ON A DENSITY THEOREM OF YU. V. LINNIK

by

P. TURÁN

§ I.

1. Let Δ be positive integer, χ runs over the characters belonging to the moduls Δ, $w = \sigma + it$ and $L(w, \chi)$ the L-function belonging to χ, $(l, \Delta) = 1$ and $P(\Delta, l)$ the least prime $\equiv l \bmod \Delta$. Yu. V. LINNIK has proved in 1944 the important theorem[1]

$$P(\Delta, l) < \Delta^{c_1} \tag{1.1.1}$$

where c_1 and later $c_2, c_3, \ldots$ stand in § I for positive numerical constants. One of his two main tools in his proof for (1.1.1) is the following theorem.[2]

If $2 \leqq \lambda \leqq \frac{1}{10} \log \Delta$ *and* $N(\lambda, \Delta)$ *stands for the number of zeros of all* $L(w, \chi)$*'s mod* Δ *in the domain*

$$1 - \frac{\lambda}{\log \Delta} \leqq \sigma \leqq 1, \qquad |t| \leqq \frac{e^\lambda}{\log \Delta}, \tag{1.1.2}$$

then for $\Delta > c_2$ *the inequality*

$$N(\lambda, \Delta) \leqq e^{c_3 \lambda} \tag{1.1.3}$$

holds.

The aim of this note is to offer an alternative, rather short proof for this theorem. More exactly we shall prove the following theorem.

Denoting by $N(\lambda, \Delta, t_0)$ *the number of the zeros of all L-functions mod* Δ *in the domain*

$$1 - \frac{\lambda}{\log \Delta} \leqq \sigma \leqq 1, \qquad |t - t_0| \leqq \frac{e^\lambda}{\log \Delta} \tag{1.1.4}$$

we have for $|t_0| \leq \sqrt{\Delta}$ *and suitable* c_4, c_6, c_7 *and*

$$0 \leqq \lambda \leqq c_4 \log \Delta \tag{1.1.5}$$

[1] For a detailed exposition of LINNIK's very powerful proof in simplified form due to K. A. RODOSSKIJ, see the book of K. PRACHAR: *Primzahlverteilung.* (Springer, 1957) in particular p. 330–370.

[2] The theorem (1.1.2)–(1.1.3) is identical with theorem 2.1 on p. 331. of PRACHAR's book and for the proof (after some preparations in § I of Chapter VII.) see p. 331–348 of this book.

for $\varDelta > c_5$ *the inequality*

$$N(\lambda, \varDelta, t_0) \leqq c_6 e^{c_7 \lambda}. \tag{1.1.6}$$

2. Our proof (though we made no attempt to squeeze out possibly small values) gives possibility to find not too large *explicit* values for c_4, c_6 and c_7 (which have a significance in finding a possibly small value for the important constant c_1). Let namely a_1, a_2, a_3 be the following three constants (whose numerical values can be evaluated by well-known arguments.[3]

I. For $(l, \varDelta) = 1$ and $x \geqq \varDelta^3$ we have

$$S_l(x) \overset{\text{def}}{=} \sum_{\substack{n \leq x \\ n \equiv l \bmod \varDelta}} \Lambda(n) \leqq a_1 \frac{x}{\varphi(\varDelta)}. \tag{1.2.1}$$

(BRUN's method, in the simplified form of A. SELBERG gives for a_1 a value < 4.)

II. If $0 \leqq \delta \leqq 1$, then the number of zeros of each single $L(w, \chi)$ in the square

$$1 - \delta \leqq \sigma \leqq 1, \quad |t - t_1| \leqq \frac{\delta}{2} \qquad (t_1 \text{ real})$$

is at most

$$1 + a_2 \delta \log \{\varDelta(1 + |t_1|)\}. \tag{1.2.2.}$$

(a_2 can be chosen about $\frac{1}{2}$).

III. The parallelogramm

$$1 - \frac{a_3}{\log \varDelta} \leqq \sigma \leqq 1, \qquad |t| \leqq 1 \tag{1.2.3}$$

can contain at most one simple zero of $\prod_{\chi} L(w, \chi)$. (If such a ϱ-zero („Siegel-zero") exists and $L(\varrho, \chi^*) = 0$, then we shall call χ^* an „exceptional" character).

Further let ω be the smallest positive integer satisfying the following inequalities:

$$\omega \geqq e^4, \tag{1.2.4a}$$

$$\omega \geqq \frac{3}{8 a_2}, \tag{1.2.4b}$$

$$8 e(\omega + 1) \left(\frac{2}{3}\right)^{\omega} < \left(\frac{2}{3}\right)^{\omega^{3/4}}, \tag{1.2.4c}$$

$$\frac{36 a_2 \log (9 e \omega)}{\sqrt{\omega}} \leqq 1, \tag{1.2.4d}$$

$$\omega \geqq \left(\frac{8}{a_3}\right)^{\frac{2}{5}}. \tag{1.2.4e}$$

$$\omega \geqq 36 a_2^2 \tag{1.2.4f}$$

$$\omega \geqq \frac{1}{a_2^4} \tag{1.2.4g}$$

$$1 \geqq (36 \omega)^2 a_1 e^{-\frac{a_3}{8} \omega^{\frac{5}{2}}} + e^{-\frac{a_2 a_3}{2} \omega^{\frac{5}{2}}}. \tag{1.2.4h}$$

[3] All can be found in PRACHAR's book, for I. see p. 44, for II. see p. 331, for III. see p. 118. 120 and 122.

