ON THE MINIMAL NUMBER OF VERTICES REPRESENTING THE EDGES OF A GRAPH

by

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Introduction

In this paper we will only consider non-directed graphs which do not contain loops and where two vertices are connected by at most one edge[1] (see [1] and [7]). We permit isolated points and we do not exclude the empty graph i. e. the graph without vertices and edges. $\pi(G)$ and $\nu(G)$ denotes the number of vertices respectively of edges of the graph G. $G' \subseteq G$ denotes that G' is a subgraph of G. (If $G' \subseteq G$ and $G' \neq G$, we write $G' \subset G$.)

We shall say that the vertices $P_1, \ldots, P_k (k \geqq 1)$[2] *represent* the edges $e_1, \ldots, e_j (j \geqq 1)$ of G if every edge $e_i (1 \leqq i \leqq j)$ contains at least one the points $P_h (1 \leqq h \leqq k)$. If the vertices $P_1, \ldots, P_k$ represent all edges of G we call $R = \{P_1, \ldots, P_k\}$ a *representing system of* G and say that R *represents* G. We denote by $\mu(G)$ the minimal number of vertices representing every edge of G (i. e. we can find $\mu(G)$ vertices in such a way that every edge of G containts at least one of these vertices, but there do not exist $\mu(G)$—1 vertices with this property). If G has no edge, then by definition $\mu(G) = 0$. The chief object of this paper will be to give various estimations from above of $\mu(G)$.

In § 1 we shall obtain estimates for $\mu(G)$ in terms of $\pi(G)$, $\nu(G)$ and other characteristic data of G. One of our results (Theorem (1.7)) which will be an easy consequence of a result of TURÁN states that

$$\mu(G) \leqq \frac{2}{\dfrac{1}{\frac{1}{2}\pi(G)} + \dfrac{1}{\nu(G)}}, \qquad \text{if} \quad \nu(G) > 0\,.$$

In § 2, 3 and 4 we shall estimate $\mu(G)$ in terms of $\mu(G')$ where G' runs through certain subgraphs of G. Our principal results are:

If $\mu(G') \leqq p$ *for all* $G' \subseteq G$ *with* $\pi(G') \leqq 2p + 2$, *then* $\mu(G) \leqq p$. (Theorem (3.5)).

[1] Every edge "contains" exactly two vertices, which are "connected" by it.

[2] Numbers which are denoted by letters are always assumed to be non negative integers.

Let $h \geq 2$, $p > p_0(h)$. Assume that $\pi(G) \geq 2p-h+3$ and that G has no isolated vertices, further assume that for every $G' \subseteq G$ with $\pi(G') \leq p+h$ we have $\mu(G') \leq p$. Then $\mu(G) \leq 2p-h$ (Theorem (2.2)).

In general the above results are best possible.

In § 5 we generalise our problems to ,,multidimensional graphs''. Instead of graphs we consider sets of k-tuples ($k \geq 2$) and study the minimal number of elements which represent each of our given k-tuples.

§ 1.

(1.1) First of all we need some definitions and notations.

G will always denote a graph, and if in the following it is not explicitely indicated to which graph some symbols and notations belong, we always assume that they refer to the graph denoted by G.

$\alpha(M)$ will always denote the number of elements of the finite set M.

We shall denote by PQ the edge connecting the vertices P and Q. The graph which consists of the vertices P and Q and the edge PQ will also be called an *edge*. The graph which consists of the vertices P_1, P_2, P_3 and the edges P_1P_2, P_2P_3, P_3P_1 will be called a *triangle* and will be denoted by $P_1P_2P_3$.

If P is a vertex of G, then we call the number of edges of G which are incident to P the *valency of* P (in G).

If any two vertices of G are connected by an edge G will be called *complete*. The graph consisting of one point will be called complete too.

If G is complete and $\pi(G) = n$ we shall call G a *complete n-graph*.

Assume that G has at least two vertices. The *complementary* graph $\overline{G}$ of G is defined as follows: $\overline{G}$ has the same vertices as G and two vertices of $\overline{G}$ are connected if and only if they are not connected in G.

For the definition of *path* and *circuit* see [7] (path = Weg, circuit = = Kreis).

A graph G — having at least two vertices — is said to be *connected* if any two of its vertices are on a path of G. The graph having one vertex is called connected.

The *components* of (the non empty) G are its maximal connected subgraphs.

Denote by S the set of vertices of G. Let $M \subseteq S$. We denote by $[M]$ the subgraph of G whose vertices are the elements of M and whose edges are all the edges of G which have both vertices in M.

If $M \subseteq S$ and $N \subseteq S$ then we call the edges one vertex of which is in M and the other in N the *MN-edges*.

$[M, N]$ denotes the subgraph of G whose vertices are the elements of $M \cup N$ and whose edges are the MN-edges of G.

G is *even* if there is an M and N for which $M \cup N = S$, $M \cap N = \emptyset$ and $[M, N] = G$.

Let $P \in S$. $G-P$ denotes the graph which we obtain by omitting from G the vertex P and all the edges incident to P.

The vertices $P_1, \ldots, P_j (j > 1)$ of G are called *independent* (in G) if no two of them are connected by an edge (in G). One vertex is always called independent. $\overline{\mu}(G)$ denotes the maximal number of the independent vertices of G. If G is empty, then by definition $\overline{\mu}(G) = 0$.

The edges $e_1, \ldots, e_j$ $(j>1)$ are called *independent* if they have no common vertex. One edge is always called independent. The maximum number of the independent edges of G is denoted by $\varepsilon(G)$. If G has no edges we have by definition $\varepsilon(G) = 0$.

We shall call G *k-fold connected* $(k \geq 1)$ if in case $k = 1$ G is connected and for $k>1$ if $\pi(G) \geqq k + 1$ and G remains connected after the omission of any $k-1$ of its vertices (and all the edges incident to them).

(1.2) It follows from our definitions that if $G_1, \ldots, G_j$ $(j \geqq 1)$ are the components of G then if $\varphi = \pi, \nu, \mu, \bar{\mu}$ or ε

$$\varphi(G) = \sum_{i=1}^{j} \varphi(G_i)\,.$$

(1.3) It is easy to see that (see [6], p. 134.)

$$\mu(G) + \bar{\mu}(G) = \pi(G)\,. \tag{1}$$

If G is non empty then $\bar{\mu}(G) \geqq 1$, equality here holds if and only if G is complete. From this remark and (1) we obtain

(1.4) *If G is non empty then $\mu(G) \leq \pi(G) - 1$. Equality holds if and only if G is complete.*

If we make special assumptions about G we can improve the above estimation. Thus the following trivial inequalities hold:

(1.5) *If G is even* $\mu(G) \leqq \frac{1}{2}\pi(G)$.

If we assume that G does not contain a triangle (or a complete k-graph $(k>3)$) then the problem of giving a sharp upper bound for $\mu(G)$ in terms of $\pi(G)$ is difficult and will not be discussed in this paper. Because of (1.3) (1) this is really RAMSAY's problem ([8], [5]).

(1.6) *$\mu(G) \leqq \nu(G)$ is trivial. Equality holds if and only if no two edges of G have a common vertex.*

We can obtain non trivial upper estimates of $\mu(G)$ using both $\pi(G)$ and $\nu(G)$.

Theorem (1.7). *Assume that G has edges.* Then

$$\mu(G) \leq \frac{2\,\nu(G)\,\pi(G)}{2\,\nu(G) + \pi(G)}$$

or in other words: *$\mu(G)$ is less than or equal to the harmonic mean between $\frac{1}{2}\pi(G)$ and $\nu(G)$. Equality holds if and only if G is a complete graph, or if each component of G is a complete graph each of which has the same number of vertices.*

Proof. Our theorem is an easy consequence of a result of TURÁN. TURÁN proved ([9], p. 26.) that if $\pi(G) = n$ and G does not contain a complete $(j+1)$-graph but contain a complete j-graph, then

$$\nu(G) \leqq \frac{j-1}{2j}(n^2 - r^2) + \binom{r}{2} \tag{1}$$

where $n = jt + r$ $(0 \leqq r < j)$. If $r = 0$ equality occurs if and only if $\overline{G}$ ($\overline{G}$ is the complement of G) has j components and each of them are complete t-graphs[3].

