UNSOLVED PROBLEMS

We start with this number a new section of this journal: that of unsolved problems. In this section unsolved problems will be proposed, at the same time some information about previous results in the direction of the problem in question will be given. Problems for this section as well as comments on published problems should be sent to G. ALEXITS, editor of the section, to the address of the redaction of the journal (Budapest, V. Reáltanoda u. 13—15.).

НЕРАЗРЕШЕННЫЕ ПРОБЛЕМЫ

Начиная с настоящего номера помещается новый раздел в нашем журнале: неразрешенные проблемы. В этом разделе мы публикуем неразрешенные проблемы и одновременно указываем на более ранные результаты, связанные с данной проблемой. Проблемы, предназначенные для этого раздела, а также замечания, связанные с сообщаемыми проблемами просим направить по адресу редакции журнала (Budapest, V. Reáltanoda utca 13—15.) для редактора раздела G. ALEXITS.

SOME UNSOLVED PROBLEMS

by

PAUL ERDŐS

In this paper I shall discuss some unsolved problems in number theory. combinatorial analysis, set theory, elementary geometry, analysis and probability. The choice of problems is purely subjective, I discuss problems on which I worked myself or which interested me and it is certainly not claimed that all or most of the problems discussed here are very important, but I hope the reader will find them challenging and amusing; most of them will have a combinatorial character. Classical and wellknown problems are avoided as much as possible.

I gave several talks on unsolved problems at various places (Moscow, Leningrad, Peking, Singapore, Adelaide). In the autumn of 1959 I gave a series of talks on unsolved problems at the Mathematical Institute of the Hungarian Academy of Sciences and most of the problems discussed here were discussed in my lectures.

My first talk on unsolved problems was given on November 16, 1957 at Assumption University Windsor, Ontario, Canada, a paper on this talk appeared in the *Michigan Mathematical Journal* **4** (1957), 291—300, and there is a considerable overlap between this paper and the present one.

c, c_1, c_2, . . . , C will denote positive absolute constants. i. o. is an abreviation for infinitely often.

I. Problems in number theory

First some problems on prime numbers.

1) Denote by $\pi(x)$ the number of primes not exceeding x. It has been conjectured that

$$\pi(x+y) \leq \pi(x) + \pi(y). \tag{I. 1.1}$$

It is easy to verify (I. 1.1) for small values of y (P. UNGÁR informed me that he verified it for $y \leq 41$). For $x > x_0$, $x = y$ (I. 1.1) was proved by

LANDAU. HARDY and LITTLEWOOD proved by BRUN's method that

$$\pi(x+y) - \pi(x) < \frac{cy}{\log y} \tag{I. 1.2}$$

and A. SELBERG proved that

$$\pi(x+y) - \pi(x) < 2\frac{y}{\log y} + O\left(\frac{y \log\log y}{(\log y)^2}\right). \tag{I. 1.3}$$

A conjecture weaker than (I. 1.1) but stronger than (I. 1.3) would be: To every $\varepsilon > 0$ there exists a y_0 so that for $y > y_0$

$$\pi(x+y) - \pi(x) < (1+\varepsilon)\frac{y}{\log y}. \tag{I. 1.4}$$

The replacement in (I. 1.3) of 2 by a smaller constant would be of great importance.

Instead of considering $\pi(x+y) - \pi(x)$ one could define $f(x, y)$ as the greatest integer k so that there exist k integers $x < a_1 < a_2 < \ldots < a_k \leqq x + y$ satisfying $(a_i, a_j) = 1$. The proof of HARDY and LITTLEWOOD gives $f(x, y) < cy/\log y$ (trivially $f(x, y) \geqq \pi(x+y) - \pi(x)$) and one could conjecture that $f(x, y) \leqq \pi(y)$ or that $f(x, y) < (1+\varepsilon)\frac{y}{\log y}$ for $y > y_0$. Following HARDY and LITTLEWOOD put

$$\varrho(y) = \limsup_{x=\infty} \left(\pi(x+y) - \pi(x)\right).$$

One would conjecture that $\lim_{y=\infty} \varrho(y) = \infty$ and perhaps even

$$\varrho(y) > (1-\varepsilon)\, y/\log y \text{ for } y > y_0,$$

but it is not even known that $\varrho(y) \geqq 2$ for $y > y_0$.

G. H. HARDY and J. E. LITTLEWOOD: "Some problems of partitio numerorum." *Acta Mathematica* **44** (1923) 1–70.

E. LANDAU: *Handbuch der Lehre von der Verteilung der Primzahlen*. Vol. 1.

A. SELBERG: „*On elementary methods in prime number theory and their limitations.*" Den 11-te Skandinaviske Matematikerkongress (1952) 13–22.

2) Denote by $2 = p_1 < p_2 < \ldots$ the sequence of prime numbers. Put $d_n = p_{n+1} - p_n$. TURÁN and I proved that for infinitely many n and m, $d_n > d_{n+1}$ and $d_{m+1} > d_m$. It is not known if $d_n = d_{n+1}$ holds i. o. We could not prove that i. o. $d_n > d_{n+1} > d_{n+2}$, in fact we could not even prove that i. o. either $d_n > d_{n+1} > d_{n+2}$ or $d_n < d_{n+1} < d_{n+2}$.

