ON SYSTEMS OF EQUATIONS CONTAINING ONLY ONE NONLINEAR EQUATION

by

Béla Hajtman

§ 1. Introduction

The systems to be considered here consist of several linear equations and of one equation of higher degree. Our purpose is to find conditions of solvability and the number of solutions, and to give an explicit formula for the solutions. This is achieved by giving explicitly an equation in one unknown the knowledge of whose roots reduces the given system to a linear system. We call this equation „the eliminant of the system". We are giving this eliminant in terms of symbolic determinants. Such ones were applied in elimination theory first by Clebsch who generalized (in his paper [3]) the use of symbols introduced by Aronhold [1].

Our method is — essentially — the application of Poisson's one [6] to this special system. It seems this has more advantages here than any other method. Namely, the known methods (those of Bézout, Cayley, Kronecker, etc.) give us procedures of constructing the eliminant, but do not give for it an explicit formula, which we can do here without any difficulty. Our method enables us to discuss certain properties of the solutions such as the number of the different roots, the multiplicity of the roots, the rank of the system of equations. We are able to establish explicit relations between eliminants belonging to different unknowns. Among the eliminants of the classical theory such relations can be found only after the application of Liouville's transformation (see e. g. Netto [5] §§ 359, 387. Bd. II.) which is rather inconvenient. Our method has the practical advantage that we may restrict ourselves to determining a preassigned unknown if we are interested in only one unknown.

The greatest part of this paper deals with systems in which the number of independent equations is equal to that of the unknowns. The general case will be considered in the last section. We devote a separate section to the behaviour of the solutions and that of the eliminants under linear transformation.

Similar systems have been dealt with by Clebsch [2], but he has considered only systems in which the number of equations is greater than that of unknowns and he has discussed merely the existence of a common root.

G. Freud was so kind as to raise the question dealt with in this paper and he proposed to choose this way of solving. In the preparation of this paper I have got many helps from Prof. L. Fuchs too. I am very much obliged to both of them.

§ 2. The solution of the system of equations

Let us consider a system of n equations in n unknowns, where $n-1$ equations are linear and one equation is of degree $k>1$. The coefficients of the system of equations are supposed to be elements of a field F of characteristic 0 or some prime $p>k$. The $(n-1)\times n$ rectangular matrix $\mathfrak{A}$ of the coefficients of the linear equations is assumed to be of rank $n-1$. We are looking for the solutions of this system.

Writing the equation of degree k in the usual way in homogeneous and symmetric form, we have the following system of equations:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i=1,2,\ldots,n-1)\,, \tag{1}$$

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\cdots\sum_{i_k=0}^{n} c_{i_1 i_2\ldots i_k}x_{i_1}x_{i_2}\ldots x_{i_k}=0\,, \tag{2}$$

$$x_0 = 1 \tag{2a}$$

where a_{ij}, b_i, $c_{i_1 i_2\ldots i_k}\in F$, the $c_{i_1 i_2\ldots i_k}$ are independent of the permutation of their indices and the matrix $[a_{ij}]=\mathfrak{A}$ is of rank $n-1$.

Two types of determinants are needed in what follows. We obtain the determinant A_j from $\mathfrak{A}$ by omitting the column of index j and prefixing the sign $(-1)^{j-1}$. The other, $A_j^{(l)}$ may be derived from A_j by putting the elements b_i (the right members of the linear equations) in place of the column of index l. Then obviously $A_j^{(l)}=-A_l^{(j)}$. It is useful to agree to put $A_0=0\,;\ -A_0^{(j)}=$ $=A_j^{(0)}=A_j\,;\ A_j^{(j)}=0$.

Next we write the coefficients $c_{i_1 i_2\ldots i_k}$ as the product of the symbols $c_{i_1}, c_{i_2},\ldots,c_{i_k}$. These symbols $c_j (j=0,1,\ldots,n)$ may be regarded as indeterminates over F which permute among themselves, but it is to be emphasized that we shall consider only formulas in which every term contains no c_j or the product of exactly k symbols.