Applying this theorem we obtain

$$\nu(\overline{G}) \geqq \binom{n}{2} - \left\{\frac{j-1}{2j}(n^2 - r^2) + \binom{r}{2}\right\} = \frac{(n-r)(n-j+r)}{2j}, \tag{2}$$

where $\pi(\overline{G}) = n$, $\overline{\mu}(\overline{G}) = j$ and $n = jt + r$ $(0 \leqq r < j)$. Further if $r = 0$ equality occurs if and only if all components of $\overline{G}$ are complete t-graphs.

Let $\mu(\overline{G}) = k$. By (1.3) $j = n - k$, thus from (2)

$$\nu(\overline{G}) \geqq \frac{(n-r)(k+r)}{2(n-k)}. \tag{3}$$

From $0 \leqq r < n - k$ we have $k < n - r \leqq n$. Thus

$$(n-r)(k+r) \geqq nk, \tag{4}$$

equality only if $r = 0$. From (3) and (4) we obtain, assuming that $\nu(\overline{G}) = m > 0$

$$k \leqq \frac{2}{\frac{2}{n} + \frac{1}{m}}.$$

Equality can hold only if we have equality both in (4) and in (2). This completes our proof since every graph G with $\pi(G) \geqq 2$ is the complementary graph of a certain graph.

From (1.7) we easily obtain

Theorem (1.8)

$$\mu(G) \leqq \frac{\pi(G) + \nu(G)}{3}. \tag{1}$$

Equality holds if and only if G is empty or if the components of G are edges and triangles.

Proof. If G is empty the theorem is trivial, henceforth we shall assume $\pi(G) > 0$. It follows from (1.2) that it will suffice to prove our theorem for

[3] TURÁN gave also in the case $r > 0$ the necessary and sufficient condition for equality in (1).

connected graphs and that equality can hold for G only if it holds for every component of G.

Henceforth we shall assume that G is connected. Put $\pi(G) = n$, $\nu(G) = m$.

For $n = 1$ (1) clearly holds with the sign $<$. Thus we can assume $m \geqq 1$. From (1.7) we have

$$\mu(G) \leqq \frac{2mn}{2m+n}, \tag{2}$$

equality holds if and only if G is complete. For positive m and n the inequality $2mn/(2m + n) \leqq (m + n)/3$ is equivalent to

$$0 \leqq (m - n)(2m - n). \tag{3}$$

Therefore if $m \geqq n$, (1) is implied by (2) and (3), further we can deduce that equality holds if and only if $m = n$ and G is a complete n-graph. But this is possible only if $n = 3$.

If $m < n$, then since G is connected, $m = n - 1$ and G is is a tree (see [7], p. 51.). Since every tree is even, we have by (1.5)

$$\mu(G) \leqq \frac{1}{2} n .$$

For $n \geqq 2$ we have $(m + n)/3 = (2n - 1)/3 \geqq 1/(2n)$, equality only for $n = 2$. This proves (1) for $m < n$ and shows that equality holds if and only if G consits of a single edge. This completes the proof of our theorem.

(1.9) Next we estimate $\mu(G)$ in terms of $\varepsilon(G)$.

Assume $\nu(G) \geqq 1$ and let $P_1 P_1', \ldots, P_s P_s' (s = \varepsilon(G) \geqq 1)$ be a maximal system of independent edges of G. Clearly the vertices $P_1, \ldots, P_s, P_1', \ldots, P_s'$ represent the edges of G. On the other hand we clearly need at least s vertices for the representation of the edges of G. Thus we obtain the following trivial inequality

$$\varepsilon(G) \leqq \mu(G) \leqq 2\varepsilon(G). \tag{1.10}$$

(1.10) trivially holds for $\nu(G) = 0$ too.

The following theorem which we will often use is due to KÖNIG ([7], p. 233.).

(1.11) (KÖNIG). *For even graphs* $\mu(G) = \varepsilon(G)$.

For the upper bound in (1.10) we have the following

Theorem (1.12). $\mu(G) = 2\varepsilon(G)$ *holds if and only if G is empty or each component G_i of G is complete and $\pi(G_i)$ is odd.*

Proof. The sufficiency of the above conditions is evident. To prove the necessity observe that because of (1.2) it will be sufficient to show that for a connected G satisfying $\pi(G) \geqq 2$, $\mu(G) = 2\varepsilon(G)$ holds only if G is complete and $\pi(G) = 2\varepsilon(G) + 1$. This immediately follows from (1.4) and from the following

Theorem (1.13). *Let G be k-fold connected ($k \geqq 1$). Assume $\pi(G) > 2\varepsilon(G) + 1$, then $k \leqq \varepsilon(G)$ and*

$$\mu(G) \leqq 2\,\varepsilon(G) - k\,.$$

The above bound for $\mu(G)$ is best possible.

Our proof of theorem (1.13) uses the theory of alternating paths. The proof can be deduced easily from the properties of alternating paths stated in § 4 of [4]. We do not give the details of the proof.

We remark that one can give a simple proof of (1.12) without using (1.13).

The following example shows that the bound $2\,\varepsilon(G) - k$ in theorem (1.13) is best possible: Let G_0 be a complete k-graph and G_i a complete $(2a_i + 1)$-graph ($k \geqq 1$, $a_i \geqq 0$, $i = 1, \ldots, l$, $l > k + 1$). The graphs G_0 and G_i have no common vertex. The vertices of G are the vertices of G_0 and those of the G_i ($i = 1, \ldots, l$), the edges of G are the edges of G_0, the edges of G_i ($i = 1, \ldots, l$), and every edge which connects a vertex of G_0 with a vertex of G_i ($1 \leqq i \leqq l$). We have

$$\varepsilon(G) = k + \sum_{i=1}^{l} a_i\,, \qquad \mu(G) = k + \sum_{i=1}^{l} 2\,a_i\,,$$

$$\pi(G) = k + \sum_{i=1}^{l} (2\,a_i + 1) = l - k + 2\,\varepsilon(G) > 2\,\varepsilon(G) + 1\,.$$

G is k-fold connected, $\mu(G) = 2\,\varepsilon(G) - k$. Observe that in our example $\pi(G)$ can be made arbitrarily large for given $\varepsilon(G)$.

Remark. If G satisfies $\pi(G) > 3\,\varepsilon(G) - 2$ $\big(\varepsilon(G) \geqq 1\big)$ and is connected then we can prove

$$\mu(G) \leqq 2\,\varepsilon(G) - d \tag{1}$$

where d is the minimum of the valency of the vertices of G. If G is k-fold connected and $\pi(G) > 1$, then clearly $d \geqq k$, thus (1) is a sharpening of (1.13). The proof of (1) is similar to that of (1.13) and will be suppressed.

Finally we obtain bounds for $\mu(G)$ in terms of $\varepsilon(G)$, $\nu(G)$ and $\pi(G)$.

Theorem (1.14)

$$\mu(G) \leqq \varepsilon(G) + \frac{\nu(G) - \varepsilon(G)}{2}\,, \tag{1}$$

$$\mu(G) \leqq \varepsilon(G) + \frac{\pi(G) - 2\,\varepsilon(G)}{2} + \frac{\nu(G) - \varepsilon(G)}{4}\,. \tag{2}$$

Remarks. These bounds are best possible. For (1) we see this by considering a graph whose components are edges and triangles, and it is not difficult to see that this is the only case of equality.

For (2) the situation is more complicated. The only connected graphs $\big($with $\nu(G) > 0\big)$ known to us for which there is equality in (2) are: 1.) an edge, 2.) a triangle, 3.) a complete 4-graph, 4.) two triangles connected by an edge. It is possible that there are no other cases. Clearly if all the components of G are the above ones then G satisfies (2) with the sign of equality.

Proof. We use induction for $\nu(G)$. (1) and (2) are trivial if $\nu(G) \leqq 1$. Let $m > 1$ and assume that (1) and (2) holds for every G^* satisfying $\nu(G^*) < m$. In what follows assume that G is an arbitrary graph for which $\nu(G) = m$. We are going to show that (1) and (2) holds for G too.

We clearly can assume that G has no isolated points. If G is not connected, let its components be $G_1, \ldots, G_j$ ($j \geqq 2$). Clearly $\nu(G_i) < m$ ($i = 1, \ldots, j$). Thus by our induction hypothesis and (1.2) it follows that G satisfies (1) and (2).