It seems very likely that the sequence $d_n/\log n$ is everwhere dense and that it has a distribution function (in other words the density of integers n satisfying $d_n/\log n < c$ exists and if we denote it by $f(c)$ then $f(0) = 0$, $f(\infty) = 1$). RICCI and I proved that the set of limit points of $d_n/\log n$ has positive measure, but ∞ is the only known limit point (theorem of WESTZYNTHIUS). Analogous questions can be asked about d_n/d_{n+1}.

P. ERDŐS and P. TURÁN: "On some sequences of integers." *Bull. Amer. Math. Soc.* **54** (1948) 371–378.

P. ERDŐS: "On the difference of consecutive primes." Ibid. 885–889.

P. ERDŐS and A. RÉNYI: "Some problems and results on consecutive primes." *Simon Stevin* **27** (1950) 115–125.

G. RICCI: "Recherches sur l'allure de la suite $p_{n+1} - p_n/\log p_n$." Colloque sur la Théorie des nombres, Bruxelles (1955) 93–106.

E. WESTZYNTHIUS: »Über die Verteilung der Zahlen die zu den ersten Primzahlen teilerfremd sind." *Commentationes Phys.-mat. Soc. Sci. fenn.* 5, Nr. 25, 1–37.

3) Sharpening the result of WESTZYNTHIUS I proved that i. o.

$$d_n > c\frac{\log n \log\log n}{(\log\log\log n)^2} \tag{I. 3.1}$$

and RANKIN proved that i. o.

$$d_n > c\frac{\log n \log\log n \log\log\log\log n}{(\log\log\log n)^2}\,. \tag{I. 3.2}$$

It seems to be very difficult to improve (I. 3.2).

INGHAM proved $d_n < n^{5/8}$ ($d_n < n^{1-\varepsilon}$ was first proved by HOHEISEL for $\varepsilon = 32999/33000$) and the Riemann hypothesis would imply $d_n < n^{1/2+\varepsilon}$. CRAMER conjectured that

$$\limsup d_n/(\log n)^2 = 1. \tag{I. 3.3}$$

The old conjecture on prime twins states that i. o. $d_n = 2$, but it is not even known that

$$\liminf d_n/\log n = 0\,. \tag{I. 3.4}$$

I proved using BRUN's method that

$$\liminf d_n/\log n < 1. \tag{I. 3.5}$$

I further proved that

$$\limsup \min (d_n, d_{n+1})/\log n = \infty\,, \tag{I. 3.6}$$

but I can not prove that

$$\liminf \max (d_n, d_{n+1})/\log n < 1 \text{ or } \limsup \min \frac{(d_n, d_{n+1}, d_{n+2})}{\log n} = \infty\,, \tag{I. 3.7}$$

also I can not prove

$$\lim \frac{d_n + d_{n+1} + \ldots + d_{n+k-1}}{k \log n} < 1 - c$$

where c does not depend on n.

P. ERDŐS: "On the difference of consecutive primes." *Quarterly Journal of Math.* **6** (1935) 124–128. See also T. H. CHANG: "Über aufeinanderfolgende Zahlen, von denen jede mindestens einer von n linearen Kongruenzen genügt, deren Moduln die ersten n Primzahlen sind." *Schriften Math. Sem. u. Inst. Angew. Math. Univ.* **4** (1938) 35–55.

R. A. RANKIN: „The difference between consecutive prime number." *Journal London Math. Soc.* **13** (1938) 242–247.

A. E. INGHAM: "On the difference between consecutive primes." *Quarterly Journal of Math.* **8** (1937) 255–266.

H. CRAMER: "On the order of magnitude of the difference between consecutive prime numbers." *Acta Arithmetica* **2** (1936) 23–46.

G. HOHEISEL: „Primzahlprobleme in der Analysis." Sitzungsber. der Preuss. Akad. der Wiss phys. Math. Klasse, (1930), 580–588.

P. ERDŐS: "The difference of consecutive primes," *Duke Math. Journal* **6** (1940) 438–441.

R. A. RANKIN, *Proc. Amer. Math. Soc.* **1** (1950) 143–150.

P. ERDŐS: "Problems and results on the differences of consecutive primes." *Publ. Math. Debrecen* **1** (1949) 37–39.

4) RÉNYI and I proved by BRUN's method that to every c_1 there exists a c_2 so that there exists $r > c_2 \log n$ $d's$ $d_k, \ldots, d_{k+r}$ satisfying

$$k < n,\ d_{k+i} > c_1, \qquad 0 \leqq i \leqq r, \tag{I. 4.1}$$

but we can not prove that (I. 4.1) holds for every c_1 and c_2 if $n > n_0(c_1, c_2)$.

Denote by $a_1 < a_2 < \ldots$ the sequence of integers having not more than two prime factors. I proved that

$$\limsup\ (a_{k+1} - a_k)/\log k > c \tag{I. 4.2}$$

but can not prove that the lim sup in (I. 4.2) is infinite, I can not prove that the limit in (I. 4.2) is positive if the a's are the integers having not more than three prime factors. (ERDŐS—RÉNYI, see problem 2.) (I. 4.2) was a problem in *Elemente der Mathematik*, 1955.

