SOME REMARKS ON THE RANDOM ERGODIC THEOREM, II.

by

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Introduction

Let $\mathcal{T}$ be a measurable space of measure preserving transformations each of which is defined on a measure space $\{X, \mathcal{S}, \mu\}$. Let further $\mathbf{P}_1, \mathbf{P}_2, \ldots$ be a sequence of probability measures defined on $\mathcal{T}$. We denote the product space

$$\mathcal{T}_1 \times \mathcal{T}_2 \times \ldots \qquad (\mathcal{T}_i = \mathcal{T};\ i = 1, 2, \ldots)$$

by $\mathcal{T}^*$ and the product measure

$$\mathbf{P}_1 \times \mathbf{P}_2 \times \ldots$$

defined on $\mathcal{T}^*$ by $\mathbf{P}^*$. $(T_1, T_2, \ldots)$ $(T_i \in \mathcal{T}_i)$ means a point of $\mathcal{T}^*$.

Let H_1 denote the space of those measurable functions defined on X for which

$$\int_X f^2(x)\, d\mu < \infty \qquad \int_X f(x)\, d\mu = 0 .$$

We assume in this paper that the function $f(T_k T_{k-1} \ldots T_1 x)$ is measurable and integrable on the space $\mathcal{T}^* \times X$ for every $f \in H_1$.

In [1] the following theorem is proved.

Statistical random ergodic theorem. *If*[1]

$$\left\| \int_{T \in \mathcal{T}} f(Tx)\, d\mathbf{P}_i \right\| = \| \mathbf{M}_{\mathbf{P}_i} f(Tx) \| \leqq m_i \| f(x) \|$$

for every $f(x) \in H_1$ *where*

$$m_i = 1 - \frac{C}{i^{1-\varepsilon}}$$

C is an arbitrary positive constant and $0 < \varepsilon \leqq 1$.

Then for every $f(x) \in H_1$ *we have*

$$\mathbf{P}^* \left\{ \left\| \frac{1}{n} \sum_{k=1}^{n} f(T_k \ldots T_1 x) \right\| \to 0 \right\} = 1 .$$

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[1] Here and in what follows $\|g\|^2 = \int_X g^2(x)\, d\mu$.

In § [1] we give some conditions under which the sequence $f(T_k \ldots T_1 x)$ forms a Markov chain. In § 2. an individual random ergodic theorem is proved and in § 3. as an application of our results we obtain a strong law of large numbers for those Markov chain which can be represented in the form $f(T_k \ldots T_1 x)$.

§ 1. The probabilistic behaviour of the sequence $f(T_k \ldots T_1 x)$.

A natural question is the following: Is the sequence $f(T_k \ldots T_1 x)$ a Markov chain for every (or almost every) x? We shall show in an example that generally $f(T_k \ldots T_1 x)$ is not a Markov chain.

Let X be the space of the numbers 0, 1, 2, 3, 4. We define the measure μ, the function f and the measure preserving transformations $T^{(1)}$, $T^{(2)}$ on X as follows

$$\mu(0) = \mu(1) = \mu(2) = \mu(3) = \mu(4) = 1;$$

$$f(0) = 0,\ f(1) = 1,\ f(2) = 2,\ f(3) = f(4) = 3;$$

$$T^{(1)}\,0 = 1, \quad T^{(1)}\,1 = 3, \quad T^{(1)}\,2 = 0$$

$$T^{(1)}\,3 = 4, \quad T^{(1)}\,4 = 2;$$

$$T^{(2)}\,0 = 2, \quad T^{(2)}\,1 = 3, \quad T^{(2)} 2 = 4$$

$$T^{(2)}\,3 = 0, \quad T^{(2)}\,4 = 1.$$

On the set $\mathcal{T} = \{T^{(1)}, T^{(2)}\}$ we define the probability measure $\mathbf{P}$ by

$$\mathbf{P}(T^{(1)}) = \mathbf{P}(T^{(2)}) = {}^1\!/_2\,.$$

Now, it is easy to see that

$$\mathbf{P}\{f(T_3 T_2 T_1 0) = 2 \mid f(T_2 T_1 0) = 3\} > 0,$$

but

$$\mathbf{P}\{f(T_3 T_2 T_1 0) = 2 \mid f(T_2 T_1 0) = 3,\ f(T_1 0) = 1\} = 0.$$

which proves that $\{f(T_n \ldots T_1 0)\}$ is not a Markov chain.