Henceforth we shall assume that G is connected.

Assume first that G has a vertex P of valency 1 and let PQ be the edge incident to P. There clearly exists another edge incident to Q say QQ' ($Q' \neq P$). Omit the edge QQ' from G, and denote the graph thus obtained by G'. Let R be a representing system of G' with $\alpha(R) = \mu(G')$. Clearly R contains P or Q, hence we can assume $Q \in R$. But then R is a representing system of G too, thus $\mu(G) = \mu(G')$. A simple argument further shows that $\varepsilon(G') = \varepsilon(G)$ (i. e. if a set of independent edges of G contains QQ', we can replace QQ' by QP and obtain a set of independent edges of G'). From this and from $\pi(G') = \pi(G)$, $\nu(G') = \nu(G) - 1$ and from the induction hypothesis we obtain (1) and (2).

Henceforth we are going to assume that the valency of every vertex of G is $\geqq 2$.

If $\pi(G) - 2\varepsilon(G) = 0$, then (2) clearly implies (1). Next we show that (2) implies (1) also if $\pi(G) - 2\varepsilon(G) = j > 0$. Let $P_i P_i'$ ($i = 1, \ldots, s$; $s = \varepsilon(G)$) be a maximal system of independent edges of G. Further put $N = \{P_1, \ldots, P_s, P_1', \ldots, P_s'\}$, $\bar{N} = S - N$ (S denotes the set of vertices of G), $[N] = G'$. By our assumptions

$$1 \leqq \varepsilon(G) \leqq \nu(G') < \nu(G) . \tag{3}$$

The vertices of $\bar{N}$ are independent (in G) and all of them have valency $\geqq 2$. Thus we have

$$\nu(G) \geqq \nu(G') + 2j$$

and hence

$$\frac{j}{2} \leqq \frac{\nu(G) - \nu(G')}{4} \leqq \frac{\nu(G) - \varepsilon(G)}{4} , \tag{4}$$

which shows that (2) implies (1).

Thus it will suffice to prove (2).

Assume for the time being that $\pi(G) - 2\,\varepsilon(G) = j > 0$ and let us use our above notations. Clearly if R is a representing system of G' then $R \cup \bar{N}$ represent all edges of G, thus $\mu(G) \leqq \mu(G') + j$. Further clearly $\varepsilon(G') = \varepsilon(G)$ and $\pi(G) = \pi(G') + j$. These equalities together with (3) and (4) imply (2) by the induction hypothesis.

Henceforth we can assume $\pi(G) = 2\varepsilon(G)$.

Assume first that G contains a path with the edges P_1P_2, P_2P_3, P_3P_4, where P_2 and P_3 have valency 2 in G. Let $G' = (G - P_2) - P_3$. If G contains the edge P_1P_4 put $G'' = G'$, if not G'' is obtained from G' by adding the edge P_1P_4 to it. It is easy to see that

(5) $\pi(G'') = \pi(G) - 2,\ \nu(G'') \leq \nu(G) - 2,\ \varepsilon(G'') = \varepsilon(G) - 1,\ \mu(G'') = \mu(G) - 1.$

(5) and our induction hypothesis implies (2).

Henceforth assume that G does not contain a path of the above type.

Let $P_i P_i'$ $(i = 1, \ldots, s;\ s = \varepsilon(G))$ be a maximal system of independent edges of G. By our assumptions the valency of both P_i and $P_i'(i = 1, \ldots, s)$ are greater than one and by our last assumption they can not both be two. Thus without loss of generality we can assume that the valency of P_i is $\geq 3\ (i = 1, \ldots, s)$. Assume that for some $i\ (1 \leq i \leq s)$ the sum of the valencies of P_i and P_i' is greater than 5. Put $G^* = (G - P_i) - P_i'$. Thus

(6) $\pi(G^*) = \pi(G) - 2,\ \nu(G^*) \leq \nu(G) - 5,\ \varepsilon(G^*) = \varepsilon(G) - 1,\ \mu(G^*) \geq \mu(G) - 2.$

(6) and our induction hypothesis proves (2).

Thus finally we can assume that the valencies of the vertices P_i are all 3 and the valencies of the vertices P_i' are all $2 (i = 1, \ldots, s)$. But then P_i' and $P_j'(i \neq j,\ 1 \leq i \leq s,\ 1 \leq j \leq s)$ can not be connected by an edge, since otherwise G would contain the path with the edges $P_i P_i'$, $P_i' P_j'$, $P_j' P_j$ where P_i' and P_j' having valency 2 in G, but this contradicts our assumptions.

Hence we see that the vertices $P_i (i = 1, \ldots, s)$ represent all edges of G, which clearly proves (2).

Thus the proof of Theorem (1.14) is complete.

§ 2.

(2.1) $\varepsilon(G) \leq p$ is equivalent to the statement that $\mu(G') \leq p$ for every $G' \subseteq G$ with $\nu(G') \leq p + 1$. Thus the trivial relation $\mu(G) \leq 2\varepsilon(G)$ can be restated in the following form:

Assume that for every $G' \subseteq G$ with $\nu(G') \leq p + 1$ we have $\mu(G') \leq p$. Then $\mu(G) \leq 2p$.

It is now a natural question to ask: what can be said about $\mu(G)$ if for every $G' \subseteq G$ with $\nu(G') \leq q$ $(q > p + 1)$ $\mu(G') \leq p$? Here we prove

Theorem (2.2). *Let $h \geq 2$. Then there exists a smallest integer $p_0(h)$ with the following properties :* If $p > p_0(h)$ *and G is a graph with $\pi(G) \geq 2p - h + 3$ which has no isolated points, and for every $G' \subseteq G$ with $\nu(G') \leq p + h$ we have $\mu(G') \leq p$, then*

(1) $$\mu(G) \leq 2p - h.$$

Before proving our theorem we make some remarks.

1.) $2p-h$ is best possible. To show this let G_1 be a complete $(2p-h)$-graph. The graph G_2 is defined as follows: Its vertices are the vertices of G_1, another vertex P, and the vertices of a set M (which may be empty, but which does not contain P and the vertices of G_1). The edges of G_2 are the edges of G_1 and every edge which connects P with a vertex of G_1 or M. It is easy to see that $\mu(G_2) = 2p-h$. Now we show that for every $G' \subseteq G_2$ (which does not contain an isolated vertex) satisfying $\nu(G') \leq p + h$ we have $\mu(G') \leq p$. To see this observe that if G' does not contain P we have $\pi(G') \leq$

$\leq 2p-h$ and therefore by theorem (1.8) $\mu(G') \leq p$. If G' contains P then the number of the not isolated vertices of $G'-P$ is not greater than $2p-h$, $\nu(G'-P) \leq p+h-1$. Thus from theorem (1.8) $\mu(G'-P) \leq p-1$ or $\mu(G') \leq p$ which completes the proof.

We remark that in our example $2\varepsilon(G_2)$ equals one of the values $2p-h$, $2p-h+1$, $2p-h+2$. This is not an accident, since if $2\varepsilon(G) \leq 2p-h$, then because $\mu(G) \leq 2\varepsilon(G)$ (1) trivially holds, equality only if $2\varepsilon(G) = 2p-h$. Further a simple modification of our proof of Theorem (2.2) shows that if $2\varepsilon(G) > 2p-h+2$ we can improve $\mu(G) \leq 2p-h$ to $\mu(G) < 2p-l$ where l tends to infinity with p but is of much lower order than p, we can give only very rough estimates for $l = l(p, h)$.

2.) In (2.3) we shall show that if p is not "sufficiently large" compared to h then (1) does not always hold. More precisely we shall show that if c is an arbitrary constant and $h > h_0(c)$ then $p_0(h) > ch$.

3.) If $h = 2$ our proof could be simplified considerably, and we can show $p_0(2) = 2$.

Proof. of (2.2). (I) According to a well known theorem of RAMSAY) (see [8] and [5]) to every k there exists a $\varphi(k)$ so that every G with $\pi(G) \geq \varphi(k)$ either contains a complete k-graph or G has k independent points (i. e. $\bar{\mu}(G) \geq k$). Clearly $\varphi(k) \geq k$.

We are going to show that

$$p_0(h) < h + \varphi(\varphi(2h+4)). \tag{2}$$

Clearly

$$h + \varphi(\varphi(2h+4)) \geq 3h+4. \tag{3}$$

Our proof will be indirect. We are going to show that the following conditions lead to a contradiction:

(4) G has no isolated point.