5) CRAMER (see problem 3) proved, assuming the Riemann hypothesis that

$$\sum_{p_k < x} (p_{k+1} - p_k)^2 < cx(\log x)^3 . \tag{I. 5.1}$$

It is possible that

$$\sum_{p_k < x} (p_{k+1} - p_k)^2 < cx \log x \tag{I. 5.2}$$

holds. $\left(\text{Perhaps even } \lim \frac{1}{x \log x} \sum_{p_k < x} (p_{k+1} - p_k)^2 \text{ exists}\right)$. (I. 5.2) seems hopeless at present, but perhaps the following conjecture of mine can be attacked. Let $1 = a_1 < a_2 < \ldots < a_{\varphi(n)}$ be the integers relatevely prime to n. Then

$$\sum_{k=1}^{\varphi(n)-1} (a_{k+1} - a_k)^2 < C \frac{n^2}{\varphi(n)} . \tag{I. 5.3}$$

I can not even prove that

$$\sum_{k=1}^{\varphi(n)-1} (a_{k+1} - a_k)^2 < C\, n(\log\log n)^{c_1} \tag{I. 5.4}$$

((I. 5.4) follows easily by BRUN's method with $n\,(\log n)^{c_2}$.

SIVASANKARANARAYAMA PILLAI conjectured that

$$\sum_{\substack{k\equiv 0 \pmod 2 \\ k<n}} d_k = \left(\frac{1}{2} + o(1)\right) p_n . \tag{I. 5.5}$$

(I. 5.5) seems very difficult but again one can conjecture that

$$\sum_{\substack{k\equiv 0 \pmod 2 \\ k<\varphi(n)}} (a_{k+1} - a_k) = \left(\frac{1}{2} + o(1)\right) n . \tag{I. 5.6}$$

We mentioned already (the probably hopeless) conjecture that $d_n/\log n$ has a distribution function. Let n_i be the product of the first i primes. Denote by $f(c, i)$ the number of solutions of $a_{k+1}^{(i)} - a_k^{(i)} < \frac{c\, n_i}{\varphi(n_i)}$ ($a_k^{(i)}$, $1 \leqq k \leqq \varphi(n_i)$ are the integers $\leqq n_i$ relatively prime to n_i). Is it true that

$$\lim_{i=\infty} f(c, i)/\varphi(n_i) = g(c)$$

exists? It is not difficult to show that the numbers

$$\frac{a_{k+1}^{(i)} - a_k^{(i)}}{n_i/\varphi(n_i)} , \; 1 \leqq k < \varphi(n_i) , \; 1 \leqq i < \infty$$

are everywhere dense in $(0, \infty)$.

6) Let $f(n)$ be a real valued multiplicative function, i.e. $f(a.b) = f(a)\cdot f(b)$ if $(a, b) = 1$. Assume $|f(n)| = 1$. Is it true that

$$\lim_{n=\infty} \frac{1}{n} \sum_{k=1}^{n} f(k) \tag{I . 6.1}$$

always exists? It is easy to prove that if

$$\sum_{f(p)=-1} \frac{1}{p} < \infty , \tag{I. 6.2}$$

hen the limit (I. 6.1) always exsists and is different from 0. It can be conectured that if (I. 6.2) diverges, then (I. 6.1) is 0. If $f(p^a) = -1$, then the conjecture is equivalent with the prime number theorem. I conjectured (I. 6.1) about 20 years ago, but quite possibly the conjecture is much older.

WINTNER observed that if $|f(n)| = 1$ can be complex valued, the limit (I. 6.1) does not have to exist.

A WINTNER: "*The theory of measure in arithmetical semigroups.* Baltimore, 1944. See also N. G. TCHUDAKOFF: "Theory of the characters of number semigroups." *Journal Indian Math. Soc.* **20** (1956) 11—15.

7) OSTMANN conjectured that there do not exist two sequences of integers $a_1 < a_2 < \ldots$; $b_1 < b_2 < \ldots$ each having at least two elements so that all but a finite number of primes are of the form $a_i + b_j$. and there are only a finite number of composite numbers of this form.

HORNFECK proved, using BRUN's method that both sequences must be infinite.

It seems certain that OSTMANN's conjecture is true, but the proof may well be difficult.

8) A. WINTNER once asked me if I can prove the existence of an infinite sequence of primes p_i, $1 \leqq i < \infty$ so that if $a_1 < a_2 < \ldots$ are the integers composed of the p's, then $\lim_{i=\infty} (a_{i+1} - a_i) = \infty$. I was unable to prove the existence of such a sequence of primes. A well-known theorem of Pólya states that if the a's are all composed of $p_1, p_2, \ldots, p_k$ then $\lim_{i=\infty} (a_{i+1} - a_i) = \infty$.

For several problems and conjectures on prime numbers see A. SCHINZEL and W. SIERPINSKI: "Sur certaines hypothèses concernant les nombres premiers." *Acta Arithmetica* **4** (1958) 185—207.