The following theorem holds.

Theorem 1. *Let* $f(x)$ a measurable function defined on X, $\mathcal{A}$ *the smallest σ-algebra of those subsets of X with regards to which $f(x)$ is measurable. If for every $A \in \mathcal{A}$ and $T \in \mathcal{T}$ we have*

$$TA \in \mathcal{A} \quad \textit{and} \quad T^{-1} A \in \mathcal{A}\,.$$

Then for almost every x the sequence $f(T_k \ldots T_1 x)$ is a Markov chain.

For the proof of the above theorem we need the following

Lemma. *Suppose that $f(x)$ and $\mathcal{T}$ satisfy the conditions of Theorem* 1. *If for a pair x, y of elements of X*

$$f(x) = f(y)$$

then for every $T \in \mathcal{T}$ we have

$$f(Tx) = f(Ty).$$

Proof. For the sake of simplicity we denote by α the common value of $f(x)$ and $f(y)$, $\alpha = f(x) = f(y)$. Introduce furthermore the notations

$$f^{-1}(\alpha) = A \qquad (A \in \mathcal{A}).$$

Since $TA \in \mathcal{A}$ there exists a Borel set $\mathfrak{B}$ on the real line such that

$$f^{-1}(\mathfrak{B}) = TA.$$

In order to prove the lemma it is sufficient to show that $\mathfrak{B}$ has only one element. In contradiction of our statement suppose that $\mathfrak{B}$ has more than one point. Let $\mathfrak{C}$ be a point of $\mathfrak{B}$ then $C = f^{-1}(\mathfrak{C})$ is a proper subset of B and $A \subset T^{-1}C$ which is a contradiction.

Proof of Theorem 1. Obviously our lemma implies that the conditional probabilities

$$\mathbf{P}\{f(T_n \ldots T_1 x) \in \mathfrak{B} \mid f(T_{n-1} \ldots T_1 x) = a_{n-1}, \ldots, f(T_1 x) = a_1\},$$
$$\mathbf{P}\{f(T_n \ldots T_1 x) \in \mathfrak{B} \mid f(T_{n-1}, \ldots T_1 x) = a_{n-1}\}$$

both are the $\mathbf{P}_n$ measures of those transformations $T \in \mathcal{T}_n$ which map the set $f^{-1}(a_{n-1})$ into the set $f^{-1}(\mathfrak{B})$ which proves the theorem.

The following question arises: given a Markov chain of real valued random variables $\zeta_1, \zeta_2, \ldots$ what is the condition which we have to impose on the transition probability functions that makes possible the construction of a measure space $\{X, \mathcal{S}, \mu\}$ a measurable function $f(x)$ defined on X, a measurable set $\mathcal{T}$ of the measures preserving transformations defined on X and a sequence $\mathbf{P}_1, \mathbf{P}_2, \ldots$ of probability measures definened on $\mathcal{T}$ such that

$$\text{(1)} \quad \mathbf{P}^*\{f(T_n \ldots T_1 x) \in \mathfrak{B} \mid f(T_{n-1} \ldots T_1 x) = \alpha_{n-1}, \ldots, f(T_1 x) = \alpha_1\} = \mathbf{P}^*\{\zeta_n \in \mathfrak{B} \mid \zeta_{n-1} = \alpha_{n-1}, \ldots, \zeta_1 = \alpha_1\}$$

for a special x and for every n and for every sequence $\alpha_1, \alpha_2, \ldots$ of real numbers. If the above construction is possible then we say that the Markov chain $\zeta_1, \zeta_2, \ldots$ can be represented in the form $f(T_k \ldots T_1 x)$. In the sequel we shall give a sufficient condition ensuring the representation in the form $\{f(T_k \ldots T_1 x)\}$. of a Markov chain. First we mention some definitions and two theorems.