(5) $h \geq 2$.

(6) $p > h + \varphi(\varphi(2h+4))$.

(7) $\pi(G) \geq 2p-h+3$.

(8) If $G' \subseteq G$ and $\nu(G') \leq p+h$ then $\mu(G') \leq p$.

(9) $\mu(G) > 2p-h$.

Let G satisfy the above conditions and put

$$\pi(G) = n, \quad \varepsilon(G) = s.$$

It is easy to deduce from our conditions and (3) that for every $h \geq 2$

$$p \geq 11, \; n \geq 21, \quad \mu(G) \geq 19, \quad s \geq 9.$$

From (8) it follows that $s \leq p$. Let

$$p = s + a.$$

Clearly $a \geqq 0$. (9) implies because of $\mu(G) \leq 2s$ that

$$2a \leq h-1 \,. \tag{10}$$

In the most important cases we will obtain the contradiction by showing that G contains a subgraph G' whose components are triangles and edges and for which $\nu(G') \leq p + h$ and $\mu(G') = p + 1$ (these facts contradict (8)). Assume that such a G' has $x + y$ components, x triangles and y edges.

Clearly

$$\nu(G') = 3x + y \leqq p + h \quad \text{and} \quad \mu(G') = 2x + y = p + 1.$$

Thus

$$x \leq h-1 \,. \tag{11}$$

Conversely if (11) is satisfied then because of (3) and $2x + y = p + 1$ we obtain $y > 0$. G' further clearly satisfies

$$x + y \leqq s \,.$$

Thus from $y = p + 1 - 2x$

$$x \geqq a + 1 \,.$$

(From (5) and (10) $a + 1 \leq h-1$.)

In the following we will only use the G' for which x and y takes on the following values:

(12) In case $2a \leq h-3$ $\quad x = 2a + 2, \quad y = s-(3a + 3)$.

(13) In case $2a \leq h-2$ $\quad x = 2a + 1, \quad y = s-(3a + 1)$.

(14) $\quad x = 2a, \quad y = s-(3a-1)$.

(15) $\quad x = a + 1, \quad y = s-(a + 1)$.

(II) Let $e_i = P_iP'_i\,(i = 1, \ldots, s)$ be a maximal system of independent edges. These edges will be considered fixed during the rest of the proof. Let

$$M = \{P_1, \ldots, P_s\},\ M' = \{P'_1, \ldots, P'_s\},\ N = M \cup M',\ G_\varepsilon = [N].$$

$$\overline{N} = S-N \text{ (S is the set of vertices of G.)}$$

If $\overline{N}$ is non empty (i. e. $n > 2s$), then put

$$\overline{N} = \{Q_1, \ldots, Q_{n-2s}\}.$$

From the fact that $s = \varepsilon(G)$ it trivially follows that

(16) the vertices of $\overline{N}$ are independent,

(17) the edges P_kQ_i and P'_kQ_j ($P_k \in M$, $P'_k \in M'$, $i \neq j$, $\{Q_i, Q_j\} \subseteq \overline{N}$) can not both occur in G,

(18) if P_iQ_k and P_jQ_l are in G ($i \neq j$, $k \neq l$, $\{P_i, P_j\} \subseteq M$, $\{Q_k, Q_l\} \subseteq \overline{N}$), then $P'_iP'_j$ is not in G.

From (4) and (16) we obtain

(19) every vertex of $\overline{N}$ is incident to $N\overline{N}$-edges.

From (17) and (18) it follows that

(20) if both P_i and $P_i'(1 \leqq i \leqq s)$ are incident to an $N\overline{N}$-edge then P_iP_i' and these two $N\overline{N}$-edges form a triangle (this means that there can be only one $N\overline{N}$-edge incident to P_i and P_i').

(III) We prove that

(21) $$\overline{\mu}(G_\varepsilon) \leqq 2h-3a-2\,.$$

If $\overline{N}$ is empty then $G_\varepsilon = G$, $n = 2s$ and because of (9)

(22) $$\overline{\mu}(G_\varepsilon) = n-\mu(G) \leqq 2s - (2p-h+1) = h-2a-1.$$

In this case from (22), (5) and (10) follows (21).

For the rest of (III) we assume that $\overline{N}$ is non empty. Put $G_0 = [N, \overline{N}]$. G_0 is an even graph which, because of (19), is non empty. Thus by the theorem (1.11) of KÖNIG

(23) $$\mu(G_0) = \varepsilon(G_0).$$

Let $e_1', \ldots, e_{s_0}'(s_0 = \varepsilon(G_0))$ be a maximal system of independent edges of G_0. By (17) we can assume that

$$e_i' = P_iQ_i \qquad (i = 1, \ldots, s_0).$$

Put $M_1' = \{P_1', \ldots, P_{s_0}'\}$. By (18) the vertices of M_1' are independent and because (20) if $P_i' \in M_1'$ then the only vertex of $\overline{N}$ with which P_i' can be connected by an edge is Q_i. Denote by M_2' the vertices of M_1' which are connected with the corresponding Q_i and put $\alpha(M_2') = t$.

Assume $t \geqq a+1$, without loss of generality we have $M_2' = \{P_1', \ldots, P_t'\}$. Let $\Delta_i = P_iP_i'Q_i(i = 1, \ldots, t)$. Then the triangles $\Delta_i(i = 1, \ldots, a+1)$ and the edges $e_{a+2}, \ldots, e_s$ form a subgraph G' of G whose existence because of (15) contradicts (8).

Assume next $t \leqq a$. The vertices of $M_3' = M_1' - M_2'$ are independent (assuming that M_3' is non empty) and the only edges incident to them belong to G_ε. Therefore the vertices of $N_1 = M \cup (M'-M_3')$ represent the edges of G. Thus

$$\mu(G) \leqq \alpha(N_1) = 2s - (s_0-t) \leqq 2p - (a+s_0).$$

Thus from (9)

(24) $$s_0 \leqq h-a-1.$$

Let R_0 respectively R_ε be a representing system of G_0 respectively G_ε having minimal number of elements. $R_0 \cup R_\varepsilon$ clearly represents G and thus by (23) $s_0 + \mu(G_\varepsilon) \geqq \mu(G)$. Thus from (9) and (24) we obtain $\mu(G_\varepsilon) \geqq 2p-2h+ a+2$. Thus by (1.3) (1) we obtain (21).

From now on the triangles Δ_i and the sets M_1', M_2' will not occur any more. Thus we will use these symbols and the symbols used for their vertices, for other purposes.

(IV) Now we shall show that both $[M]$ and $[M']$ contain suitably related complete graphs having sufficiently many vertices. From (6) and (10) we have

$$\pi([M]) = s > \varphi(\varphi(2h+4)).$$

By $\bar{\mu}([M]) \leqq \bar{\mu}(G_\varepsilon)$ we have from (21)

$$\bar{\mu}([M]) < 2h + 4 \leqq \varphi(2h+4).$$

Thus by RAMSAY's theorem there is an $M_1 \subseteq M$ so that $[M_1]$ is complete and

$$\pi([M_1]) = \varphi(2h+4).$$

Let

$$M_1 = \{P_1, \ldots, P_u\},\ M_1' = \{P_1', \ldots, P_u'\} \quad (u = \varphi(2h+4)).$$

By (21) $\bar{\mu}([M_1']) < 2h + 4$. Thus by $\pi([M_1']) = \varphi(2h+4)$ we obtain from RAMSAY's theorem that there exists an $M_2' \subseteq M_1'$ so that $[M_2']$ is complete and $\pi([M_2']) = 2h + 4$. Put

$$M_2' = \{P_1', \ldots, P_{2h+4}'\}.$$

(10) implies $3(a+2) < 2h+4$. Thus since $[M_1]$ and $[M_2']$ are complete, the triangles

$$\Delta_i = P_{3i-2}P_{3i-1}P_{3i},\quad \Delta_i' = P_{3i-2}'P_{3i-1}'P_{3i}' \quad (i = 1, \ldots, a+2)$$

are all subgraphs of G.

By (10) $2a \leqq h-1$. Now we distinguish three cases, $2a \leqq h-3$, $2a = h-2$ and $2a = h-1$.