Now we consider some problems on additive number theory.

9) Can one give $k + 2$ integers $1 \leqq a_1 < a_2 < \ldots < a_{k+2} \leqq 2^k$ so that the sums $\sum_{i=1}^{k+2} \varepsilon_i a_i$, $\varepsilon_i = 0$ or 1, are all distinct? The sequence 2^i, $0 \leqq i \leqq k$ shows that one can give $k + 1$ such integers and 3, 5, 6, 7 shows that $a_{k+1} < 2^k$ is possible. Very recently CONWAY and GUY answered this question affirmatively, independently of each other. The problem, wether one can find $k + 3$ such integers $\leqq 2^k$ remains open.

More generally one can ask what is the maximum number of integers $a_1 < a_2 < \ldots < a_{k_x} < x$ so that the sums $\sum_{i=1}^{k} \varepsilon_i a_i$, $\varepsilon_i = 0$ should be all different? MOSER and I proved that

$$k_x \leqq \frac{\log x}{\log 2} + (1 + \varepsilon) \frac{\log\log x}{2 \log 2}. \tag{I. 9.1}$$

Probably (I. 9.1) is very far from being best possible, $k_x = \frac{\log x}{\log 2} + O(1)$. is quite possibly true.

P. ERDŐS: "Problems and results in additive number theory." Colloque sur la théorie des nombres, Bruxelles (1955) 136—137.

10) Denote by $f(n)$ the maximum number of positive integers $a_1 < a_2 < \ldots$ not exceeding n for which the sums $a_i + a_j$ are all different. SIDON asked to estimate $f(n)$. TURÁN and I proved that

$$f(n) < n^{1/2} + n^{1/4}, \tag{I. 10.1}$$

and SINGER proved that for infinitely many n

$$f(n) > n^{1/2}. \tag{I. 10.2}$$

It is possible that $f(n) = n^{1/2} + O(1)$.

SINGER's proof is based on his construction of a perfect difference set i.e. a set of residues $a_1, a_2, \ldots, a_{k+1} \pmod n$ so that every residue mod n except 0 can be uniquely represented in the form $a_i - a_j$. Clearly a perfect difference set is only possible if $n = k^2 + k + 1$ and SINGER proved that a perfect difference set exists if k is a power of a prime. It has been conjectured

if k is not the power of a primes a perfect set can not exist. Special cases of this conjecture have been proved by BRUCK and RYSER. The case $k = 10$ is not yet decided.

From (I. 10.1) and SINGER's result one can in fact deduce

$$f(n) = (1 + o(1))n^{1/2} . \tag{I. 10.3}$$

Denote by $f_3(n)$ the maximum number of a's not exceeding n so that all the sums $a_i + a_j + a_l$ are all distinct. BOSE recently asked me if I can prove analogously to (I. 10.1) and (I. 10.3)

$$f_3(n) = (1 + o(1))n^{1/3} \tag{I.10.4}$$

The proof of (I.10.4) seems difficult, the method we used in the proof of (I.10.1) does not work.

SIDON also asked what can be said about an infinite sequence for which the sums $a_i + a_j$ are all different. TURÁN and I proved that for such a sequence

$$\limsup a_k/k^2 = \infty \quad (\text{or } \liminf f(n)/\sqrt{n} = 0), \tag{I.10.5}$$

but we constructed a sequence for which $\liminf a_k/k^2 < \infty$.

One can show that there exists such a sequence for which

$$a_k < ck^3 \qquad \text{for all } k . \tag{I.10.6}$$

There is a considerable gap between (I.10.5) and (I.10.6), which at present I can not fill.

RÉNYI and I proved by using probabilistic methods that to every ε there exists an $l = l(\varepsilon)$ and a sequence $a_1 < a_2 < \ldots$ for which $a_k < k^{2+\varepsilon}$ and the number of solutions of $n = a_i + a_j$ is less than l.

ERDŐS—TURÁN: "On the problem of Sidon in additive number theory and on some related problems." *Journal London Math. Soc.* **16** (1941) 212—215.

J. SINGER: "A theorem in finite projective geometry and some applications to number theory." *Trans. Amer. Math. Soc.* **43** (1938) 377—385.

R. H. BRUCK and H. J. RYSER: "The nonexistence of certain finite projective planes." *Canadian Journal of Math.* **1** (1949) 88—93.

P. ERDŐS and A. RÉNYI: „Additive properties of random sequences of positive integers." *Acta Arithmetica* **6** (1940) 83—110.

For the problems considered in 10. and 11. see also A. STÖHR "Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe, I. and II." *Journal für die reine und angewandte Math.* **194** (1955) 40—65 and 111—140, many interesting problems can be found in this paper.

Of the many problems discussed in STÖHR's paper I just wish to mention the following problem of ROHRBACH: What is the smallest number of integers $a_1 < a_2 < \ldots < a_{k_n}$ so that every integer $\leq n$ should be of the form $a_i + a_j$. The estimate $k_n > \sqrt{2n}$ is trivial and ROHRBACH improves this to $(1+\varepsilon)\sqrt{2n}$ for a fixed $\varepsilon > 0$. Recently MOSER obtained a better value for ε (L. MOSER, *Acta Arithmetica* **6** (1960) 11—13). Trivially $k_n \leq 2\sqrt{n}$ and perhaps $k_n = 2\sqrt{n} + O(1)$.