Then we assert that with this ω

(1.2.5) $$c_4 = \frac{1}{2\omega^{\frac{5}{2}}}, \qquad c_6 = \frac{6}{a_3}, \qquad c_7 = \left(\omega^{\frac{5}{2}} + 1\right)$$

can be chosen in (1.1.5) resp. (1.1.6).

3. For the sake of orientation I remark that for

$$\log\log \varDelta \leqq \lambda \leqq \frac{1}{2}\log \varDelta .$$

LINNIK's estimation (1.1.2)—(1.1.3) is easy[4] and the same holds in the whole range $2 \leqq \lambda \leqq \frac{1}{2}\log \varDelta$ about the proof of the inequality

(1.3.1) $$N(\lambda, \varDelta) \leqq e^{c_3\lambda} \log \varDelta \,;$$

the principal difficulty lies in eliminating the logarithmic factor in (1.3.1) for the range $a_3 \leqq \lambda \leqq \log\log \varDelta$. My proof of the theorem (1.1.5)—(1.1.6) will turn out to be essentially an application of my second main theorem in the following special form.[5]

If m *is a positive integer,*

(1.3.2) $$\max_{j=j,\ldots n} |z_j| \geqq 1$$

and $n \leqq N_1$, *then for a suitable integer* ν_1 *with*

(1.3.3) $$m + 1 \leqq \nu_1 \leqq m + N_1$$

we have

(1.3.4) $$|z_1^{\nu_1} + z_2^{\nu_1} + \ldots + z_n^{\nu_1}| \geqq \left(\frac{N_1}{8\,e(m + N_1)}\right)^{N_1}.$$

The paper apart from this is self-contained. The only new feature (compared to other applications of this theorem) is, apart from the unu-

---

[4] For a very short and simple proof see my paper »Über die Wurzeln der Dirichletschen L.-Functionen.« *Acta Scient. Szeged* T. **10** (1943) 3—4 p. 188—201 (manuscr. received 27. Aug. 1941).

[5] See my book: *Eine neue Methode in der Analysis und deren Anwendungen* Akadémiai Kiadó, Budapest, 1953; a completely rewritten English edition will appear among the Interscience Tracts. As to this sharper form of the theorem see our paper with Vera T. Sós "On some new theorems in the theory of diophantine approximations." *Acta Math. Hung.* T. **6** (1955) 3—4, p. 241—254.

sual choice of some parameters, an estimation of the quantity $|R(n)|$ (see lemma I) which was dispensable in the previous applications.[6]

Theorem (1.1.2)—(1.1.3) enables one to prove (1.1.1) for at least the half of the progressions mod Δ and also to prove (1.1.1) for all progressions mod Δ with a „small" c_1 under the supposition that there is no exceptional character mod Δ in the sense of III. (which as well-known is generally the case, apart from a „very few" Δ's). We shall not treat these here. In the case when an exceptional character exists mod Δ, the situation is met by Linnik's second theorem (theorem 3.1. in PRACHAR's book). I think that also this theorem can be proved along the lines of this paper.

As we shall show in 9. of § II the theorem (1.1.4)—(1.1.5)—(1.1.6) can be quickly deduced from the following.

Theorem. *For* ω *satisfying (*1.2.4*),* $\Delta > c_8$*, for*

(1.3.5)
$$1 - \frac{1}{2\,\omega^{5/2}} \leqq \gamma \leqq 1$$

and $|t_2| \leq \Delta^{3/4}$ *the total-number of zeros of all L-functions belonging to modulus* Δ *in the square*

(1.3.6)
$$\gamma \leqq \sigma \leqq 1\,, \qquad |t - t_2| \leqq \frac{1-\gamma}{2}$$

cannot exceed

(1.3.7)
$$4\,\Delta^{\omega^{5/2}(1-\gamma)}\,.$$

Since in the proof we want to make dependent c_4, c_6 and c_7 upon the quantities a_1, a_2 and a_3, we shall make the distinction in the constants denoting by $b_1, b_2, \ldots$ positive numerical constants with values independent of a_1, a_2, a_3 and by $d_1, d_2, \ldots$ constants depending only upon a_1, a_2 and a_3 (and ω).

§ II.

1. Now we shall prove the formulated theorem (1.3.5)—1.3.6)—(1.3.7). Fixing ω according to (1.2.4) let β be an arbitrary positive number satisfying

(2.1.1)
$$\frac{a_3\,\omega^{\frac{5}{2}}}{2\log\Delta} \leqq \beta \leqq \frac{1}{4}$$

(which has a sense for $\Delta > d_1$) and fixed. Then we define

(2.1.2)
$$A = \left[\frac{\sqrt{\omega}}{\beta}\right] (> 28)\,.$$

[6] The useful kernel $K_1(w)$ in (2.1.6) and the quantity $R(n)$ appeared for the first time in my paper "On the so-called density-hypothesis in the theory of zeta-function of Riemann." *Acta Arith.* **4** (1958) 1. 31—56.