(V) Assume $2a \leqq h-3$. Then the pairs of triangles (Δ_i, Δ_i') $(i=1, \ldots, a+1)$ and the edges $e_{3a+4}, \ldots, e_s$ form a subgraph of G which by (12) contradicts (8).

(VI) Assume next $2a = h-2$. By (7) $n \geqq 2s+1$, thus $\bar{N}$ is non empty. By (19) there are $N\bar{N}$-edges. Now the following statement holds:

(25) Any two vertices of N which are not incident to $N\bar{N}$-edges are connected by an edge.

For if two such vertices would not be connected, the other vertices of N would represent the edges of G. Thus $n \leqq 2s-2 = 2p-h$, which contradicts (9). Thus (25) is proved.

Assume first that there is a j $(1 \leqq j \leqq s)$ so that both P_j and P_j' are incident to $N\bar{N}$-edges. By (20) the vertices of these $N\bar{N}$-edges which are in N must coincide. Denote this common vertex by Q_1. Consider the triagles (Δ_i, Δ_i') $(i = 1, \ldots, a+2)$ defined in (IV). We can find a of these pairs in such a way that none of them should have a common vertex with e_j. These pairs of triangles together with the triangle $P_j P_j' Q_1$ and together with all the edges e_i $(1 \leqq i \leqq s,\ i \neq j)$ which have no common vertex with our a triangle-pairs form a subgraph of G whose existence by (13) contradicts (8).

For the rest of part (VI) we can assume that no vertex of M' is connected (by an edge) to a vertex of $\bar{N}$. Thus we obtain by (25) that $[M']$ is a complete graph. No we prove the following statement:

(26) Assume that G contains an edge P_jQ_l $(1 \leq j \leq s,\ Q_l \in \overline{N})$, assume further $k \neq j (1 \leq k \leq s)$, then the edge $P_j' P_k$ is not in G.

If (26) would be false, then since $[M']$ is complete the triangle $P_j' P_k P_k'$ is a subgraph of G. From the triangle-pairs (Δ_i, Δ_i') $(i = 1, \ldots, a+2)$ we can again find a of them so that none of them have a common vertex with e_j or e_k. These triangles together with the triangle $P_j' P_k P_k'$, the edge P_jQ_l, and all the edges $e_i (1 \leq i \leq s,\ i \neq j,\ i \neq k)$ which have no common vertex with our a triangle-pairs form a subgraph of G whose existence by (13) contradicts (8).

We now show that every vertex of M is incident to $N\overline{N}$-edges. To see this observe that if $P_k (1 \leq k \leq s)$ would be a vertex which is not incident to an $N\overline{N}$-edge, then by (25) this would be connected to every vertex of M'. Among these vertices there clearly is a vertex P_j' so that the corresponding P_j is incident to an $N\overline{N}$-edge, which contradicts (26).

From (18) and from the fact that $[M']$ is complete it follows that the $N\overline{N}$-edges incident to the vertices of M are all incident to the same vertex Q_1. Therefore by (19) $\overline{N} = \{Q_1\}$. From (26) we further deduce that the only vertex of M' to which P_j can be connected is $P_j' (1 \leq j \leq s)$.

Next we show that no two vertices of M are connected. To see this assume that G contains the edge $P_j P_k (j \neq k,\ \{P_j, P_k\} \subset M)$. Choose a of the triangle-pairs (Δ_i, Δ_i') $(i = 1, \ldots, a+2)$ so that none of them contain a common vertex with the edges e_j and e_k. These triangle-pairs together with the triangle $Q_1 P_j P_k$ and together with all the edges $e_i (1 \leq i \leq s,\ i \neq j,\ i \neq k)$ which have no common vertex with one of our a triangle-pairs form a subgraph of G which by (13) contradicts (8).

From what has been said it follows that the set $R = M' \cup \overline{N}$ represents G, further $\alpha(R) = s + 1 \leq 2p - h$ and this contradicts (9).

(VII) Finally assume $2a = h - 1$. Then by (7) $n \geq 2s + 2$, or $\overline{N}$ contains at least two vertices. Every vertex of N is incident to $N\overline{N}$-edges. For if N would have a vertex which is not incident to an $N\overline{N}$-edge then the other vertices of N would represent G, their number is $2s - 1 = 2p - h$ which contradicts (9).

By (19) and (20) there is a j and $k (1 \leq j \leq s,\ 1 \leq k \leq s,\ j \neq k)$ for which the triangles $\Delta' = Q_1 P_j P_j'$ and $\Delta'' = Q_2 P_k P_k'$ are subgraphs of G. We now select from the triangle-pairs (Δ_i, Δ_i') $(i = 1, \ldots, a+2)$ $a-1$ pairs so that none of them contain a common vertex with e_j or e_k. These pairs together with Δ', Δ'' and with all the edges $e_i (1 \leq i \leq s,\ i \neq j,\ i \neq k)$ which have no vertex in common with the selected pairs form a subgraph of G. By (14) this contradicts (8).

This completes the proof of Theorem (2.2).

Now we show that if c $(c > 1)$ is any constant and $h > h_0(c)$ then $p_0(h) > ch$. More precisely we shall show

Theorem (2.3). *Let c $(c > 1)$ be any constant, then there exists an $h_0(c)$ so that for every $h > h_0(c)$ there exists an integer $p > ch$ and a graph G satisfying the following conditions:*

1.) *G contains no isolated vertex.*

2.) $\pi(G) \geq 2p - h + 3$.

3.) *For every* $G' \subseteq G$ *which satisfies* $\nu(G') \leq p + h$ *we have* $\mu(G') \leq p$.
4.) $\mu(G) > 2p - h$.

Proof. (I) A theorem of ERDŐS ([3], p. 34. (4)) implies that to every $c (c > 1)$ there is an $n_0(c)$ so that for every $n > n_0(c)$ there exists a graph G, having no isolated vertices, for which

$$(1) \qquad \pi(G) = n, \qquad \bar{\mu}(G) < \frac{3}{56} \cdot \frac{n}{c}$$

and for which

(2) every circuit contains more than $28c$ vertices.

We are going to show that

$$(3) \qquad h_0(c) = \max \left(28, \frac{n_0(c)}{c} \right)$$

satisfies the requirements of our theorem.

Let $h > h_0(c)$, and choose p so that

$$(4) \qquad ch < p < \frac{7}{6} ch .$$

Let further n satisfy

$$(5) \qquad 2p - \frac{3}{4c} p < n < 2p - \frac{6}{7} \cdot \frac{3}{4c} p .$$

Let G be a graph having no isolated vertices and satisfying (1) and (2) with the above choices of c and n. We shall show that G satisfies the conditions 1.), 2.), 3.) and 4.) of Theorem (2.3).

Conditions 1.), 2.) and 4.) are clearly satisfied. Thus to complete our proof we only have to show that 3.) is satisfied.

(II) Let $G' \subseteq G$, $\nu(G') \leq p + h$. We shall prove that

$$(6) \qquad \mu(G') \leq p.$$

To prove (6) we define by recursion for every $k \geq 0$ a subgraph G_k of G' as follows: $G_0 = G'$. If G_k has no vertex of valency > 2 we put $G_{k+1} = G_k$. If G_k has a vertex of valency > 2, let P_k such a vertex and put $G_{k+1} = G_k - P_k$. Since G was finite there is a smallest k say l so that $G_{l+1} = G_l$. G_l has no vertex of valency greater than 2, and we obtained G_l from G' by the omission of l vertices of valency ≥ 3. Thus from (4), (5) and $\nu(G') \leq p + h$ we obtain

$$(7) \qquad \pi(G_l) = n - l < 2p - \frac{9}{14c} p - l ,$$

$$(8) \qquad \nu(G_l) = \nu(G') - 3l < p + \frac{p}{c} - 3l .$$

Since all vertices of G_l have valency ≤ 2, the components of G_l can only be circuits, paths and isolated vertices. Assume that there are j circuits among the components of G_l. By (2) every circuit of G_l contains more than $28c$ vertices, thus by (7)

$$j \cdot 28\, c < \pi(G_l) < 2p$$

or

$$j < \frac{p}{14\, c}\,. \tag{9}$$

The edges of a circuit or path of k vertices can always be represented by $[k/2]$ or $[k/2] + 1$ vertices respectively. Thus from (7) and (9)

$$\mu(G_l) \leq \frac{1}{2}\pi(G_l) + j < p - \frac{p}{4\, c} - \frac{l}{2}\,.$$

The edges of G' which do not occur in G_l we represent by the l vertices which do not occur in G_l. Thus we obtain

$$\mu(G') \leq \mu(G_l) + l < p - \frac{p}{4\, c} + \frac{l}{2}\,.$$

Thus if $p/(4c) \geq l/2$ we obtain $\mu(G') < p$. If $p/(4c) < l/2$, then by $\mu(G_l) \leq \nu(G_l)$ and by (8) we have

$$\mu(G') \leq \mu(G_l) + l < p,$$

which proves 3.) and thus the proof of Theorem (2.3) is complete.