For a review of additive number theory see H. H. OSTMANN: *Additive Zahlentheorie*, Ergebnisse der Math. Heft 7. (two volumes).

11) Another problem of SIDON asked if there exists an infinite sequence of integers so that if $g(n)$ denotes the number of solutions of $n = a_i + a_j$,

then to every $\varepsilon > 0$ there exists an n_0 so that for $n > n_0$

$$0 < g(n) < n^{\varepsilon}. \tag{I.11.1}$$

I proved by probabilistic arguments that such a sequence exists, in fact I proved the existence of a sequence with

$$c_1 \log n < g(n) < c_2 \log n. \tag{I.11.2}$$

The existence of a sequence with $g(n)/\log n = c > 0$ is an open problem. An older conjecture of TURÁN and myself stated that if $g(n) > 0$ for all $n > n_0$ then lim sup $g(n) = \infty$ (perhaps even $g(n) > c \log n$ for infinitely many n, which would show that (I.11.2) is best possible). Our conjecture seems rather difficult. A stronger conjecture would be: if $a_k < ck^2$ for all k then lim sup $g(n) = \infty$. This would imply our original conjecture, but is perhaps easier to attack; all we can show is that lim sup $g(n) > 1$ (see STÖHR's paper quoted in 10.1).

TURÁN and I conjectured that if $a_1 < a_2 < \ldots$ is any infinite sequence of integers then

$$\sum_{k=1}^{n} g(k) = cn + O(1) \tag{I.11.3}$$

is impossible. FUCHS and I proved the following stronger theorem:

$$\sum_{k=1}^{n} g(k) = cn + o\left(\frac{n^{1/4}}{(\log n)^{1/2}}\right) \tag{I.11.4}$$

is impossible for $c > 0$. In the case $a_k = k^2$ HARDY and LANDAU proved that

$$\sum_{k=1}^{n} g(k) - \frac{\pi}{4} n \neq o\left((n \log n)^{1/4}\right). \tag{I.11.5}$$

In the case $a_k = k^2$ (this is the classical problem of the lattice points in the circle) it has been conjectured that for every $\varepsilon > 0$

$$\sum_{k=1}^{n} g(k) = \frac{\pi}{4} n + O(n^{1/4+\varepsilon}). \tag{I.11.6}$$

(I.11.6) is very deep. It is very likely that (I.11.4) is very close to being best possible, but we have not been able to prove this. Very recently JURKAT improved the error term in (I.11.4) to $o(n^{1/4})$.

It would be of interest to show that the number of solutions of $a_i + a_j + a_r \leqq n$ can not be of the form $cn + O(1)$, but this, and possible generalizations in the direction of (I.11.4) have not yet been done.

Very recently H. E. RICHERT proved the following result:

Let $a_1 < a_2 < \ldots$ be any sequence. Then

$$\sum_{kl \leqq n} a_k a_l = n \log n + cn + O(n^{\alpha}) \tag{I.11.5}$$

and

$$\sum_{k=1}^{n} a_k = n + O(n^{\alpha}) \tag{I.11.6}$$

can not both hold if $\alpha < \frac{1}{4}$. Perhaps the condition (I.11.6) is superfluous [perhaps the error term in (I.11.5) has to be changed].

P. Erdős: "On a problem of Sidon in additive number theory," *Acta Szeged* **11** (1954) 255—259, see also the paper quoted in I. 9).

P. Erdős and W. H. J. Fuchs: "On a problem of additive number theory." *Journal London Math. Soc.* **31** (1956) 67—73.

E. Landau: *Vorlesungen über Zahlentheorie*, Vol. 2.

12) Lorenz proved the following conjecture of Straus and myself: to every infinite sequence of integers $a_1 < a_2 < \ldots$ there exists a sequence of density 0, $b_1 < b_2 < \ldots$ so that every sufficiently large integer can be expressed in the form $a_i + b_j$. In particular he proved that if the a's are the primes then the b's can be chosen so that $B(x) < c\,(\log x)^3$. I improved this to $B(x) < c\,(\log x)^2$ $\left(B(x) = \sum_{b_i \leq x} 1\right)$. Perhaps such a sequence exists satisfying $B(x) < c \log x$. From the prime number theorem $c \geqq 1$. I can not prove that $c > 1$. This would follow from the following general conjecture of H. Hanani (oral communication): Let $a_1 < a_2 < \ldots$; $b_1 < b_2 < \ldots$ be two infinite sequences of integers so that every sufficiently large n can be written in the form $a_i + b_j$. Then

$$\limsup_{x=\infty} A(x)\,B(x)/x > 1\,. \tag{I.12.1}$$

Does there exist a sequence $b_1 < b_2 < \ldots$ satisfying $B(x) < \frac{cx}{\log x}$ so that every sufficiently large integer can be written in the form $2^k + b_i$? Lorenz's result only gives $B(x) < cx\,\mathrm{loglog}x/\log x$.