Owing to (1.2.2) the number of zeros of each single $L(w, \chi)$ in the square

$$1 - \frac{4}{A} \leqq \sigma \leqq 1, \qquad |t - t_1| \leqq \frac{2}{A} \tag{2.1.3}$$

with $|t_1| \leq \Delta$ is for $\Delta > d_2$ at most

$$1 + a_2 \frac{4}{A} \log \Delta (1 + \Delta) < 2 + \frac{8\,a_2}{A} \log \Delta \overset{\text{def}}{=} N. \tag{2.1.4}$$

The integer k should be restricted at present only by

$$\omega N \leqq k \leqq (\omega + 1) N; \tag{2.1.5}$$

then we have evidently $k > 98$. Let further

$$K(w) \overset{\text{def}}{=} e^{2w} \frac{e^w - e^{-w}}{2\,w}, \qquad K_1(w) \overset{\text{def}}{=} K(Aw), \tag{2.1.6}$$

further for $n = 1, 2, \ldots$

$$R(n) \overset{\text{def}}{=} \frac{1}{2\pi i} \int\limits_{(2)} K_1(w)^k e^{-w \log n} dw \tag{2.1.7}$$

and

$$g_0(x) \overset{\text{def}}{=} \frac{1}{x} e^{-\frac{k}{5}\left(2 - \frac{\log x}{kA}\right)^2}. \tag{2.1.8}$$

2. As to $R(n)$ we shall need the

Lemma I. *We have for $n \geqq e^{3kA}$ or $n \leqq e^{kA}$*

$$R(n) = 0, \tag{2.2.1}$$

further for $e^{kA} \leqq n \leqq e^{3kA}$

$$|R(n)| \leqq \frac{1}{A} \sqrt{\frac{7}{k}}\, n g_0(n). \tag{2.2.2}$$

Since the integrand in (2.1.7) is an entire function tending to 0 sufficiently quickly on every vertical line, Cauchy's theorem gives

$$R(n) = \frac{1}{2\pi i} \int\limits_{(-1)} K_1(w)^k e^{-w \log n} d\mathrm{w}. \tag{2.2.3}$$

Applying the binomial formula in (2.1.7) resp. (2.2.3) and the well-known formulas

$$\frac{1}{2\pi i} \int\limits_{(2)} \frac{e^{\lambda w}}{w^k} dw = 0, \quad \text{resp} \quad \frac{1}{2\pi i} \int\limits_{(-1)} \frac{e^{\lambda w}}{w^k} dw = 0, \tag{2.2.4}$$

valid for $\lambda \leqq$ resp. $\lambda \geqq 0$ we get (2.2.1) at once. To prove (2.2.2) we remark that writing

$$(2.2.5)\qquad 2-\frac{\log n}{kA}=\vartheta\,,\qquad -1\leqq\vartheta\leqq+1$$

we have from (2.1.7)

$$(2.2.6)\qquad R(n)=\frac{1}{2\pi i A}\int\limits_{\left(-\frac{\vartheta}{4}\right)}\left(e^{\vartheta w}\frac{e^{w}-e^{-w}}{2w}\right)^{k}dw\overset{\text{def}}{=}J_1+J_2$$

where J_1 means the integral with $|t|\leqq 4$ and J_2 the rest. For the second we have trivially

$$(2.2.7)\qquad |J_1|\leqq\frac{1}{\pi A}\int\limits_4^{\infty}\left(e^{-\frac{\vartheta^2}{4}}\frac{e^{\frac{|\vartheta|}{4}}}{t}\right)^{k}dt\leqq\frac{4^{1-k}e^{\frac{k}{16}}}{\pi A(k-1)}<\frac{e^{-\frac{k\vartheta^2}{5}}}{4A\sqrt{k}}\,.$$

For J_1 we have

$$(2.2.8)\qquad |J_1|\leqq\frac{1}{\pi A}e^{-\frac{\vartheta^2 k}{4}}\int\limits_0^4\left(\frac{(e^{\frac{|\vartheta|}{4}}-e^{-\frac{|\vartheta|}{4}})^2+\sin^2 t}{\frac{\vartheta^2}{4}+4t^2}\right)^{\frac{k}{2}}dt\,.$$

Since, as easy to see,

$$(e^{\frac{|\vartheta|}{4}}-e^{-\frac{|\vartheta|}{4}})^2\leqq\left\{\frac{|\vartheta|}{2}\left(1+\frac{\vartheta^2}{90}\right)\right\}^2<\frac{\vartheta^2}{4}e^{\frac{\vartheta^2}{45}}\,,$$

we get from (2.2.8)

$$(2.2.9)\qquad |J_1|\leqq\frac{e^{k\vartheta^2\left(\frac{1}{90}-\frac{1}{4}\right)}}{\pi A}\int\limits_0^4\left(\frac{1+\frac{16}{\vartheta^2}e^{-\frac{\vartheta^2}{45}}\sin^2 t}{1+\frac{16}{\vartheta^2}t^2}\right)^{\frac{k}{2}}dt\,.$$