§ 3.

(3.1) In connection with the general problem raised in (2.1) the following questions can be asked:

Does there exist to every p a smallest $f(p)$ so that if G has the property that for every $G' \subseteq G$ with $\nu(G') \leq f(p)$ we have $\mu(G') \leq p$, then $\mu(G) \leq p$?

This question can be answered affirmatively. From the Theorem (3.5) we easily deduce

Theorem (3.2). *Assume that for every* $G' \subseteq G$ *with* $\nu(G') \leq \binom{2\,p+2}{2}$ *we have* $\mu(G') \leq p$. *Then* $\mu(G) \leq p$.

The estimate $f(p) \leq \binom{2\,p+2}{2}$ seems to be a poor one.

Conjecture (3.3).

$$f(p) = \binom{p+2}{2}\,.$$

We can prove our conjecture for $p \leq 4$ (see the remark 1. made to Theorem (3.10)). The example of the complete $(p+2)$-graphs shows that $f(p) \geq \binom{p+2}{2}$, since if G is a complete $(p+2)$-graph for every proper subgraph G' of it we have $\mu(G') \leq p$ and $\nu(G') \leq \binom{p+2}{2} - 1$, but $\mu(G) = p + 1$.

(3.4) Now we ask the following question:

Assume that for every $G' \subseteq G$ satisfying $\pi(G') \leq q$ we have $\mu(G') \leq p$ what upper bound can be given for $\mu(G)$?

If $q = 2p + 1$, $\mu(G)$ can be arbitrarily large. To see this consider the following even graph G^*: The vertices of G^* are $P_1, \ldots, P_m, Q_1, \ldots, Q_n$ and its edges are $P_iQ_j (i = 1, \ldots, m;\ j = 1, \ldots, n)$. Clearly $\mu(G^*) = \min(m, n)$, but a simple argument shows that for every $G' \subseteq G^*$ with $\pi(G') \leq 2p + 1$ we have $\mu(G') \leq p$. Here we have for $m = n$ $\pi(G^*) = 2n$, $\mu(G^*) = n$. The more complicated examples given in [2] and [3] show that a graph G with $\pi(G) = n$, $\mu(G) > n - o(n)$ exists so that for every $G' \subseteq G$ with $\pi(G') \leq 2p + 1$ we have $\mu(G') \leq p$.

On the other hand we are going to prove that for $q = 2p + 2$ we have $\mu(G) \leq p$ (which is clearly best possible).

Theorem (3.5). *Assume that for every $G' \subseteq G$ with $\pi(G') \leq 2p + 2$ we have $\mu(G') \leq p$. Then $\mu(G) \leq p$.*

We will prove Theorem (3.5) in § 4. It is curious to observe the sharp change between $q = 2p + 1$ and $q = 2p + 2$. This change can be seen also in the order of magnitude of the number of edges.

If $q = 2p + 2$ (3.5) immediately gives

$$\nu(G) \leq p\big(\pi(G) - 1\big). \tag{1}$$

(1) is best possible. To see this let the vertices of G be $P_1, \ldots, P_p$, $Q_1, \ldots, Q_{n-p}$ (the set of the Q's may be empty). The edges of G connect each of the vertices $P_1, \ldots, P_p$ with all the other vertices of G. Clearly $\mu(G) = p$ and $\nu(G) = p\,(n-1)$.

If $q = 2p + 1$ then G^* shows that $\nu(G)$ can be as large as $\left[\left(\frac{\pi(G)}{2}\right)^2\right]$ $\left(\text{for } m = \left[\frac{\pi(G)}{2}\right],\ n = \left[\frac{\pi(G)+1}{2}\right]\right)$. For sufficiently large values of $\pi(G)$ this is best possible. Here we have

Theorem (3.6). *Let $\pi(G) \geq 4(p + 1)$. Assume that for every $G' \subseteq G$ with $\pi(G') \leq 2p + 1$, $\mu(G') \leq p$. Then*

$$\nu(G) \leq \left[\left(\frac{\pi(G)}{2}\right)^2\right].$$

Disregarding the condition $\pi(G) \geq 4\,(p + 1)$, for $p = 1$ this theorem is identical with TURÁN's theorem ([9], p. 26.) for $j = 2$. The proof of Theorem (3.6) uses this special case of TURÁN's theorem. We supress the details.

Perhaps we can digress for a moment and call attention to the following interesting class of problems. Let $\pi(G) = n$ and assume that for every $G' \subseteq G$ with $\pi(G') \leq q$ we have $\mu(G') \leq p$. Denote by $g(n, p, q)$ the maximum value of $\nu(G)$. We wish to determine or estimate $g(n, p, q)$. The cases $q \leq p + 1$ are trivial since there trivially $g(n, p, q) = \binom{n}{2}$. $q \geq 2p + 2$ implies by (3.5) $g(n, p, q) = p(n-1)$. The interesting range is $p + 2 \leq q \leq 2p + 1$. The case $q = 2p + 1$ is settled by Theorem (3.6). $q = p + 2$ means that G does not contain a complete $(p + 2)$-graph and is thus settled by TURÁN's theorem.

The determination of $g(n, p, q)$ for general p and q seems to be a difficult problem and we made very little progress with it. ERDŐS can show that for sufficiently large n

$$g(n, p, 2p) = \left[\frac{(n+1)^2}{4}\right] - 1 . \tag{1}$$

The methods required for the proof of (1) and Theorem (3.6) are quite different than those used in this paper.

It is easy to see that conjecture (3.3) and theorem (3.5) can be restated in the following form:

(3.7) Conjecture. *If* $\mu(G) > p$ *then there is a* $G' \subseteq G$ *for which* $\mu(G') > p$ *and* $\nu(G') \leq \binom{p+2}{2}$.

(3.8) *If* $\mu(G) > p$ *then there is a* $G' \subseteq G$ *for which* $\mu(G') > p$ *and* $\pi(G') \leq 2p + 2$.

A graph G is said to be *edge-critical* if it has edges and for every $G' \subset G$, $\mu(G') < \mu(G)$.

G is *point-critical* if it has edges and for every $G' \subset G$ for which $\pi(G') < \pi(G)$ we have $\mu(G') < \mu(G)$.

Clearly every G which has edges has subgraphs G' which are edge-, respectively point-critical and for which $\mu(G') = \mu(G)$.

(3.7) and (3.8) are substantially equivalent to the following statements:

Conjecture (3.9). *For every edge-critical graph G we have* $\nu(G) \leq \binom{\mu(G)+1}{2}$.

Theorem (3.10) *For every point-critical G we have* $\pi(G) \leq 2\,\mu(G)$.

The proof of the equivalence is left to the reader. The proof of (3.10) will be given in § 4.

Remarks. 1.) Conjecture (3.9) holds for $\mu(G) \leq 4$.

2.) In § 4 we shall show that in (3.10) equality can hold only if $2\varepsilon(G) = \pi(G)$.

3.) From (3.10) and from the fact that an edge-critical graph is also point-critical we obtain that for an edge-critical graph G we have $\nu(G) \leq \binom{2\,\mu(G)}{2}$.

§ 4.

In this § we are going to prove Theorem (3.10) (and thus also Theorem (3.5)).

Our definitions trivially imply

(4.1) *A point-critical graph can have no isolated vertices. If G is non-empty and not point-critical, then it has a vertex P with* $\mu(G-P) = \mu(G)$.