1) A matrix $A = \{a_{ik}\}_{i,k=1}^{\infty}$ is called doubly stochastic if

$$a_{ik} \geqq 0 \qquad (i, k = 1, 2, \ldots)$$

$$\sum_{k=1}^{\infty} a_{ik} = 1 \quad (i = 1, 2, \ldots), \qquad \sum_{i=1}^{\infty} a_{ik} = 1 \quad (k = 1, 2, \ldots).$$

2) A matrix $A = \{a_{ik}\}_{i,k=1}^{\infty}$ is called weakly doubly stochastic (WDS) if

$$a_{ik} \geqq 0 \qquad (i, k = 1, 2, \ldots)$$

$$\sum_{k=1}^{\infty} a_{ik} = 1 \quad (i = 1, 2, \ldots), \qquad \sum_{i=1}^{\infty} a_{ik} \leqq 1 \quad (k = 1, 2, \ldots).$$

3) A matrix Π having only zeros and ones as elements is called a permutation matrix (resp. weak permutation matrix) if it is a doubly stochastic matrix (resp. WDS matrix). We denote the set of all permutation matrices (resp. weak permutation matrices) by Ω (resp. by Ω^*).

4) Let Ω_{ik} (resp. Ω^*_{ik}) be the subset of those elements of Ω (resp. Ω^*) in which the k-th element of the i-th row is 1.

The following theorems are proved in [2].

Theorem 2. *Let $A = \{a_{ik}\}_{i,k=1}^{\infty}$ be a WDS matrix. Then we can find a σ-algebra $\mathcal{S}^*$ of subsets of Ω^* and a probability measure $\mathbf{P}^*$ on $\mathcal{S}^*$ such that $\Omega^*_{ik} \in \mathcal{S}^*$ $(i = 1, 2, \ldots, k = 1, 2, \ldots)$ and $\mathbf{P}^*(\Omega^*_{ik}) = a_{ik}$.*

Theorem 3. *Let $A = \{a_{ik}\}_{i,k=1}^{\infty}$ be a doubly stochastic matrix. Then we can find a σ-algebra $\mathcal{S}$ of subsets of Ω and a probability measure $\mathbf{P}$ on $\mathcal{S}$ such that $\Omega_{ik} \in \mathcal{S}$ $(i = 1, 2, \ldots; k = 1, 2, \ldots)$ and $\mathbf{P}(\Omega_{ik}) = a_{ik}$.*

These two theorems together imply the following

Consequence. *Let $x \in l^2$ the i-th coordinate of which is denoted by $x|_i$. Consider every doubly stochastic (WDS) matrix as a linear bounded operator in the space l^2. Then $\Pi x|_i$ $(\Pi \in \Omega)$. (resp. $\Pi \in \Omega^*$) is a measurable function defined on the measurable space $\{\Omega, \mathcal{S}\}$ (resp. $\{\Omega^*, \mathcal{S}^*\}$). Now, if A is a doubly stochastic (WDS) matrix, then we can find a probability measure $\mathbf{P}$ ($\mathbf{P}^*$) on the σ-algebra $\mathcal{S}$ (resp. $\mathcal{S}^*$) satisfying.*

$$Ax|_i = \mathbf{M}(\Pi x|_i) = \int_{\pi \in \Omega} \Pi x|_i \, d\mathbf{P}$$

resp.

$$Ax|_i = \mathbf{M}(\Pi x|_i) = \int_{\pi \in \Omega^*} \Pi x|_i \, d\mathbf{P}^* .$$

These formulae can be written in the concise form

$$Ax = \mathbf{M}(\Pi x).$$

Applying these results we prove the following two theorems

Theorem 4. *Let $\zeta_1, \zeta_2, \ldots$ be a discrete Markov chain having $\alpha_1, \alpha_2, \ldots$ as possible values. Suppose that the matrix*

$$A_n = \{a_{ik}^{(n)}\}_{i,k=1}^{\infty}, \quad a_{ik}^{(n)} = \mathbf{P}\{\zeta_n = \alpha_k | \zeta_{n-1} = \alpha_i\}$$

is doubly stochastic. Under these conditions the Markov chain $\zeta_1, \zeta_2, \ldots$ can be represented in the form $f(T_k \ldots T_1 x)$.

Proof. Let X be the set of the positive integers, $\mathcal{S}$ the set of all subsets of X. We define the function $f(x)$ and the measure μ on X as follows

$$\begin{aligned} f(i) &= \alpha_i \\ \mu(i) &= 1. \end{aligned} \qquad (i = 1, 2, \ldots)$$

Let further $\mathcal{T}$ denote the set of all measure preserving transformations defined on X. Every $T \in \mathcal{T}$ there can be found a permutation matrix Π_T as follows if

$$Ti = k_i \quad (i = 1, 2, \ldots)$$

(k_i is a permutation of the positive integers) then

$$\Pi_T = \{\pi_{ik}\}_{i,k=1}^{\infty} \quad \pi_{ik} = \begin{cases} 1 & \text{if} \quad k = k_1 \\ 0 & \text{if} \quad k \neq k_1. \end{cases}$$