Since for $0\leqq t\leqq 4$ we have

$$(0\leqq)\sin t\leqq t-\frac{t^3}{30},\qquad\text{i.e.}\quad \sin^2 t\leqq t^2-\frac{t^4}{30},$$

further for $-1\leqq\vartheta\leqq+1$

$$\left(\frac{1}{45}-\frac{1}{30.16}\right)\vartheta^2-\frac{\vartheta^4}{30.45.16}\geqq\frac{1}{30.45}\cdot 16\,\vartheta^2\geqq\frac{\vartheta^2\cdot t^2}{30.45}$$

i. e.

$$\left\{\left(\frac{1}{45}-\frac{1}{30.16}\right)\vartheta^2-\frac{\vartheta^4}{30.45.16}\right\}t^2-\frac{\vartheta^2 t^4}{30.45}\geqq 0$$

we obtain

$$\sin^2 t \leq \left\{\left(1 - \frac{\vartheta^2}{30.16}\right) t^2 - \frac{t^4}{30}\right\} \left(1 + \frac{\vartheta^2}{45}\right) < e^{\frac{\vartheta^2}{45}} \left\{\left(1 - \frac{\vartheta^2}{30.16}\right) t^2 - \frac{t^4}{30}\right\}$$

i. e.

$$1 + \frac{16}{\vartheta^2} e^{-\frac{\vartheta^2}{45}} \sin^2 t \leq \left(1 + \frac{16}{\vartheta^2} t^2\right) \left(1 - \frac{t^2}{30}\right).$$

Putting it into (2.2.9) we obtain

$$|J_1| \leq \frac{e^{-\frac{k\vartheta^2}{5}}}{\pi A} \int_0^4 \left(1 - \frac{t^2}{30}\right)^{\frac{k}{2}} dt < \frac{e^{-\frac{k\vartheta^2}{5}}}{\pi A} \int_0^\infty e^{-\frac{kt^2}{60}} dt = \sqrt{\frac{15}{\pi k}} \frac{1}{A} e^{-\frac{k\vartheta^2}{5}}.$$

This and (2.2.7) prove the lemma.

3. Let b be real and

$$J(\chi, b) \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \int_{(2)} K_1(w)^k \frac{L'}{L} (w + 1 + ib, \chi)\, dw\,. \tag{2.3.1}$$

Then we have the simple

Lemma II. *We have independently of b*

$$S \stackrel{\text{def}}{=} \sum_\chi |J(\chi, b)|^2 < 112\, a_1^2\,.$$

We have namely from (2.1.7) and (2.2.1)

$$J(\chi, b) = \sum_{e^{kA} \leq n \leq e^{3kA}} \frac{\Lambda(n)\, R(n)}{n^{1+ib}}\, \chi(n)\,. \tag{2.3.2}$$

Writing

$$\Lambda(n)\, R(n)\, n^{-1-ib} = e_n$$

shortly, we get

$$S = \sum_\chi \sum_{n_1, n_2} e_{n_1} \bar{e}_{n_2} \chi(n_1)\, \overline{\chi(n_2)}\,,$$

where n_1 and n_2 run independently over $[e^{kA}, e^{3kA}]$. Changing the order of summations we obtain

$$\begin{aligned} S = \varphi(\Delta) \sum_{(l,\Delta)=1} \Big| \sum_{\substack{e^{kA} \leq n \leq e^{3kA} \\ n \equiv l \bmod \Delta}} e_n \Big|^2 \leq \varphi(\Delta) \sum_{(l,\Delta)=1} \Big(\sum_{\substack{e^{kA} \leq n \leq e^{3kA} \\ n \equiv l \bmod \Delta}} \frac{\Lambda(n)}{n} |R(n)| \Big)^2 \leq \\ \leq \frac{7\,\varphi(\Delta)}{A^2 k} \sum_{(l,\Delta)=1} \Big(\sum_{\substack{e^{kA} \leq n \leq e^{3kA} \\ n \equiv l \bmod \Delta}} \Lambda(n)\, g_0(n) \Big)^2 = \frac{7\,\varphi(\Delta)}{A^2 k} \sum_{(l,\Delta)=1} \Big(\int_{e^{kA}}^{e^{3kA}} g_0(x)\, d\, S_l(x) \Big)^2, \end{aligned} \tag{2.3.3}$$

using (2.2.2) and (1.2.1). The last integral is

$$(2.3.4) \qquad = [g_0(x)\, S_l(x)]_{e^{kA}}^{e^{3kA}} - \int_{e^{kA}}^{e^{3Ak}} S_l(x)\, |g_0'(x)\, dx\,.$$

One sees at once from (2.1.8) that the (non-negative) $g_0(x)$ assumes for $x \geqq 0$ its only maximum at

$$x_0 = e^{kA\left(2-\frac{5A}{2}\right)} < e^{kA},$$

i. e. the expression in (2.3.4)

$$(2.3.5) \qquad = [g_0(x)\, S_l(x)]_{e^{kA}}^{e^{3kA}} + \int_{e^{kA}}^{e^{3kA}} S_l(x)\, |g_0'(x)|\, dx\,.$$

Since in the range of integration we have, using (2.1.5), (2.1.4) and (1.2.4b) the inequality

$$\log x \geqq kA \geqq \omega N A > \omega\, 8\, a_2 \log \varDelta \geqq 3 \log \varDelta,$$