(4.2) Let S be the set of vertices of G, further let

(1) $S_i \subseteq S$ $(i = 1, \ldots, k;\ k \geq 2)$; $S_i \cap S_j = \emptyset$ $(i \neq j,\ i, j = 1, \ldots, k)$ and

$$\bigcup_{i=1}^{k} S_i = S .$$

Let R be a set of $\mu(G)$ vertices which represent every edge of G. Clearly $R \cap S_i$ represents all edges of $G_i = [S_i]$, thus

$$\sum_{i=1}^{k} \mu(G_i) \leq \mu(G) . \tag{2}$$

If there exists a decomposition of S into *non empty* subsets S_i satisfying (1) for which

$$\sum_{i=1}^{k} \mu(G_i) = \mu(G)$$

holds, then we say that G is *decomposable* and we call the set $\{G_1, \ldots, G_k\}$ a *decomposition of* G. The following two statements trivially follow from our definitions:

(4.3) *If G is decomposable we have $\pi(G) > 1$ and G has a decomposition $\{G_1, \ldots, G_k\}$ where all the $G_i (1 \leq i \leq k)$ are indecomposable.*

(4.4) *If (the non-empty) G is not connected, it is decomposable.*

(4.5) *If $\pi(G) > 1$ and G is indecomposable, then G is point-critical.*

Proof. If (4.5) would be false, there would exist by (4.1) a $P \in S$ so that for $G_1 = G - P$ we would have $\mu(G_1) = \mu(G)$. Clearly neither G_1 nor $G_2 = [P]$ are empty and $\mu(G_2) = 0$. Thus $\mu(G_1) + \mu(G_2) = \mu(G)$, but then $\{G_1, G_2\}$ would be a decomposition of G.

(4.6) *Let G be point-critical and $\{G_1, \ldots, G_k\}$ a decomposition of G. Then the G_i $(i = 1, \ldots, k)$ are also point-critical.*

Proof. Assume say that G_1 is not point-critical. Since G_1 is non empty it has by (4.1) a vertex P so that $\mu(G_1 - P) = \mu(G_1)$. But then by (4.2) (2)

$$\mu(G - P) \geq \mu(G_1 - P) + \sum_{i=2}^{k} \mu(G_i) = \sum_{i=1}^{k} \mu(G_i) = \mu(G) ,$$

which is a contradiction since G was assumed to be point-critical.

Theorem (4.7). *If $\pi(G) > 1$ and G is indecomposable, then*

$$\pi(G) \leq 2\mu(G)$$

where equality stands only if G consists of a single edge.

Proof. (I) Because of (4.4) G is connected and therefore it has no isolated vertex. If G consists of a single edge $\pi(G) = 2\mu(G)$ trivially holds. Henceforth we assume $\pi(G) > 2$. Let R be a set of $\mu(G) = r$ vertices which represent every edge of G. Put $S - R = T$. Clearly neither R nor T are empty and the vertices of T are independent. Thus every vertex of T is incident to TR-edges.

Consider the graph $G' = [R, T]$. Clearly G' is even and contains edges. Put $\mu(G') = r'$. Clearly $0 < r' \leqq r$.

We are going to show in (II) that the only representing system of G' with r' elements is T. This easily implies $\pi(G) < 2\mu(G)$, since R is a representing system of G' and therefore $r > r'$, or $\pi(G) = r + r' < 2r$ as stated.

(II) Let R' be any representing system of G' which has r' elements. R' is non empty. Put

$$R' \cap R = R_1, \quad R' \cap T = T_1.$$

Assume that R_1 is empty. Then from $R' \subseteq T$ and from the fact that every vertex of T is incident to TR-edges it follows that $R' = T$.

Thus to complete our proof we only have to show that the assumption $\alpha(R_1) = r_1 > 0$ leads to a contradiction.

By theorem (1.11) of König G' contains r' independent edges, say $e_i = P_i P_i'$ ($P_i \in R$, $P_i' \in T$, $i = 1, \ldots, r'$). Each of these edges is incident to exactly one vertex of R'. Denote by $e_1, \ldots, e_{r_1}$ the edges incident to the vertices of R_1 and put $\{P_1', \ldots, P_{r_1}'\} = T_2$. We evidently have $T_1 \cap T_2 = \varnothing$. Let $R - R_1 = \overline{R}_1$, $T - T_2 = \overline{T}_2$. G' clearly does not contain an $\overline{R}_1 T_2$-edge. Put

$$G_1 = [R_1 \cup T_2], \quad G_2 = [\overline{R}_1 \cup \overline{T}_2].$$

a) Assume first that G_2 is empty. Then

$$r_1 = r' = r = \alpha(R) = \alpha(T).$$

Since $\pi(G) > 2$ we have $r > 1$. Then if $G_3 = [\{P_1, P_1'\}]$ and $G_4 = [S - \{P_1, P_1'\}]$ we have $\mu(G_3) = 1$ and $\mu(G_4) \geqq r - 1$ (since G_4 contains $e_2, \ldots, e_r$). Thus $\{G_3, G_4\}$ is a decomposition of G and this is a contradiction.

b) Assume now G_2 non empty. The vertices of R_1 represent all edges of G_1 and since G_1 contains the edges $e_1, \ldots, e_{r_1}$ we obtain

$$\mu(G_1) = r_1. \tag{1}$$

$\mu(G_2) \geqq r - r_1$ is impossible since $\{G_1, G_2\}$ would then be a decomposition of G. But $\mu(G_2) < r - r_1$ is also impossible, for in this case if R_2 would be a representing system of G_2 having $\mu(G_2)$ elements, then $R_1 \cup R_2$ would represent all edges of G and thus

$$\mu(G) \leqq \mu(G_1) + \mu(G_2) < r,$$

which is impossible. This completes the proof of (4.7).

Finally we prove (3.10) and our remark 2.) belonging to it.

(4.8) *If G is point-critical then $\pi(G) \leqq 2\mu(G)$, equality can hold only if $2\varepsilon(G) = \pi(G)$.*

Proof. If G consists of an edge, (4.8) it trivial. We can therefore assume that $\pi(G) > 2$. If G is indecomposable, then by (4.7) $\pi(G) < 2\mu(G)$. Assume now that G is decomposable and let $\{G_1, \ldots, G_k\}$ be a decomposition of G

where all the G_i $(1 \leqq i \leqq k)$ are indecomposable. By (4.6) G_i $(i = 1, \ldots, k)$ is point-critical and thus

$$\pi(G) = \sum_{i=1}^{k} \pi(G_i) \leqq 2 \sum_{i=1}^{k} \mu(G_i) = 2\,\mu(G)\,.$$

Equality occurs if and only if every G_i $(1 \leqq i \leqq k)$ consits of a single edge In this case the edges of $G_1, \ldots, G_k$ are independent, which implies $2\varepsilon(G) = \pi(G)$.

§ 5.

(5.1) In this § we generalise our problems to "graphs of several dimensions" i. e. to *k-tuples*. Let S be a set (its elements we will call points) and H a certain *finite* set of k-tuples formed from the elements of S. (For $k = 2$ H was G and the points of S which occur in the 2-tuples of H, i. e. in the edges of G were called the vertices of G. This G has no isolated vertices.) Denote by $\pi(H)$ the number of elements of S which occur in the k-tuples of H and by $\nu(H)$ the number of k-tuples of H. If $R \subseteq S$ and if every k-tuple of H containts at least one point of R we say that the points of R *represent* H or that R is a *representing system of* H. Denote by $\mu(H)$ the minimal number of points which represent H.

Generalising the problems considered in (3.1) and (3.5) (i. e. in (3.7) and (3.8)) we wish to determine the smallest values $f(k, p)$ and $g(k, p)$ which satisfy the following conditions:

Every H for which $\mu(H) > p$ contains a subset H' and a subset H'' for which $\mu(H') > p$, $\mu(H'') > p$ and $\nu(H') \leqq f(k, p)$, $\pi(H'') \leqq g(k, p)$.

We now obtain upper estimates for $f(k, p)$ and $g(k, p)$ further we determine $f(k, 1)$ respectively $g(k, 1)$ for every $k \geqq 2$.