The equation (2) may be written as

$$\left(\sum_{j=0}^{n} c_j x_j\right)^k = 0\,. \tag{3}$$

The explicit construction of the solution of the systems above depends on the following lemma:

Lemma. *If* $\{\xi_1, \xi_2, \ldots, \xi_n\}$ *is a solution of the system (1)—(2), then* ξ_i *satisfies the equation*

$$(Px_i - R_i)^k = 0 \qquad (i=1,2,\ldots,n), \tag{4}$$

where P *and* R_i *are symbolic determinants defined by*

$$P=\begin{vmatrix} c_1 & c_2 & \ldots c_n \\ a_{11} & a_{12} & \ldots a_{1n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n-11} & a_{n-12} & \ldots a_{n-1n}\end{vmatrix}\,;\ R_i=\begin{vmatrix} c_1 & \ldots c_{i-1} & -c_0 & c_{i+1} & \ldots c_n \\ a_{11} & \ldots a_{1i-1} & b_1 & a_{1i+1} & \ldots a_{1n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{n-11} & \ldots a_{n-1i-1} & b_{n-1} & a_{n-1i+1} & \ldots a_{n-1n}\end{vmatrix}. \tag{5}$$

The equations (4) will be called the *eliminants* of the system (1)—(2).

Proof. By Cramer's rule we obtain from the equations (1):

$$x_j A_i = x_i A_j + A_i^{(j)}. \tag{6}$$

This equation is valid also in the cases $i = j$, $i = 0$, $j = 0$.

In order to verify the lemma, let us multiply the equation (3) by the kth power of A_i and substitute (6). It is enough to perform the calculation for a single factor. We obtain

$$A_i \sum_{j=0}^{n} c_j x_j = \sum_{j=0}^{n} c_j (x_i A_j - A_j^{(i)}) = P x_i - R_i ,$$

for, denoting by Σ^i the sum from which the index i is omitted, we have

$$\begin{gathered} \sum_{j=0}^{n} c_j A_j = \sum_{j=1}^{n} c_j A_j = P, \\ \sum_{j=0}^{n} c_j A_j^{(i)} = \sum_{j=1}^{n}{}^{i} c_j A_j^{(i)} + c_0 A_0^{(i)} + c_i A_i^{(i)} = \sum_{j=1}^{n}{}^{i} c_j A_j^{(i)} - c_0 A_i = R_i . \end{gathered} \tag{7}$$

This proves the lemma.

We shall call the unknown x_j (and also the index j) *singular* if $A_j = 0$. For nonsingular unknowns we can state also the converse of the lemma above:

Theorem 1. *Let x_i be a nonsingular unknown. To every root $x_i = \xi_i$ of the eliminant*

$$(P x_i - R_i)^k = 0 \tag{8}$$

there exists one and only one solution $\{\xi_1, \xi_2, \ldots, \xi_n\}$ of the system (1)—(2) which may be got by the formulas (6) and in this way every solution of (1)—(2) can be obtained.

(Thus the solution of (1)—(2) can be got by solving first (8) and then using (6)).

Since $A_i \neq 0$, the values ξ_j got from (6) obviously satisfy the system (1), whatever ξ_i may be (Cramer's rule!). If we substitute them into (2), it seems immediately that they satisfy this too provided that ξ_i is a root of the equation (8). And this is what we wished to prove.

By the consideration of the equation (6) it will be seen at once that (4) for singular unknowns is not suitable to solve the system of equations: to each of them (6) gives only one value. (If there exist more solutions than one then the singular unknowns have the same value in each one.) However, the equation (4) is often not adequate to the determination of this single value. Concerning this case we may state:

Theorem 2. *Let x_s be a singular unknown. If $P^k \neq 0$, the equation (4) for x_s has a single root with multiplicity k. If $P^k = 0$, then every coefficient of the polynomial $(P x_s - R_s)^k$ is equal to zero.*

Proof. Let x_p and x_r be nonsingular unknowns. From (6) we have

$$\xi_s = \frac{A_p^{(s)}}{A_p} = \frac{A_r^{(s)}}{A_r} \tag{9}$$

since $A_s = 0$. Multiply the first equation of (7) by ξ_s to get:

$$P\,\xi_s = \sum_{j=0}^{n} c_j\,A_j\,\xi_s = \sum_{j=0}^{n} c_j\,A_j^{(s)} = R_s\,. \tag{10}$$