BRUN's estimation (1.2.1) is applicable and hence the expression in (2.3.5) is

$$\leqq [g_0(x)\, S_l(x)]_{e^{kA}}^{e^{3kA}} + \frac{a_1}{\varphi(\varDelta)} \int_{e^{kA}}^{e^{3kA}} x\, |g_0'(x)|\, dx = [g_0(x)\, S_l(x)]_{e^{kA}}^{e^{3kA}} - \frac{a_1}{\varphi(\varDelta)} \int_{e^{kA}}^{e^{3kA}} x g_0'(x)\, dx =$$

$$= \left[g_0(x) \left(S_l(x) - \frac{a_1 x}{\varphi(\varDelta)}\right)\right]_{e^{kA}}^{e^{3kA}} + \frac{a_1}{\varphi(\varDelta)} \int_{e^{kA}}^{e^{3kA}} g_0(x)\, dx < g_0(e^{kA}) \frac{a_1}{\varphi(\varDelta)} e^{kA} +$$

$$+ 2 \frac{a_1 kA}{\varphi(\varDelta)} \int_0^\infty e^{-\frac{k}{5}t^2}\, dt < \frac{a_1}{\varphi(\varDelta)} \left(1 + A \sqrt{5 \pi k}\right) < \frac{4 a_1}{\varphi(\varDelta)} A \sqrt{k}.$$

Putting it into (2.3.3) lemma II follows.

4. It follows from lemma II that with the β in (2.1. 1) the inequality

$$(2.4.1) \qquad |J(\chi, b)| \leqq 11\, a_1 \varDelta^{-\frac{\beta}{2}}$$

holds for all χ's mod $\varDelta$, with the exception of $\varDelta^\beta$ „bad'' characters at most (which may depend upon b.). Putting, if necessary, the principal character, and the (perhaps existing) „exceptional'' character among the „bad'' ones, their number is

$$(2.4.2) \qquad \varDelta^\beta + 2$$

at most. Since as well-known

$$\left| \frac{L'}{L}\left(-\frac{1}{2} + it, \chi\right) \right| \leqq b_1 \log(\varDelta(1 + |t|))$$

and from the definition of $K_1(w)$ in the left half-plane the inequality

$$(2.4.3)\qquad |K_1(w)| = e^{\frac{A\sigma}{2}} \left| \frac{e^{\frac{5}{2}Aw} - e^{\frac{A}{2}w}}{2wA} \right| \leqq \frac{e^{\frac{A\sigma}{2}}}{A\sqrt{\sigma^2+t^2}}$$

follows, we have

$$(2.4.4)\qquad \left| \frac{1}{2\pi i} \int_{(-\frac{3}{2})} K_1(w)^k \frac{L'}{L}(w+1+ib,\chi)\,dw \right| < \frac{b_1}{2\pi} \int_{-\infty}^{\infty} \log\left(\Delta(1+|t|)\right) \cdot \frac{e^{-\frac{3}{4}kA}}{\left(\frac{9}{4}+(t+b)^2\right)^{\frac{k}{2}}}\,dt\,.$$

Owing to (2.1.5), (2.1.4), (1.2.4b), for

$$-\Delta \leqq b \leqq \Delta$$

the right-side expression in (2.4.4) is

$$< b_2 \frac{\log \Delta}{\Delta^{\frac{9}{4}}};$$

hence a routine application of Cauchy's integral theorem to the integral defining $J(\chi, b)$ leads from (2.4.1) to the

Lemma III. *For $\Delta > d_2$ for each „b-good" characters* mod Δ *(i. e. with exception of $(\Delta^\beta + 2)$ characters at most, depending perhaps upon* b*) we have*

$$(2.4.5)\qquad \left|\sum_\varrho K_1(\varrho - 1 - ib)^k\right| < 12\, a_1 \Delta^{-\frac{\beta}{2}},$$

where ϱ runs over the zeros of the respective L(w, χ) *in the strip* $0 \leqq \sigma < 1$.

5. We define

$$(2.5.1)\qquad \alpha_j = \frac{2j}{\omega^2 A}, \qquad (j = 0, \pm 1, \ldots, \pm \Delta)$$

and the squares

$$(2.5.2)\qquad D_j: \qquad 1 - \frac{2}{\omega^2 A} \leqq \sigma \leqq 1, \qquad |t - \alpha_j| \leqq \frac{1}{\omega^2 A}.$$

Then we assert the crucial

Lemma IV. *Fixing any of the α_j's in* (2.5.1) *none of the* L(w, χ)*'s belonging to an „α_j-good" character can vanish in the square D_j of* (2.5.2), *if $\Delta > d_3$.*

Suppose this would be false for $j = j_0$, and for an $L(w, \chi')$ belonging to an „α_{j_0}-good" character χ', ϱ' being a zero of $L(w, \chi')$ in D_{j_0}. In order to derive a contradiction out of this assumption we start from the inequality

(2.4.5) with $\chi = \chi'$ and $b = \sigma_{j_0}$. We consider first the contribution of the ϱ's with

$$\operatorname{Im} \varrho \geqq \frac{2 j_0}{\omega^2 A} + \frac{2}{A} . \tag{2.5.3}$$