In the space $\mathcal{T}$ we define a sequence $\mathbf{P}_1, \mathbf{P}_2, \ldots$ of probability measures as follows:

The $\mathbf{P}_1$ measure of the set $T \in \mathcal{T}$ transforming the point 1 into the point i equals $\mathbf{P}\{\zeta_1 = i\}$

The $\mathbf{P}_n$ measure of the set $T \in \mathcal{T}$ transforming the point i into the point k equals $a_{ik}^{(n)}$. (Theorem 3. states that we can define a measure in this way).

It is easy to see that for the sequences $f(T_k \ldots T_1 x)$ and $\zeta_1, \zeta_2, \ldots$ formula (1) holds.

Theorem 5. *Let $\zeta_1, \zeta_2, \ldots$ be a discrete Markov chain. Let the values of $\zeta_1, \zeta_2, \ldots$ be the real numbers $\alpha_1\ \alpha_2, \ldots$ Suppose that the matrix*

$$A_n = \{a_{ik}^{(n)}\}_{i,k=1}^{\infty} \qquad a_{ik}^{(n)} = \mathbf{P}\{\zeta_n = k \mid \zeta_{n-1} = i\}$$

is a WDS matrix. Then there exists a measurable set $\mathcal{T}'$ of measurable transformations defined on the positive integers and a sequence $\mathbf{P}_1, \mathbf{P}_2, \ldots$ of probability measures defined ox $\mathcal{T}$ such that

$$\mathbf{P}^* \{f(T_k \ldots T_1 1) = \alpha_{j_k} | f(T_{k-1} \ldots T_1 1) = \alpha_{j_{k-1}}, \ldots f(T_1 1) = \alpha_{j_1}) =$$
$$= \mathbf{P}\{\zeta_k = \alpha_{j_k} | \zeta_{k-1} = \alpha_{j_{k-1}}, \ldots \zeta_1 = \alpha_1\}$$

and

$$\mu(T^{-1}E) < C\mu(E) \qquad (C > 0)$$

for every $T \in \mathcal{T}$ where $f(x)$ is the function

$$f(i) = \alpha_i.$$

The proof of this theorem is similar to the proof of Theorem 4. the only one change is the application of Theorem 2. instead of Theorem 3.

§ 2. An individual random ergodic theorem

Let H_1^* be the space of those bounded measurable functions defined on X for which

$$\int_X f^2(x)\, d\mu < \infty; \quad \int_X |f(x)|\, d\mu < \infty; \quad \int_X f(x)\, d\mu = 0.$$

In this section we prove the following

Individual random ergodic theorem. *If for every $f(x) \in H_1^*$*

$$(1) \qquad \| \int_{T \in \mathbf{T}} f(Tx)\, d\mathbf{P}_i \| = \| \mathbf{M}_{\mathbf{P}_i} f(Tx) \| \leq m_i \|f(x)\|$$

where

$$m_i = 1 - \frac{C}{i^{1-\varepsilon}}.$$

C is an arbitrary positive constant and $0 < \varepsilon \leq 1$ then

$$(2) \qquad \mathbf{P}^* \left\{ \frac{1}{n} \sum_{k=1}^{n} f(T_k \ldots T_1 x) \to 0 \text{ for almost every } x \right\} = 1$$

for every $f(x) \in H_1^*$.

Proof. In [1] it is proved that

$$\int_X \mathbf{M}\,[(S_n f)^2]\, d\mu = O(1/n^{1/\alpha}) \tag{3}$$

where $S_n f = \frac{1}{n} \sum_{k=1}^{n} f(T_k \ldots T_1 x)$ and $\alpha = [1/\varepsilon] + 1$. (3) implies that

$$\sum_{n=1}^{\infty} \int_X \mathbf{M}\,[(S_{n^{2\alpha}} f)^2]\, d\mu < \infty \,. \tag{4}$$