Hence (4) has the form

$$(Px_s - R_s)^k = (Px_s - P\,\xi_s)^k = P^k(x_s - \xi_s)^k = 0\,,$$

which completes the proof.[1]

§ 3. Properties of the eliminants and the solutions

The eliminants for nonsingular unknowns are suitable not only for getting the solutions, but also for establishing certain properties of them, e. g. the number of solutions, the multiplicity of a solution. Starting with a fixed nonsingular unknown x_i, it is evident that the number of all solutions is equal to the effective degree g_i of the eliminant for x_i, unless the eliminant vanishes identically (and so g_i is undefined). Let a root ξ_i of this eliminant have the multiplicity μ_i, then — according to the usual definition (e. g. see NETTO [5] § 349. Bd. II.) — we say the multiplicity of the solution $\{\xi_1, \xi_2 \ldots, \xi_n\}$ (belonging to ξ_i by (6)) in the system (1)—(2) is μ_i.

However, — as it is clear from the proof of Theorem 1 — there cannot be made any distinction between the different nonsingular unknowns.[2] Indeed, we shall prove that g_i is characteristic for the system itself, i. e. for every nonsingular index its value is independent of the index. (Consequently, the index i may be omitted.) It will also be proved that in the eliminants for nonsingular unknowns the multiplicities μ_i of the elements ξ_i of a fixed solution $\{\xi_1, \xi_2, \ldots, \xi_n\}$ are equal. Both of these assertions follow immediately from the next theorem.

Theorem 3. *Let x_r and x_t be nonsingular unknowns. If (by (6)) we apply the linear transformation*

$$x_r = \frac{A_r}{A_t}\,x_t + \frac{A_t^{(r)}}{A_t} \tag{11}$$

to the eliminant belonging to x_r and then multiply by $\left(\frac{A_t}{A_r}\right)^k$, we obtain the eliminant belonging to x_t.

---

[1] Since ξ_s is the root of the $(k-1)$st derivative of (4), we have

$$\xi_s = \frac{P^{k-1}\,R_s}{P^k}$$

(if $P^k \neq 0$). By (10), this is the same value as that got by (9).

[2] The conditions required at the start of our discussion guarantee the existence of only one nonsingular unknown.

Thus the eliminants belonging to nonsingular unknowns differ essentially only by a linear transformation.

Proof. Clearly, it is enough to show that (11) implies

$$A_r(Px_t - R_t) = A_t(Px_r - R_r)\,.$$

First we shall verify the relation

$$A_t R_r = A_t^{(r)} P + A_r R_t\,. \tag{12}$$

Using (7), the coefficient of c_j in (12) is:

$$\begin{aligned} A_t A_j^{(r)} &= A_t x_r A_j - A_t x_j A_r = \\ &= A_t^{(r)} A_j + A_r x_t A_j - A_r x_j A_t = A_t^{(r)} A_j + A_r A_j^{(t)} \end{aligned}$$

where we have applied (6) again and again. This proves (12). Now by (11) and (12)

$$\begin{aligned} A_t(Px_r - R_r) &= P(x_t A_r - A_r^{(t)}) - A_t R_r = \\ &= A_r P x_t - (A_r^{(t)} P + A_t^{(r)} P + A_r R_t) = A_r(Px_t - R_t) \end{aligned}$$

which completes the proof.

Corollary 1. *The eliminants (4) for nonsingular indices are of the same effective degree g.*

Corollary 2. *If $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is a solution of (1)—(2), then for every nonsingular index i the multiplicity of ξ_i in the ith eliminant is independent of i.*[3]

For singular indices the effective degree of the eliminants is equal to g if and only if $g = k$, — otherwise the eliminants are identically zero. Similarly, the multiplicity of a root of some singular eliminant is either k or undefined, — as we have seen.

We discuss the case, when $g > 0$ is not true, separately. If $g = 0$ (i. e. among the coefficients of (8) only R_i^k is different from zero), the system (1)—(2) is inconsistent. If every coefficient vanishes (g is undefined), our system has infinitely many solutions (it is indeterminated).

