and thus (11a) and therefore (11b) follows. Thus our lemma is proved.

Hence if for some domains D hypothesis A is valid, then our lemma can be applied and thus it follows that

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{M(x, y)}{xy} = \alpha = \alpha(D) \tag{23}$$

exists and obviously must be a constant with value between 0 and 1.

Writing down inequality (11a) for $M(x, y)$ it follows also that

$$| M(x, y) - \alpha(D)\, xy | \leq A(x + y) . \tag{24}$$

Thus $M(x, y)$ may be written in the form

$$M(x,y) = \alpha(D)\,xy + m(x,y) \tag{25}$$

where $m(x, y) = o(xy)$ for $x \to \infty$, $y \to \infty$.

§ 3. The value of $\alpha = \lim\limits_{\substack{x\to\infty \\ y\to\infty}} \dfrac{M(x,y)}{xy}$ if D is a rectangle

Let us suppose now that D is the rectangle of unit area S_q with sides q and $\dfrac{1}{q}$ $(q>0)$ which are parallel to the x resp. y axis. If $M_q(x, y)$ denotes the mean value of the number of such rectangles which can be placed in a rectangle with sides x and y, then by a similar transformation one obtains

$$M_q(x,y) = M_1\left(\frac{x}{q}, yq\right). \tag{26}$$

It follows that

$$\alpha(S_q) = \lim_{\substack{x\to\infty \\ y\to\infty}} \frac{M_q(x,y)}{xy} = \lim_{\substack{x\to\infty \\ y\to\infty}} \frac{M_1\left(\dfrac{x}{q}, yq\right)}{\dfrac{x}{q}\cdot yq} = \alpha(S_1). \tag{27}$$

Thus $\alpha(S_q) = \alpha$ does not depend on q. By a heuristic argument one is led to conjecture that

$$M_1(x,y) \sim M(x)\,M(y) \tag{28}$$

where $M(x)$ is the mean number of unit intervals which can be placed on the interval $(0, x)$, mentioned in the introduction. We hope to return to the proof of (28) in an other paper.

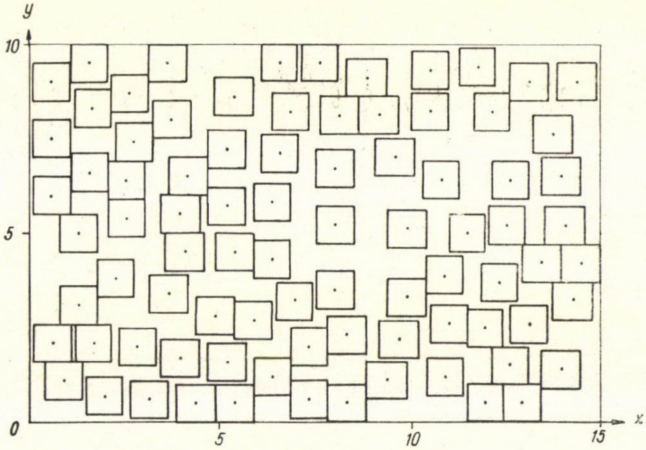

Figure 1.

We conjecture that (28) holds, from which it follows that $\alpha(D_1)$ is numerically equal to c^2 where c is defined by (2), and thus is approximately equal to 0,56.... Thus if our conjecture is valid, then if x and y are sufficiently large in an average $0{,}56xy$ small rectangles can be placed at random on the rectangle T_{xy} with their sides parallel to those of T_{xy}.

To check this we have carried through experiments by the Monte Carlo method in order to determine the approximate value of $M(x, y)$ (see Figure 1). In the rectangle T with sides x and y the point P was determined by means of a table of random numbers [3] in such a way that the coordinates t and u were chosen independently ($^1/_2 \leqq t \leqq x - {}^1/_2$), ($^1/_2 \leqq u \leqq y - {}^1/_2$). This point P was considered as the centre of a unit square if this was possible, i. e. if this unit square did not intersect with any of the former ones. Otherwise the point P was rejected and another point was chosen in the same way. This procedure was continued as long as the rectangle T was filled with unit squares. The results of this experiment are contained in the following table:

Length of the sides of rectangle T		Area of rectangle T xy cm²	Number of unit squares placed	56% of the area of rectangle
x cm	y cm			
5	15	75	42	42
10	15	150	84	84
15	15	225	126	126
20	15	300	167	168

Thus the results of the experiment are in very good agreement with our conjecture.