By (4) and the Beppo-Levi theorem

$$\mathbf{P}^* \{S_{n^{2\alpha}} f \to 0 \text{ for almost every } x\} = 1\,.$$

If n is an arbitrary positive integer then there exists a k such that

$$k^{2\alpha} \leqq n < (k+1)^{2\alpha}\,.$$

Using this fact we can write

$$|S_n f| \leqq \left| \frac{f(T_1 x) + f(T_2 T_1 x) + \ldots + f(T_{k^{2\alpha}} \ldots T_1 x)}{k^{2\alpha}} \right| +$$

$$+ \left| \frac{f(T_{k^{2\alpha}+1} \ldots T_1 x) + \ldots + f(T_n \ldots T_1 x)}{k^{2\alpha}} \right|.$$

By supposition there exists a K for which $|f(x)| \leqq K$. Thus

$$|S_n f| \leqq |S_{k^{2\alpha}} f| + \frac{K}{k}$$

and the theorem follows.

Remark 1. It is worth while to mention that Theorem states the validity of (2) for every $f(x) \in H_1^*$. If we want to prove the validity of (2) only for a special function $f(x) \in H_1^*$ then it is enough to assume that Condition (1) holds for the functions

$$g_k(x) = \mathbf{M}\{f(T_k \ldots T_1 x)\} = \int_{\mathcal{T}_k} \ldots \int_{\mathcal{T}_1} f(T_k \ldots T_1 x)\, d(\mathbf{P}_1 \times \mathbf{P}_2 \times \ldots \times \mathbf{P}_k)$$

$$(k = 1, 2, \ldots)$$

Remark 2. In our theorem we can substitute the assumption that the elements of $\mathcal{T}$ are measure preserving transformations by the following ones.

1) for every sequence $T_1, T_2 \ldots$ of elements of $\mathcal{T}$ and for every $E \in \mathcal{S}$ the inequality

$$\frac{1}{n} \sum_{k=1}^{n} \mu(T_1^{-1} T_2^{-1} \ldots T_k^{-1} E) \leqq C\, \mu(E)$$

holds.

2) the condition (1) is fulfilled for the functions

$$g_k(x) = \mathbf{M}\big(f(T_k \ldots T_1 x)\big)$$

We omit the proofs of these remarks, because they can be proved in similar way to the proof of the Individual random ergodic theorem.

§ 3. An example

In this § a strong law of large numbers for a special class of the inhomogenous Markov chains is proved.

Theorem 6. *Let $\zeta_1, \zeta_2, \ldots$ be a discrete Markov chain with the state space $\{\alpha_1, \alpha_2, \ldots\}$. We introduce the following conditions*

$$1) \quad \sum_{i=1}^{\infty} \alpha_i = 0\,, \qquad \sum_{i=1}^{\infty} |\alpha_i| < \infty\,.$$

2) *The matrices*

$$A_n = \{a_{ik}^{(n)}\}_{i,=1}^{\infty} \quad a_{ik}^{(n)} = \mathbf{P}\{\zeta_n = \alpha_k | \zeta_{n-1} = \alpha_i\}$$

are WDS matrices.

3) *There exists a $C > 0$ and a $0 < \varepsilon \leqq 1$ such that for every n*

$$\|A_n\| \leqq 1 - \frac{C}{n^{1-\varepsilon}} \tag{5}$$

where the norm of a matrix A is defined by

$$\|A\| = \sup_{X \in H_1} \frac{\|Ax\|}{\|x\|}$$

and H_1 contains those points $x = (x_1, x_2, \ldots)$ of the space l^2 for which $\sum_{i=1}^{\infty} x_i = 0$ and $\sum_{i=1}^{\infty} |x_i| < \infty$.

Under these conditions

$$\mathbf{P}\left\{\frac{\zeta_1 + \ldots + \zeta_n}{n} \to 0\right\} = 1\,.$$

Proof. We can represent the Markov chain $\zeta_1, \zeta_2, \ldots$ in the same way in the form $f(T_k \ldots T_1 x)$ as it has been made in the proof of Theorem 5. due to Condition 2. The corollary of Theorem 4 shows that (5) implies (1) Hence follows the theorem.