G. G. Lorenz: „On a problem of additive number theory." *Proc. Amer. Math. Soc.* **5** (1954) 838—841.

P. Erdős: „Some results on additive number theory." Ibid. 847—853., see also my paper quoted in I. 9).

W. Narkiewicz: „Remarks on a conjecture of Hanani in number theory." *Coll. Math* **7** (1960) 161—165.

13) A sequence $b_1 < b_2 < \ldots$ was called by Khintchine an essential component if for every $a_1 < a_2 < \ldots$ of positive density α the Schnirelmann sum of the two sequences has density greater than α. By density we mean here Schnirelmann density i.e. the greatest lower bound of $A(n)/n$, $1 \leqq n < \infty$. The Schnirelmann sum of a_i and b_j, $1 \leqq i, j < \infty$ is the set of integers of the form $\{a_i, b_j, a_i + b_j\}$. I proved, extending previous results of Khintchine and Buchstab, that every basis is an essential component, a sequence $b_1 < b_2 < \ldots$ is called a basis if there exists an integer k so that every integer is the sum of k or fewer b's. Linnik proved that an essential component does not have to be a basis, in fact he constructed an essential component for which $B(x) = o(x^\varepsilon)$ for every $\varepsilon > 0$. Linnik informed me that he can construct an essential component satisfying $B(x) < \exp\left[(\log x)^{1-c}\right]$. It seems to me that if $b_{i+1}/b_i > c > 1$ then the sequence b_i can not be an essential component, but I have not been able to show this (it is easy to show this for $b_i = 2^i$). Perhaps $B(x)/\log x \to \infty$ holds for every essential component.

Does there exist an essential component b_i for which there does not exist a function $f(\alpha)$, satisfying $f(\alpha) > 0$ for $0 < \alpha < 1$, so that if a_i has Schnirelmann density α the Schnirelmann sum of the two sequences

has density $\geqq \alpha + f(\alpha)$? (I was recently informed by E. WIRSING that he proved in his unpublished dissertation 10 years ago that such an essential component does not exist).

P. ERDŐS: "On the arithmetical density of the sum of two sequences one of which forms a basis for the integers." *Acta Arithmetica* **1** (1936) 197—200.

U. V. LINNIK: "On Erdős's theorem on the addition of numerical sequences." *Mat. Sbornik* **10** (1942) 67—78, see also A. STÖHR and E. WIRSING: "Beispiele von wesentlichen Komponenten die keine Basen sind.« *Journal reine und angewandte Math.* **196** (1956) 96—98.

14) ROMANOFF proved that for every integer $a > 1$ the density of integers of the form $p + a^k$ is positive (p runs through the primes). L. KALMÁR asked me a few years ago if for every $A > 1$ the density of integers of the form $p + [A^k]$ is > 0. The answer no doubt is affirmative, but I have not been able to prove it.

I proved that if $g(n)$ denotes the number of solutions of $p + 2^k = n$, then $\limsup g(n) = \infty$, in fact $g(n) > c \log\log n$ i. o. It seems that 105 is the largest integer n for which all the integers $n - 2^k$, $2 \leqq 2^k < n$ are primes.

Let now $1 \leqq a_1 < a_2 < \ldots$ be a sequence of integers satisfying $A(x) > c \log x$. Denote by $g(n)$ the number of solutions of $a_i + p = n$. Is it true that $\limsup g(n) = \infty$? Clearly analogous questions could be asked if the primes are replaced by other sequences.

N. P. ROMANOFF: »Über einige Sätze der additiven Zahlentheorie.« *Math. Annalen* **109** (1934) 668—678.

P. ERDŐS: "On integers of the form $2r + p$ and some related problems." *Summa Brasil. Math.* **2** (1947—51) 113—123.

15) Denote by $A_2(x)$ the number of distinct integers not exceeding x which are of the form $a_i + a_j$. I conjectured that if $\lim A(x)/x = 0$ then

$$\limsup A_2(x)/A(x) \geqq 3. \tag{I.15.1}$$

It is easy to see that (I.15.1) holds with 2 instead of 3 and that if (I.15.1) is true it is best possible.

H. MANN: "A refinement of the fundamental theorem on the density of the sum of two sets of integers." *Pacific Journal of Math.* **10** (1960) 909—915.

16) ROTH conjectured that there exists an absolute constant c so that to every k there exists an $n_0 = n_0(k)$ which has the following property: Let $n > n_0$, split the integers not exceeding n into k classes $\{a_i^{(j)}\}$, $1 \leqq j \leqq k$. Then the number of distinct integers not exceeding n which for some j, $1 \leqq j \leqq k$ can be written in the form $a_{i_1}^{(j)} + a_{i_2}^{(j)}$ is greater than cn.

17) Let $(a, b) = 1$. I conjectured and BIRCH proved that every sufficiently large integer can be expressed as the sum of distinct integers of the form $a^k b^l$, $0 \leqq k, l < \infty$.

Let $a_1 < a_2 < \ldots$ be an infinite sequence satisfying $a_{i+1}/a_i \to 1$, I conjectured that if every arithmetic progression contains infinitely many integers which are the sum of distinct a's then every sufficiently large integer is the sum of distinct a's. This was disproved by CASSELS, who also proved a weaker sufficient condition that every integer should be the sum of distinct a's.