§ 4. The three-dimensional case

The analogous problem is in this case the following: let us place at random with uniform distribution in a parallelepiped K_{xyz} having the edges x, y, z disjoint domains which are congruent and parallel to the convex domain D of unit volume till the parallelepiped K_{xyz} is filled with domains.

Let us denote by $M(x, y, z)$ the mean of the number of such domains placed in the parallelepiped K_{xyz}. Let us suppose that the following is true:

Hypothesis B. The following inequalities are valid:

$$|M(x_1 + x_2, y, z) - M(x_1, y, z) - M(x_2, y, z)| \leqq B\, yz$$

$$|M(x, y_1 + y_2, z) - M(x, y_1, z) - M(x, y_2, z)| \leqq B\, xz$$

and

$$|M(x, y, z_1 + z_2) - M(x, y, z_1) - M(x, y, z_2)| \leqq B\, xy .$$

Then one can deduce in the same way as shown above that the limit

$$\lim_{\substack{x \to +\infty \\ y \to +\infty \\ z \to +\infty}} \frac{M(x, y, z)}{xyz} = \beta(D)$$

exists. In case D is a unit cube C_1, one is led in the same way as to (28) to the conjecture that $\beta(C_1) = c^3 = 0{,}42$ where c is the constant defined by (2).

The problem can be treated similarly in general for the n-dimensional case too, which leads to the conjecture that in the average c^n % of the volume of a large n-dimensional parallelotope will be filled by the n-dimensional unit cubes, if these are placed at random with uniform distribution so that they should not intersect and their sides parallel to those of the parallelotope.

I am indebted to Professor RÉNYI for his valuable remarks.

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ПРОБЛЕМЫ ОТНОСИТЕЛЬНО СЛУЧАЙНОГО ЗАПОЛНЕНИЯ ПРОСТРАНСТВА

I. PALÁSTI

Резюме

Работа занимается следующей задачей: расположим на плоскости во внутренность прямоугольника со сторонами x и y случайным образом (с равномерным распределением) параллельно расположенные области, конгруэнтные выпуклой области D с единичной площадью, так, чтобы они не имели общих точек, и будем продолжать этой процесс, пока это возможно. Спрашивается, какую часть площади прямоугольника заполняют так расположенные области?

Пусть $M(x, y)$ означает ожидаемое число расположенных вышеуказанным образом областей. Автор высказывает относительно $M(x, y)$ очевидную гипотезу, согласно которой существует такое $B > 0$, что

$$|M(x_1 + x_2, y) - M(x_1, y) - M(x_2, y)| \leqq By$$

и

$$|M(x, y_1 + y_2) - M(x, y_1) - M(x, y_2)| \leqq Bx.$$

Автор доказывает двухмерное обобщение одной теоремы HYERS-а относительно функциональных неравенств [2] и с его помощью доказывает, что из упомянутой гипотезы следует существование предела

$$\lim_{\substack{x \to \infty \\ y \to \infty}} \frac{M(x, y)}{xy} = \alpha(D).$$

Автор высказывает гипотезу (к которой его привели эвристические соображения), что в случае, когда D есть единичный квадрат S_1, $\alpha(S_1) = c^2$, где постоянная c совпадает с постоянной, полученной в решенной RÉNYI аналогичной одномерной задаче, т. е.

$$c = 2\int_0^\infty (1 - e^{-t}) \exp\left(-2\int_0^t \frac{1 - e^{-u}}{u}\, du\right) dt = 0{,}748\ldots$$

и поэтому c^2 приблизительно равно 0,56. Согласно этой гипотезе примерно 56%-ов площади большого четырехугольника может быть покрыто случайно расположенными единичными квадратами, стороны которых параллельны сторонам прямоугольника.

Относительно n-мерной аналогичной задачи высказывается аналогичная гипотеза, согласно которой c^n-ую часть n-мерного прямоугольника заполняют случайно расположенные n-мерные единичные кубы, грани которых параллельны граням прямоугольника.