We cover this part of the critical strip by the squares

$$P_{\mu\nu}: \quad \begin{array}{ll} 1 - \dfrac{\mu+1}{A} \leqq \sigma < 1 - \dfrac{\mu}{A} & \mu = 0, 1, \ldots, A-1, \\ \dfrac{2 j_0}{\omega^2 A} + \dfrac{\nu}{A} \leqq t < \dfrac{2 j_0}{\omega^2 A} + \dfrac{\nu+1}{A}. & \nu = 2, 3, \ldots \end{array} \tag{2.5.4}$$

The absolute value of each term belonging to $P_{\mu\nu}$ cannot exceed owing to (2.4.3) the quantity

$$\nu^{-k} e^{-\frac{k\mu}{2}}$$

and the number of terms owing to (1.2.2) (applying it a bit roughly with

$$\delta = \frac{\mu+1}{A}, \qquad t_1 = \frac{2 j_0}{\omega^2 A} + \frac{\nu + \frac{1}{2}}{A}$$

cannot exceed

$$1 + a_2 \frac{\mu+1}{A} \left\{ \log \Delta + \log \left(1 + \frac{2 j_0}{\omega^2 A} + \frac{\nu + \frac{1}{2}}{A} \right) \right\}.$$

Hence this contribution is for $\Delta > d_4$

$$\leqq \sum_{\mu=0}^{A-1} \sum_{\nu=2}^{\infty} \nu^{-k} e^{-\frac{k\mu}{2}} \left(1 + a_2 \frac{\mu+1}{A} \right) \left\{ \log \Delta + \log \left(1 + \frac{2 j_0}{\omega^2 A} + \frac{\nu + \frac{1}{2}}{A} \right) \right\} \leqq$$

$$\leqq 2^{1-k} + a_2 \frac{\log \Delta}{A} \cdot 2^{1-k} + \frac{a_2 \sqrt{2}}{A} \sum_{\nu=2}^{\infty} \nu^{-k} \log (1 + \Delta + \nu) <$$

$$< 2 \left(1 + 2 a_2 \frac{\log \Delta}{A} \right) 2^{-k} .$$

The same holds for the contribution of the ϱ 's with

$$\operatorname{Im} \varrho \leqq \frac{2 j_0}{\omega^2 A} - \frac{2}{A} ;$$

hence the total contribution of the ϱ 's with

$$\left| \operatorname{Im} \varrho - \frac{2 j_0}{\omega^2 A} \right| \geqq \frac{2}{A} \tag{2.5.5}$$

does not exceed

$$4\left(1+2a_2\frac{\log \Delta}{A}\right)2^{-k}. \tag{2.5.6}$$

6. Next we consider the contribution of the zeros with

$$0 \leqq \sigma \leqq 1-\frac{4}{A}, \qquad |t-\alpha_{j0}| \leqq \frac{2}{A}. \tag{2.6.1}$$

We cover this part of the critical strip by the parallelogramms

$$Q_\mu: \quad 1-\frac{\mu+1}{A} \leqq \sigma < 1-\frac{\mu}{A}, \qquad |t-\alpha_{j0}| \leqq \frac{2}{A}. \tag{2.6.2}$$

$$(\mu = 4, 5, \ldots, A-1)$$

The absolute value of each term belonging to Q_μ cannot exceed owing to (2.4.3) the quantity

$$\mu^{-k} e^{-\frac{k\mu}{2}}.$$

and the number of ter ms owing to (1.2.2) $\left(\text{applying it with } \delta = \frac{\mu+1}{A},\ t_1 = \alpha_{j0}\right.$ cannot exceed

$$1 + a_2\frac{\mu+1}{A}\left\{\log\Delta + \log\left(1+\frac{2j_0}{\omega^2 A}\right)\right\} < 1 + 2a_2\frac{\mu+1}{A}\log\Delta.$$

Hence the contribution of the Q_μ-parallelogramms is at most

$$\sum_{\mu=4}^{\infty} \mu^{-k} e^{-\frac{k\mu}{2}}\left(1+2a_2\frac{\mu+1}{A}\log\Delta\right) < \left(1+3a_2\frac{\log\Delta}{A}\right)4^{-k}. \tag{2.6.3}$$

From (2.4.5), (2.5.6) and (2.6.3) we get for $\Delta > d_5$

$$\left|\Sigma' K_1(\varrho-1-ib)^k\right| < 12a_1\Delta^{-\frac{\beta}{2}} + 5\left(1+2a_2\frac{\log\Delta}{A}\right)2^{-k}, \tag{2.6.4}$$

where the summation refers to the ϱ's in

$$1-\frac{4}{A} \leqq \sigma \leqq 1, \qquad |t-\alpha_{j0}| \leqq \frac{2}{A}. \tag{2.6.5}$$

With ϱ 's in D_{j0} we can write (2.6.4) in the form

$$|K_1(\varrho'-1-i\alpha_{j0})|^k \left|\sum{}' \left(\frac{K_1(\varrho-1-i\alpha_{j0})}{K_1(\varrho'-1-i\alpha_{j0})}\right)^k\right| \leqq$$

$$\leqq 12a_1\Delta^{-\frac{\beta}{2}} + 5\left(1+2a_2\frac{\log\Delta}{A}\right)2^{-k}. \tag{2.6.6}$$