CASSELS'S beautiful work (which incidentally contains BIRCH's result as a special case) leads one to the following conjecture: Let $a_1 < a_2 < \ldots$

be an infinite sequence of integers satisfying

$$A(2x) - A(x) \to \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \{a_k \alpha\} = \infty, \qquad 0 < \alpha < 1 \tag{I. 17.1}$$

where $\{n\alpha\}$ is the distance of α from the nearest integer. Then every sufficiently large integer is the sum of distinct a's. CASSELS proved this under the assumption of (c sufficiently large)

$$\frac{A(2x) - A(x)}{\log \log x} > C, \quad \sum_{k=1}^{\infty} \{a_k \alpha\}^2 = \infty, \qquad 0 < \alpha < 1 \tag{I. 17.2}$$

I conjectured that for every β, $1 < \beta < 2$ every sufficiently large integer is the sum of distinct integers of the form $[\beta^k]$. CASSELS observed that this fails to be true if $[\beta^k]$ gets replaced by the nearest integer to β^k.

B. J. BIRCH: „Note on a problem of Erdős." *Proc. Cambridge Phil. Soc.* **55** (1959) 370—373.

J. W. S. CASSELS: „On the representation of integers as the sums of distinct summands taken from a fixed set." *Acta Szeged* **21** (1960) 111—124.

18) Let $a_1 < a_2 < \ldots < a_n \leqq 2n$ be n arbitrary integers. Denote by $b_1 < b_2 < \ldots < b_n$ the other integers $\leqq 2n$. Denote by M_k the number of solutions of $a_i - b_j = k$. Put

$$M = \min \max_{-2n \leqq k < 2n} M_k$$

where the minimum is taken over all sequences $a_1, a_2, \ldots, a_n$.

I proved $M > \frac{n}{4}$, SCHERK improved this to $\left(1 - \frac{1}{\sqrt{2}}\right) n$ and SWIERCZKOWSKI proved $\frac{4 - \sqrt{6}}{5} n$.

MOSER proved in a very simple and ingenious way that

$$M > \frac{\sqrt{2}}{4} (n - 1)$$

and by more complicated arguments he can prove

$$M > \sqrt{4 - \sqrt{15}}\,(n - 1) > 0{\cdot}3570\,(n - 1).$$

SELFRIDGE MOTZKIN and RALSTON showed that $M < \frac{2}{5} n$, which disproved my conjecture $M = \frac{n}{2}$. The problem of determining the exact value of M is open.

P. ERDŐS: "Some results in number theory." (In Hebrew) *Riveon Lematematika* **9** (1955) 48.

S. SWIERCZKOWSKI: "On the intersection of a linear set with the translation of its complement." *Coll. Math.* **5** (1957) 185—197.

L. MOSER: "On the minimal overlap problem of Erdős." *Acta Arith.* **5** (1959) 117—119.

T. S. MOTZKIN, K. E. RALSTON and J. L. SELFRIDGE: "Minimal overlap under translation." *Abstract. Bull. Amer. Math. Soc.* **62** (1956) 558.

Now I state various problems on different topics of number theory.

19) Denote by $r_k(n)$ the maximum number of integers not exceeding n which do not contain an arithmetical progression of k terms. The first publication on $r_k(n)$ is due to TURÁN and myself where the conjecture $r_k(n) < n^{1-\varepsilon_k}$ was enunciated (the problem may be older but I can not definitely trace it. SCHUR gave it to HILDEGARD ILLE around 1930).

SALEM and SPENCER disproved $r_k(n) < n^{1-\varepsilon_k}$. In fact they shoved

$$r_3(n) > n^{1-c/\log\log n}. \tag{I. 19.1}$$

BEHREND improved this to

$$r_3(n) > n^{1-c/\sqrt{\log n}}. \tag{I. 19.2}$$

and MOSER constructed and infinite sequence which satisfies (I.19.2) for every n. ROTH proved $r_3(n) = o(n)$, more precisely he showed

$$r_3(n) < cn/\log\log n. \tag{I. 19.3}$$

For $k > 3$ the plausible conjecture $r_k(n) = o(n)$ is still open.

The inequality, $r_k(n) < (1-\varepsilon)n/\log n$, $1 \leqq k < \infty$, $n > n_0(k)$, would imply that for every k there are k primes in an arithmetic progression. Recently W. A. GOLUBIEFF observed that $23143 + l.\ 300\ 30$ is a prime for $0 \leqq l \leqq 11$. CHOWLA proved that there are infinitely many triplets of primes in an arithmetic progression.

VAN der WAERDEN proved that to every k there exists an $f(k)$ so that if we split the integers $\leqq f(k)$ into two classes at least one of them contains an arithmetic progression of k terms. If we could show that for some n $r_k(n) < \frac{n}{2}$, we clearly would have $f(k) \leqq n$, and in fact this observation led TURÁN and myself to the problem of estimating $r_k(n)$. VAN der WAERDEN's upper estimate for $f(k)$ is very bad, and unfortunately nobody succeeded in giving a better one. RADO and I proved that $f(k) > \left((k-1)2^k\right)^{1/2}$ (W. Schmidt just showed $f(k) > 2^{k-ck^{1/2}\log k}$, see Am. Math. Soc. Notices June 1961 p. 261.)