Our system is of rank n or $n - 1$. In KRONECKER's terminology [4], a system of equations in n unknowns is of rank $n - j$ if its solutions form a j-dimensional manifold. In our case, if $g > 0$, the rank is n, while if g is undefined, it is n—1 and not lower, because the system (1) is of rank n—1.[4]

The eliminant determines too, of course, what extension of the field F is needed in computing the solutions. The extension needed in the calculation of the roots of the eliminants will obviously contain the whole solutions, since the equations (6) are linear. Because of Theorem 3, however, it is also true that the extension of the field needed in solving will be the same one for every

[3] This statement follows from a theorem of the classical theory too. That asserts (see e. g. NETTO [5] § 403. Bd. II.) that the multiplicity of a solution is equal to the product of the multiplicities occurring in the respective equations of the system. (We shall use this theorem in the fourth section.) However, our Corollary 2 shows that the eliminant (8) is equivalent to those defined in the classical theory.

[4] We note that the values got for the singular unknowns form always a 0-dimensional manifold therefore the rank of the indeterminated system will not in general be so-called pure rank.

nonsingular unknown, and we cannot narrow this field by seeking some more suitable eliminant. Accordingly, if one of the eliminants of some system (1)—(2) is not solvable by radicals (in the cases $k \geq 5$), then none of the others for nonsingular unknowns may be solved so.

§ 4. Linear transformations

We shall use the following notations. The determinant of the matrix $[\tau_{ij}]$ of the linear transformation

$$x_i = \sum_{j=1}^{n} \tau_{ij} y_j \qquad (i = 1, 2, \ldots, n) \tag{13}$$

will be denoted by T. We suppose that $T \neq 0$. Furthermore let $[\sigma_{ij}]$ be the matrix of the inverse transformation of (13):

$$y_i = \sum_{j=1}^{n} \sigma_{ij} x_j \qquad (i = 1, 2, \ldots, n). \tag{14}$$

Then $\sigma_{ij} = T_{ji} T^{-1}$, where T_{ji} is the so-called cofactor of τ_{ji} in T.

If we apply the transformation (13) to the system (1)—(2), we get a similar system of equations with unknowns y_i. The new system has again eliminants, etc. We keep the same notations for the new system with adding asterisks.

Next we discuss the behaviour of the eliminants under a linear transformation.

Theorem 4. *For the leading coefficient of the transformed eliminant we have*

$$P^{*k} = P^k T^k.$$

Proof. Let us apply the transformation (13) to the equation (3), after we have extended the matrix $[\tau_{ij}]$ by the elements $\tau_{00} = 1$, $\tau_{0j} = \tau_{i0} = 0$ $(i, j = 1, 2, \ldots, n)$. We have

$$\sum_{i=0}^{n} c_i \sum_{j=0}^{n} \tau_{ij} y_j = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} c_i \tau_{ij} \right) y_j .$$

Hence $[\tau_{ij}]$ transforms the vector $(c_0, c_1, \ldots, c_n)$ into $(c_0^*, c_1^*, \ldots, c_n^*)$. Therefore, the symbols c_j are transformed in the same manner as the coefficients a_{ij}. Consequently,

$$P^* = PT,$$

proving the theorem.

Theorem 5. *For the effective degrees we have*

$$g^* = g .$$

Proof. In view of the factorization theory of matrices (e. g. see Wellstein [7]) one can interpret every nonsingular linear transformation as a finite sequence of the elementary transformations of the following types:

a) interchange of two unknowns;
b) multiplication of one of the unknowns by some element $c \neq 0$ $(c \in F)$;
c) addition of a certain unknown to an other one.

If we consider in detail the change of the eliminant uder these elementary transformations, we obtain that the equation (8) either remains unchanged or will be one of the forms

$$c\,(Py_i - R_i)^k = 0\,,$$

$$(Py_i - R_i + R_j)^k = 0\,.$$

It is easy to see that in both cases the effective degree remains the same as the original was. (Unless y_i will be singular.) Q. e. d.