Remark 3. We show by an example that if instead of (5) we suppose only that

$$\|A_n\| \leqq 1 - \frac{1}{n}$$

then the strong law in general does not hold.

$$\mathbf{P}\{\zeta_1 = +1\} = \mathbf{P}\{\zeta_1 = -1\} = 1/2$$

$$A_n = \begin{pmatrix} 1 - 1/2\,n & 1/2\,n \\ 1/2\,n & 1 - 1/2\,n \end{pmatrix} = (a_{ik}^{(n)})_{i,k=1}^{2}\, a_{ik}^{(n)} = \mathbf{P}\{\zeta_n = \alpha_k | \zeta_{n-1} = \alpha_i\}\,.$$

It is easy to see that

$$\|A_n\| = 1 - 1/n$$

and

$$\mathbf{M}\left[\left(\frac{\zeta_1 + \zeta_2 + \ldots + \zeta_n}{n}\right)^2\right] \to 0 .$$

The variables ζ_n are bounded hence the arithmetic mean of $\zeta_1, \zeta_2, \ldots$ is also bounded. This implies that

$$\mathbf{P}\left\{\frac{\zeta_1 + \ldots + \zeta_n}{n} \to 0\right\} < 1 .$$

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НЕСКОЛЬКО ЗАМЕЧАНИЙ О СЛУЧАЙНОЙ ЭРГОДИЧЕСКОЙ ТЕОРЕМЕ

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Резюме

Пусть $\{X, \mathscr{S}, \mu\}$ есть измеримое пространство, $\mathscr{T}$ — множество определенных на пространстве X измеримых и не изменяющих меру преобразований. Обозначим через $\mathbf{P}_1$, $\mathbf{P}_2, \ldots$ последовательность определенных на $\mathscr{T}$ вероятностных мер. Пусть, далее,

$$\begin{aligned} \mathscr{T}^* &= \mathscr{T} \times \mathscr{T}_2 \times \ldots \\ \mathbf{P}^* &= \mathbf{P}_1 \times \mathbf{P}_2 \times \ldots \end{aligned} \quad (\mathscr{T}_i = \mathscr{T}) \qquad (i = 1, 2, \ldots)$$

и, наконец, через H_1^* обозначим множество тех определенных на пространстве X ограниченных интегрируемых с квадратом функций, для которых

$$\int_X f(x)\, d\mu = 0 .$$

Предположим, что для всех $f(x) \in H_1^*$ функции $f(T_k \ldots T_1 x)$ измеримы на пространстве $\mathscr{T}^* \times X$, где $(T_1, T_2, \ldots)$ обозначает некоторую точку пространства $\mathscr{T}^*$. В работе доказываются следующие теоремы.

Теорема 1. *Пусть $f(x)$ определенная на пространстве X измеримая функция. Обозначим через $\mathscr{A}$ наиболее узкую σ-алгебру, для которой $f(x)$ измерима. Предположим, что для всех $A \in \mathscr{A}$ и $T \in \mathscr{T}$*

$$TA \in \mathscr{A} \quad \text{и} \quad T^{-1}A \in \mathscr{A} .$$

Тогда почти для всех фиксированных X последовательность случайных величин $\{f(T_n \dots T_1 x\}$ образует цепь Маркова.

Теорема 2. *Пусть $\zeta_1, \zeta_2, \dots$ есть дискретная цепь Маркова, предположим, что одношаговые матрицы вероятности перехода дубльстохастичны. Тогда можно найти пространство $\{X, \mathscr{S}, \mu\}$, пространство $\mathscr{T}$ определенных на нем удерживающих меру преобразований, определенная на X измеримая функция $f(x)$ и последовательность $\mathbf{P}_1, \mathbf{P}_2, \dots$ определенных на $\mathscr{T}$ вероятностных мер так, что для некоторого X n — мерные функции распределения последовательности случайных величин $\{f(T_n \dots T_1 x\}$ совпадают с соответствующими n — мерными распределениями цепи Маркова $\zeta_1, \zeta_2, \dots$.*

Теорема 3. *Если для всех $f(x) \in H^*$*

$$\| \int_{T \in \mathscr{T}} f(Tx)\, d\, \mathbf{P}_i \| = \| \mathbf{M}_{\mathbf{P}_i} f(Tx) \| \leqq m_i \, \| f(x) \| ,$$

где

$$m_i = 1 - \frac{C}{i^{1-\varepsilon}} .$$

(C любая положительная постоянная, $0 < \varepsilon \leqq 1$), тогда

$$\mathbf{P}^* \left\{ \frac{1}{n} \sum_{k=1}^{n} f(T_k \dots T_1 x) \to 0 \text{ почти для всех } x \right\} = 1 \text{ для всех } f(x) \in H_1^*.$$

В § 3 работы с помощью теорем 2 и 3 доказывается одна теорема больших чисел относительно неоднородных цепей Маркова.