Since in the domain

$$-\frac{2}{\omega^2} \leqq \sigma \leqq 0, \qquad |t| \leqq \frac{1}{\omega^2}$$

we have the estimation

$$|K(w)|^k \geqq e^{-\frac{4k}{\omega^2}} \left(1 - \frac{1}{3!}\left(\frac{3}{\omega^2}\right)^2 - \frac{1}{5!}\left(\frac{3}{\omega^2}\right)^4 - \dots\right)^k > e^{-\frac{4k}{\omega^2}}\left(1 - \frac{2}{\omega^4}\right)^k > e^{-\frac{5k}{\omega^2}},$$

i. e. for

$$-\frac{2}{\omega^2 A} \leqq \sigma \leqq 0, \qquad |t| \leqq \frac{1}{\omega^2 A}$$

$$|K_1(w)|^k \geqq e^{-\frac{4k}{\omega^2}}$$

we have

$$|K_1(\varrho' - 1 - i\,\alpha_{j0})|^k \geqq e^{-\frac{5k}{\omega^2}}$$

and hence from (2.6.6)

$$\text{(2.6.7)} \qquad \left|\sum{}' \left(\frac{K_1(\varrho - 1 - i\,\alpha_{j0})}{K_1(\varrho' - 1 - i\,\alpha_{j0})}\right)^k\right| \leqq 12\,a_1\,\Delta^{-\frac{\beta}{2}}\,e^{\frac{5k}{\omega^2}} + \left(\frac{1}{2} + a_2 \frac{\log \Delta}{A}\right)\left(\frac{2}{3}\right)^k$$

with the summation indicated in (2.6.5).

7. We want to apply to the remaining sum the theorem (1.3.2)—(1.3.3)—(1.3.4) with

$$\text{(2.7.1)} \qquad m = [\omega N]$$

and

$$z_j = \frac{K_1(\varrho - 1 - i\,\alpha_{j0})}{K_1(\varrho' - 1 - i\,\alpha_{j0})}\,.$$

Owing to (2.1.4) and (2.6.5) we may choose

$$\text{(2.7.2)} \qquad N_1 = N\,;$$

then the interval $m + 1 \leqq x \leqq m + N_1$ is obviously contained in $\omega N \leqq x \leqq$ $\leqq (\omega + 1)\,N$ and thus we may choose as k the ν_1 in (1.3.3). Then (2.6.7) assumes the form

$$\left(\frac{1}{8\,e(\omega + 1)}\right)^N \leqq 12\,a_1\,\Delta^{-\frac{\beta}{2}}\,e^{\frac{5\nu_1}{\omega^2}} + \left(\frac{1}{2} + a_2\frac{\log \Delta}{A}\right)\left(\frac{2}{3}\right)^{\nu_1} \leqq$$

$$\leqq 12\,a_1\,\Delta^{-\frac{\beta}{2}}\,e^{\frac{6N}{\omega}} + \left(\frac{1}{2} + a_2\frac{\log \Delta}{A}\right)\left(\frac{2}{3}\right)^{\omega N}$$

or

$$1 \leqq 12\,a_1\,\Delta^{-\frac{\beta}{2}}\left(8\,e^{1+\frac{6}{\omega}}(\omega + 1)\right)^N + \left(\frac{1}{2} + a_2\frac{\log \Delta}{A}\right)\left\{8\,e(\omega + 1)\left(\frac{2}{3}\right)^{\omega}\right\}^N$$

or owing to (1.2.4c)

$$(2.7.3) \qquad 1 \leqq 12\, a_1 \Delta^{-\frac{\beta}{2}} (9\, e\, \omega)^N + \left(\frac{1}{2} + a_2 \frac{\log \Delta}{A}\right) \left(\frac{2}{3}\right)^{N\omega^{3/4}}.$$

Using (2.1.4) it would follow

$$1 \leqq (9\, e)^2\, 12\, a_1\, \omega^2 \Delta^{-\frac{\beta}{2} + \frac{8a_2}{A} \log (9 e \omega)} + \left(\frac{1}{2} + a_2 \frac{\log \Delta}{A}\right) \left(\frac{2}{3}\right)^{\frac{8a_2}{A} \omega^{3/4} \log \Delta}$$

further from (2.1.2) using the abbreviation $\beta \log \Delta = \xi$

$$1 \leqq (9\, e)^2\, 12\, a_1\, \omega^2\, e^{\xi\left(-\frac{1}{2} + \frac{9a_2 \log 9e\omega}{\sqrt{\omega}}\right)} + \left(\frac{1}{2} + \frac{2\, a_2}{\sqrt{\omega}} \xi\right) e^{-8 \log \frac{3}{2} a_2 \omega^{\frac{1}{4}} \xi} <$$

$$(2.7.4) \qquad < (36\, \omega)^2\, a_1\, e^{-\frac{\xi}{4}} + \left(\frac{1}{2} + \frac{2\, a_2}{\sqrt{\omega}} \xi\right) e^{-\frac{8}{3} a_2 \omega^{1/4} \xi},$$

taking in account (1.2.4d). Owing of the definition of ξ and (2.1.1) and (1.2.4e) we have

$$(2.7.5) \qquad \xi \geqq \frac{a_3}{2} \omega^{\frac{5}{2}} (\geqq 4)$$

and hence using this, (1.2.4f) and (1.2.4g)