By the theorem mentioned in the third footnote a solution of (1)—(2) has the same multiplicity μ in the system (1)—(2) as in the equation (2). This μ is — by definition — the multiplicity of the ith element of the solution in the ith eliminant, where i is any nonsingular index. The usual definition of multiple roots of some equation in several unknowns is the following (e. g. NETTO [5] § 351. Bd. II.): a root $\{\xi_1, \xi_2, \ldots, \xi_n\}$ of an equation in n unknowns has the multiplicity μ if the equation may be written as a homogeneous polynomial of degree μ of the factors $(x_j - \xi_j)$. (In general the coefficients of this one will be, of course, polynomials in n unknowns.)

Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be a μ-tuple solution of (1)—(2). Denote by η_j $(j = 1, 2, \ldots, n)$ the values obtained from the transformation (14) by the substitution $x_j = \xi_j$. Writing (2) in the mentioned polynomial form of $(x_j - \xi_j)$ and carrying out in this form the transformation (13), we shall see that the array $\{\eta_1, \eta_2, \ldots, \eta_n\}$ is a root of the equation (2)* which has at least the multiplicity μ, or, since the same holds for the inverse transformation, it has exactly the multiplicity μ. Thus we have proved the following:

Theorem 6. *Applying the transformation (14) to the μ-tuple solution $\{\xi_1, \xi_2, \ldots, \xi_n\}$ of the system (1)—(2), the arising array $\{\eta_1, \eta_2, \ldots, \eta_n\}$ will again be an exactly μ-tuple solution of the system of equations transformed by (13).*

§ 5. The general case

We have solved the system (1)—(2) imposing strong restrictions upon the number of equations and the rank of the matrix of coefficients. Consider now the general case.

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \qquad (i = 1, 2, \ldots, m)\,, \tag{15}$$

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \cdots \sum_{i_k=0}^{n} c_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k} = 0\,, \tag{16}$$

$$x_0 = 1\,, \tag{16a}$$

where the coefficients are taken from a field F (which is of characteristic 0 or a prime greater than k). The values of the coefficients $c_{i_1 i_2 \ldots i_k}$ are invariant under taking any permutation of the indices.

For the solvability of our system a trivial necessary condition is that the system consisting of its linear equations be solvable, i. e. the rank of the matrix of the linear system be equal to that of the augmented matrix. Then choose a maximal independent system of linear equations and suppose that (15) denotes already this, that is, the matrix $\mathfrak{A}_{mn}$ of cofficients of (15) is of rank m.

We shall distinguish three cases: $m<n-1$, $m=n-1$ and $m=n$. The second one was dealt with above in detail, now we scetch how to treat the others.

Consider first the case $m<n-1$. By hypothesis every solution (if exists) contains parameters, so the rank of the system is a priori lower than n. Now, introduce the indeterminates $b_{m+1}, b_{m+2}, \ldots, b_{n-1}$ as parameters and add to the system (15) the equations

$$x_{j_{m+2}} = b_{m+1}, x_{j_{m+3}} = b_{m+2}, \ldots, x_{j_n} = b_{n-1}, \tag{17}$$

where $j_1, j_2, \ldots, j_n$ denotes a permutation of $1, 2, \ldots, n$, such that $\mathfrak{A}_{mn}$ has a nonvanishing minor composed of its columns $j_1, j_2, \ldots, j_{m+1}$. Let f denote the number of *different* systems which may be got from (15)—(16) by permuting the indices. It is easily seen this f varies in the interval

$$(n-m) \leqq f \leqq \binom{n}{m+1}$$

according to the number of nonvanishing m-rowed minors of $\mathfrak{A}_{mn}$.

Now, choose a fixed permutation of indices and add (17) to the equations (15); it is immediately seen that the conditions required in § 2 are fulfilled. Hence the system of equations can be solved in the same way as there. The only difference is that the first m of $b_1, b_2, \ldots, b_{n-1}$ are constant, but the other $n-m-1$ are indeterminates over F.

The results of sections 2 and 3 remain valid also in this case. Obviously, the results of § 4 are valid only for linear transformations leaving invariant the linear subspace of the unknowns and that of the parameters.

Now we make some remarks.

Considering the matrix of the system (15) enlarged by (17), one sees immediately that the parameters are always singular.