Theorem (5.2)

$$f(k, p) \leqq \sum_{i=0}^{p} k^i\,.$$

Proof. For $p = 0$ our statement is trivial. Assume henceforth $p \geqq 1$. Let H be an arbitrary finite set of k-tuples with $\mu(H) > p$ and let $t_0 = \{P_1, \ldots, P_k\}$ be an arbitrary element of H. Put $H_0 = \{t_0\}$. Since $\mu(H) > p \geqq 1$ a single element can not represent H and therefore to every P_{i_1} $(1 \leqq i_1 \leqq k)$ there is a t_{i_1} in H which does not contain P_{i_1}. Let $t_{i_1} = \{P_{i_1 1}, \ldots, P_{i_1 k}\}$ $(i_1 = 1, \ldots, k)$ and put $H_1 = \{t_1, \ldots, t_k\}$. If $p \geqq 2$ we need at least three points for the representation of H and therefore we can find to every pair of points P_{i_1}, $P_{i_1 i_2}$ $(1 \leqq i_1 \leqq k,\ 1 \leqq i_2 \leqq k)$ a k-tuple $t_{i_1 i_2} = \{P_{i_1 i_2 j} \mid j = 1, \ldots, k\}$ which does not contain P_{i_1} and $P_{i_1 i_2}$. Put $H_2 = \{t_{i_1 i_2} \mid i_1, i_2 = 1, \ldots, k\}$. Continuing this process for every j $(1 \leqq j \leqq p)$ we obtain the k-tuples $t_{i_1 \ldots i_j}$ (of H) and the points $P_{i_1 \ldots i_j i_{j+1}}$ and the sets of k-tuples H_j $(i_1, \ldots, i_j, i_{j+1} = 1, \ldots, k)$. Put

$$H' = \bigcup_{j=0}^{p} H_j\,.$$

Since $\nu(H_j) \leq k^j$ we have $\nu(H') \leq \sum_{j=0}^{p} k^j$. Now we show $\mu(H') > p$. To see this let R be a representing system of H'. R must contain an element of t_0 say P_1. By our construction $P_1 \notin t_1$, thus R must contain an element of t_1 say P_{12}. If $p > 2$ then P_1 and P_{12} are not contained in t_{12} and R must contain an element P_{123} of t_{12}. This process can be continued $(p + 1)$ times and we obtain that R contains the elements $P_1, P_{12}, \ldots, P_{12 \ldots p+1}$ or $\alpha(R) > p$ as stated.

Theorem (5.3) $f(k, 1) = k + 1$ $(k \geqq 2)$.

Proof. By (5.2) $f(k, 1) \leq k + 1$. The following example shows $f(k, 1) = k + 1$. Let $S = \{P_0, \ldots, P_k\}$. H consists of the $(k + 1)$ k-tuples formed from S. Here $\mu(H) > 1$ but for every $H' \subset H$ $\mu(H') = 1$.

(5.4) In general we know little about the value of $f(k, p)$. Conjecture (3.7) states that $f(2, p) = \binom{p+2}{2}$. This and (5.3) might permit us to conjecture $f(k, p) = \binom{p+k}{k}$. In any case $f(k, p) \geqq \binom{p+k}{k}$. To see this let H consists of all the $\binom{p+k}{k}$ k-tuples formed from $p + k$ elements. Clearly $\mu(H) = p + 1$, but a simple argument shows that for every $H' \subset H$ $\mu(H') \leqq p$, which proves $f(k, p) \geqq \binom{p+k}{k}$.

A trivial argument shows that $g(k, p) \leq k f(k, p)$. Thus we have

Theorem (5.5). $g(k, p) \leq \sum_{i=1}^{p+1} k^i \quad (k \geqq 2)$.

We know only a little more about $g(k, p)$ than about $f(k, p)$. (3.8) states that $g(2, p) = 2p + 2$. Further we have

Theorem (5.6). $g(k, 1) = \left[\frac{(k+2)^2}{4}\right] \quad (k \geqq 2)$.

Proof. (I) First we show $g(k, 1) \leq [(k + 2)^2/4]$. To see this let H be a set of k-tuples for which $\mu(H) > 1$, let further t' and t'' be two k-tuples of H for which $\alpha(t' \cap t'') = a$ is minimal.

If $a = 0$, then putting $H' = \{t', t''\}$ we have $\mu(H') > 1$ and $\pi(H') = 2k \leq [(k + 2)^2/4]$. Thus we can assume $a > 0$. Put $t' \cap t'' = \{P_1, \ldots, P_a\}$. To every P_i $(1 \leq i \leq a)$ we can find a t_i of H which does not contain P_i. Put $H' = \{t', t'', t_1, \ldots, t_a\}$. Clearly $\mu(H') > 1$. Further for every i $(1 \leqq i \leqq a)$

(1) $$\alpha(t' \cap t_i) \geqq a, \quad \alpha(t'' \cap t_i) \geqq a, \quad \alpha(t' \cap t'' \cap t_i) \leqq a - 1.$$

Denote by a_i the number of elements of t_i which do not belong to $t' \cap t''$. We have by (1)

$$a_i = \alpha(t_i) - \alpha(t' \cap t_i) - \alpha(t'' \cap t_i) + \alpha(t' \cap t'' \cap t_i) \leqq k - a - 1.$$

Thus

$$\pi(H') \leqq \alpha(t' \cup t'') + \sum_{i=1}^{a} a_i \leqq 2k - a + a(k - a - 1) =$$

$$= 2k + a(k - 2 - a) \leqq 2k + \left(\frac{k-2}{2}\right)^2 = \frac{(k+2)^2}{4}$$

which proves our assertion.

(II) To show $g(k,1) \geqq [(k+2)^2/4]$ put $[k/2] = l$ and $M = \{P_1, \ldots, P_q\}$ where $q = l + 1$ if k is even and $q = l + 2$ if k is odd. Let further

$$M_i = M - \{P_i\},\quad M'_i = \{P_{i1}, \ldots, P_{il}\},\quad t_i = M_i \cup M'_i \quad (i = 1, \ldots, q),$$

and

$$H = \{t_1, \ldots, t_{l+1}\},$$

(the P's with different indices denote different points).

Here we have $\alpha(t_i) = k\ (i = 1, \ldots, l+1)$ and

$$\pi(H) = q + ql = \left[\frac{(k+2)^2}{4}\right].$$

Clearly $\mu(H) > 1$, but for $H_i = H - \{t_i\}$ $(i = 1, \ldots, l+1)$ we have $\mu(H_i) = 1$ since P_i clearly represents H_i. This completes our proof.

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О МИНИМАЛЬНОМ ЧИСЛЕ ТОЧЕК, РЕПРЕЗЕНТИРУЮЩИХ РЕБРА ГРАФА

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Резюме

В работе фигурируют лишь такие конечные ненаправленные графы, которые не содержат петлей и в которых две точки связаны не более чем одним ребром. Число точек графа G обозначается через $\pi(G)$, а число его ребер через $\nu(G)$.

Если $e_1, \ldots, e_j (j \geqq 1)$ ребра графа G и $P_1, \ldots, P_k (k \geqq 1)$ такие точки G, что любое $e_i (1 \leqq i \leqq j)$ содержит хотя бы одну из них, то мы говорим, что $P_1, \ldots, P_k$ *репрезентируют* ребра $e_1, \ldots, e_j$. Обозначим через $\mu(G)$ минимальное число точек, репрезентирующих все ребра G. Если G не содержит ребер, то полагаем $\mu(G) = 0$. Цель работы дать верхние грани для $\mu(G)$, используя различные данные и свойства G. Основные результаты:

Если G содержит ребро, то $\mu(G)$ не превосходит гармоническое среднее от $\frac{1}{2}\pi(G)$ и $\nu(G)$. Равенство имеет место лишь в том случае, если G полный граф, или если каждая компонента G есть полный граф с одним и тем же числом точек. **(Теорема** (1.7).)

Если p «достаточно велико» относительно $h (h \geqq 2)$ и для графа G, не содержащего изолированных точек, $\pi(G) \geqq 2p - h + 3$, то, если для всех подграфов G' графа G, содержащих не более $p + h$ ребер, $\mu(G') \leqq p$, то $\mu(G) \leqq 2p - h$. **(Теорема** (2.2.).)

Если для всех подграфов G' графа G, содержащих не более $2p + 2$ точек, $\mu(G') \leqq p$, то $\mu(G) \leqq p$. **(Теорема** (3.5).)

Границы, фигурирующие в этих теоремах, не могут быть улучшенны без дальнейших предположений.

§ 5 занимается «многомерным» обобщением проблем. Здесь вместо графов фигурируют k-аты, образуемые из любых элементов $(k \geqq 2)$.