$$\frac{1}{2} + \frac{2\, a_2}{\sqrt{\omega}} \xi < 1 + \frac{\xi}{3} < \frac{\xi^2}{2!} \leqq \frac{(\xi\, a_2\, \omega^{1/4})^2}{2!} < e^{a_2 \omega^{\frac{1}{4}} \xi};$$

putting it into (2.7.4)

$$1 < (36\, \omega)^2\, a_1\, e^{-\frac{\xi}{4}} + e^{-a_2 \omega^{\frac{1}{4}} \xi} < (36\, \omega)^2\, a_1\, e^{-\frac{a_3}{8} \omega^{\frac{5}{2}}} + e^{-\frac{a_2 a_3}{2} \omega^{\frac{5}{2}}}$$

which contradicts to (1.2.4h). This proves lemma IV.

8. Consider now an arbitrary of our D_j 's in (2.5.2). Owing to lemma IV, (2.4.2) and (1.2.2) the total-number of zeros of all L-functions mod Δ in D_j is at most

$$(\Delta^\beta + 2) \left\{1 + a_2 \frac{2}{\omega^2 A} \log\left(\Delta\left(1 + \frac{2\,|j|}{\omega^2\, A}\right)\right)\right\} < (e^\xi + 2) \left(1 + \frac{5\, a_2}{\omega^{5/2}} \xi\right) <$$

$$(2.8.1) \qquad < (e^\xi + 2) \left(1 + \frac{\xi}{\omega^2}\right) < (e^\xi + 2)\, e^{\frac{\xi}{\omega^2}} < 2\, e^{\left(1 + \frac{1}{\omega^2}\right)\xi} = 2\, \Delta^{\beta\left(1 + \frac{1}{\omega^2}\right)},$$

using also (2.1.2), (2.5.1) and (1.2.4f). Since from (2.1.2)

$$\frac{2}{\omega^2\, A} \geqq \frac{2}{\omega^{5/2}} \beta,$$

the estimation (2.8.1) holds a fortiori for the number of zeros of all L-functions mod Δ in each square

$$(2.8.2)\qquad 1-\frac{2}{\omega^{\frac{5}{2}}}\beta \leqq \sigma \leqq 1\,, \quad \left|t-\frac{2j}{\omega^2 \Delta}\right| \leqq \frac{\beta}{\omega^{5/2}} \quad (j=0,\pm 1,\ldots,\pm \Delta)\,.$$

This is proved under the restriction (2.1.1); if

$$0 \leqq \beta \leqq \frac{a_3\, \omega^{5/2}}{2 \log \Delta}\,,$$

then the square (2.8.2) is obviously contained in the parallelogramm (1.2.3) i. e. the estimation (2.8.1) holds trivially. Since the square in our theorem can be covered for $\Delta > d_6$ by two squares at most, both of the form (2.8.2) with

$$\beta = \frac{\omega^{\frac{5}{2}}}{2}(1-\gamma)\,,$$

the theorem (1.3.5)—(1.3.6)—(1.3.7) is proved.

9. Finally we shall deduce from (1.3.5—1.3.6)—1.3.7) the theorem of LINNIK in the form (1.1.4)—(1.1.5)—1.1.6). We may suppose

$$a_3 \leqq \lambda \leqq \frac{1}{2\,\omega^{5/2}} \log \Delta$$

in (1.1.4) owing to (1.2.3). We can cover the parallelogramm (1.1.4) by $2\left(1+\left[\frac{e^{\lambda}}{\lambda}\right]\right)$ squares of the form (1.3.6) with

$$\gamma = 1-\frac{\lambda}{\log \Delta}$$

and $|t_2| < \Delta^{3/4}$. Hence by the theorem (1.3.5)—(1.3.6)—(1.3.7) the total-number of zeros in (1.1.4) cannot exceed

$$2\left(1+\frac{e^{\lambda}}{\lambda}\right) 4\, e^{\omega^{\frac{5}{2}}\lambda} < \left(1+\frac{1}{e}\right) 4\, \frac{e^{\lambda}}{\lambda}\, e^{\omega^{\frac{5}{2}}\lambda} < \frac{6}{a_3}\, e^{(\omega^{\frac{5}{2}}+1)\lambda}\,.$$

Q. e. d.

(Received October 25, 1960.)

О ТЕОРЕМЕ ПЛОТНОСТИ Ю. В. ЛИННИКА

P. TURÁN

Резюме

В 1944-ом году Ю. В. Линник доказал, что существует такое положительное постоянное число b, что если $(k, l) = 1$, тогда существует простое число $p \equiv l \pmod{k}$ такое, что $p < k^b$; Это доказательство было упрощенно К. А. Родосскти. Трудность в доказательстве этой глубокой теоремы состоит в доказательстве двух других теорем, первой из которых является теорема плотности указанная в заголовке. В настоящей статье дается в самом себе полное, отличное от первоначального, доказательство этой теоремы на основании теоремы (1.3.2) — (1.3.3) — (1.3.4) автора, являющейся также источником многих других результатов теории чисел, с подробностями сделающими возможным эффективное определение встречающихся там существенных постоянных (1. (1.2.4)). Недавно S. Knapowski удалось на основании этого доказательства найти также доказательство второй упомянутой теоремы.