Because of the special form of the matrix of the system of equations the determinant P defined by (5) equals

$$S = \begin{vmatrix} c_{j_1} & c_{j_2} & \ldots & c_{j_{m+1}} \\ a_{1j_1} & a_{1j_2} & \ldots & a_{1j_{m+1}} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{mj_1} & a_{mj_2} & \ldots & a_{mj_{m+1}} \end{vmatrix} = (-1)^{\pi} P,$$

where π is determinated by the chosen permutation of the unknowns. However, concerning the determinant R_i $(i = j_1, j_2, \ldots, j_{m+1})$ we get

$$(-1)^{\pi} R_i = T_i - \sum_{j=m+2}^{n} b_{j-1} U_i^{(j)},$$

where T_i, resp. $U_i^{(j)}$ are obtained from S by interchanging the ith column with the column $(-c_0, b_1, \ldots, b_m)$, resp. $(c_j, a_{1j}, \ldots, a_{mj})$. Consequently, the eliminant (8) becomes

$$\left|Sx_i - T_i + \sum_{j=m+2}^{n} b_{j-1} U_i^{(j)}\right|^k = 0 .$$

The determinants S, T_i, $U_i^{(j)}$ contain no longer parameter.

Finally we discuss the case $m = n$. CLEBSCH [2] has explicitly given a product of symbolic determinants which vanishes if and only if there exists a solution. We give a further condition for the solvability:

Theorem 7. *Consider a system of equations in n unknowns consisting of one equation of degree k and n linear equations having a matrix $\mathfrak{A}_{nn}$ of coefficients of rank n. This system has a (necessarily single) solution if and only if the equation of degree k can be written as a polynomial (of degree k) of the linear equations of the system.*

Proof. It is well-known that there exists at most one solution. The sufficiency of the condition is also trivial.

In order to prove the necessity let us apply for simplicity's sake the linear transformation

$$\sum_{j=1}^{n} a_{ij} x_j = y_i \qquad (i = 1, 2, \ldots, n), \tag{18}$$

transforming the linear system into

$$y_i = b_i \qquad (i = 1, 2, \ldots, n). \tag{19}$$

The transformation can be extended also to the unknown x_0 in the trivial way. So the unknowns increase with the ,,unknown'' $y_0 = 1$ and the linear system increases with a new equation of index 0 where $b_0 = 1$.

Apply the transformation (18) also to the equation (2). Obviously we shall get a polynomial of degree k in the new unknowns y_i:

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \cdots \sum_{i_k=0}^{n} \gamma_{i_1 i_2 \ldots i_k} y_{i_1} y_{i_2} \ldots y_{i_k} = 0, \tag{20}$$

that is, the transformed equation is a polynomial of the left sides of the linear equations.

Now, if the solution of the linear system (given by the Cramer's rule) satisfies also (2), then we have

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \cdots \sum_{i_k=0}^{n} \gamma_{i_1 i_2 \ldots i_k} b_{i_1} b_{i_2} \ldots b_{i_k} = 0 . \tag{21}$$

To complete the proof it remained to show that (20) may be written as a polynomial of the $(y_i - b_i)$. This means that, applying the transformation

$$y_i = z_i + b_i ,$$

no constant member remains in (20). Transforming we see that the constant term is nothing else than the expression (21) — which vanishes. Q. e. d.

(Received October 10, 1960.)

REFERENCES

[1] Aronhold, S.: "Theorie der homogenen Functionen dritten Grades von drei Veränderlichen." *Journal für die reine und angewandte Mathematik* **55** (1858) 97—191.
[2] Clebsch, A.: »Ueber eine Classe von Eliminationsproblemen und über einige Punkte der Theorie der Polaren.« *Journal für die reine und angewandte Mathematik* **58** (1861) 273—291.
[3] Clebsch, A.: »Ueber symbolische Darstellung algebraischer Formen.« *Journal für die reine und angewandte Mathematik* **59** (1861) 1—62.
[4] Kronecker, L.: »Grundzüge einer arithmetischen Theorie der algebraischen Grössen.« *Journal für die reine und angewandte Mathematik* **92** (1881) 1—122.
[5] Netto, E.: *Vorlesungen über Algebra*, Leipzig, 1900.
[6] Poisson, D.: «Mémoire sur l'élimination dans les équations algébriques.» *Journal de l'École Polytechnique* **4** (1804) cah. 11, 199—212.
[7] Wellstein, G.: »Die Dekomposition der Matrizen.« *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen* (1909) 77—79.

О СИСТЕМАХ УРАВНЕНИЙ, СОДЕРЖАЩИХ ТОЛЬКО ОДНО НЕЛИНЕЙНОЕ УРАВНЕНИЕ

B. HAJTMAN

Резюме

Рассмотрим следующую систему уравнений:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, 2, \ldots, m)\,, \tag{1}$$

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \ldots \sum_{i_k=0}^{n} c_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k} = 0\,, \tag{2}$$

$$x_0 = 1\,, \tag{2a}$$

где a_{ij}, b_i, $c_{i_1 i_2 \ldots i_k}$ элементы тела F (характеристика которого 0 или простое число, большее чем k) и значение коэффициентов $c_{i_1 i_2 \ldots i_k}$ не зависит от перестановок их индексов. Ищутся решения этой системы уравнений и условия её разрешимости.

В § 2—4 рассматривается случай $m = n - 1$, при предположении, что ранг матрицы коэффициентов $\mathfrak{A}$ системы линейных уравнений (1) равен $n - 1$. Определителем, относящимся к неизвестному x_i, называется минор матрицы $\mathfrak{A}$, получаемый вычеркиванием i-ого столбца. Неизвестное *сингулярно*, если соответствующий минор равен нулю.

В § 2 находится решение. Важную роль играют следующие символические определители:

$$P = \begin{vmatrix} c_1 & c_2 & \ldots c_n \\ a_{11} & a_{12} & \ldots a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n-1,1} & a_{n-1,2} & \ldots a_{n-1,n} \end{vmatrix}, \quad R_i = \begin{vmatrix} c_1 & \ldots c_{i-1} & -c_0 & c_{i+1} & \ldots c_n \\ a_{11} & \ldots a_{1,i-1} & b_1 & a_{1,i+1} & \ldots a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1,1} & \ldots a_{n-1,i-1} & b_{n-1} & a_{n-1,i+1} & \ldots a_{n-1,n} \end{vmatrix}$$

где символи c_j определяются так:

$$c_{i_1} c_{i_2} \ldots c_{i_k} \equiv c_{i_1 i_2 \ldots i_k}\,.$$

Теорема I утверждает, что, если неизвестное x_i не сингулярно, то каждому корню символического уравнения

$$(Px_i - R_i)^k = 0 \tag{3}$$

соответствует одно (и только одно) решение системы уравнений (1) — (2), остальные элементы которого получается из простой системы линейных уравнений, и таким образом получаются все решения (1) — (2). Система уравнений (3) называется *элиминантом* системы.

Согласно теореме 2, если уравнение (3) записано для сингулярного неизвестного, то оно имеет единственный k-кратный корень при $P^k \neq 0$ (и это значение принимает сингулярное неизвестное во всех решениях), если же $P^k = 0$, то все коэффициенты уравнения равны нулю.

В § 3 исследуются свойства элиминантов и решений. Теорема 3 определяет соотношение между элиминантами, относящимися к несингулярным неизвестным; оказывается, что они — не считая постоянных факторов — отличаются друг от друга простым линейным преобразованием. Отсюда следует, что разрешимость, число решений, кратность отдельных решений однозначно определяется уже одним элиминантом (относительно несингулярного неизвестного).

В § 4 изучается поведение элиминантов и решений в случае линейного преобразования неизвестных. Теорема 4 занимается поведением коэффициента P^k, теорема 5 — фактической степени элиминантов, наконец, теорема 6 — кратности решений при преобразовании.

В § 5 не ставится ни каких ограничений относительно m и ранга матрицы $\mathfrak{A}$. Сначала рассматривается случай, когда решение содержит параметр. Этот случай сводится к рассмотренному в первых параграфах. Наконец, рассматривается случай, когда ранг матрицы $\mathfrak{A}$ равен n. Теорема 7 дает необходимое и достаточное условие разрешимости в этом случае.