P. ERDŐS and P. TURÁN: „On some sequences of integers." *Journal London Math. Soc.* **11** (1936). 261–264.

R. SALEM and D. C. SPENCER: "On sets of integers which contain no three terms in an arithmetic progression." *Proc. Nat. Acad. Sci. USA* **28** (1942) 561–563.

F. A. BEHREND: "On sets of integers which contain no three terms in arithmetical progression." Ibid. **32** (1946) 331–332.

L. MOSER: "On non-averaging set of integers." *Canadian Journal of Math.* **5** (1953) 245–252.

S. CHOWLA: "There exists an infinity of 3-combinations of primes in A. P.," *Proc. Lahore Philos. Soc.* **6** (1944) no. 2 15–16.

B. L. VAN DER WAERDEN: "Beweis einer Baudet'schen Vermutung." *Nieuw Archiv Viskunde* (2) **15** (1928) 212–216.

P. ERDŐS and R. RADO: "Combinatorial theorems on classifications of subsets of a given set." *Proc. London Math. Soc.* (3) **2** (1952) 438–439.

20) SCHUR proved that if we split the integers $< en!$ into n classes the equation $x + y = z$ is always solvable in integers of the same class.

Denote by $f(n)$ the smallest integer with this property. It seems likely that $f(n)$ is very much less than $en!$, in fact it has been conjectured that $f(n) < c^n$ and $f(n)^{1/n} \to C$.

TURÁN proved (unpublished) that if one splits the integers $n < k \leqq 5n + 3$ into two classes then in at least one of them the equation $x + y = z$, $x \neq y$ is solvable, and that this is not true for $n < k \leqq 5n + 2$. The analogous problem for three classes is not yet solved.

I. SCHUR: *Jahresbericht der Deutschen Math. Ver.* **25** (1916) 114.
R. RADO: "Studien zur Kombinatorik" *Math. Z.* **36** (1933) 424–480.

In his interesting paper Rado considers very much more general problems.

21) Let $f(n)$ be an arbitrary number theoretic function which only assumes the values ± 1. Is it true that to every c_1 there exists a d and an m so that

$$g(m, d) = \left| \sum_{k=1}^{m} f(kd) \right| > c_1 \, ? \tag{I. 21.1}$$

It is perhaps even true that

$$\max_{\substack{d,m \\ dm \leqq n}} g(m, d) > c_2 \log n \, . \tag{I. 21.2}$$

If we assume that $f(a.b) = f(a)f(b)$ then (I.21.1) would imply

$$\limsup_{n=\infty} \left| \sum_{k=1}^{n} f(k) \right| = \infty \, . \tag{I. 21.3}$$

This conjecture is similar to the conjecture of VAN DER CORPUT on the discrepancy of sequences. Let $|z_k| = 1$, $1 \leqq k \leqq \infty$. Denote by $N(n\,; a, b)$ the number of z_i, $1 \leqq i \leqq n$ on the arc (a, b). The discrepancy $D(z_1, z_2, \ldots, z_n)$ is defined as follows:

$$D(z_1, z_2, \ldots, z_n) = \max \left| N(n\,; a, b) - \frac{b - a}{2\pi} n \right| ,$$

where the maximum is taken over all the arcs (a, b) of the unit circle.

VAN DER CORPUT conjectured and Mrs. VAN AARDENNE—EHRENFEST proved that for every infinite sequence z_i; $1 \leqq i \leqq \infty$, $|z_i| = 1$

$$\limsup_{n=\infty} D(z_1, z_2, \ldots, z_n) = \infty \, . \tag{I. 21.4}$$

(in fact she proved that $D(z_1, z_2, \ldots, z_n) > c \log\log n / \log\log\log n$ i. o.). ROTH proved that i. o.

$$D(z_1, z_2, \ldots, z_n) > c_1 (\log n)^{1/2} . \tag{I.21.5}$$

It is easy to see that there exists an infinite sequence for which

$$D(z_1, z_2, \ldots, z_n) < c_2 \log n$$

for every n and it seems possible that in (I.21.5) $c_1(\log n)^{1/2}$ can be replaced by $c_3 \log n$.

As far as I know the following two problems are still unsolved: Let $|z_i| = 1$, $1 \leq i < \infty$ be any infinite sequence. Does there exist a fixed arc (a, b) of the unit circle so that

$$\text{(I. 21.6)} \qquad \limsup \left| N(n; a, b) - \frac{b-a}{2\pi} n \right| = \infty ?$$

Is it true that

$$\text{(I. 21.7)} \qquad \limsup_{n=\infty} \max_{|z|=1} \prod_{i=1}^{n} |z - z_i| = \infty ?$$

If (I.21.7) and (I.21.6) hold one could try to determine how fast the left sides tend to infinity.

N. G. TCHUDAKOFF, quoted in problem 6.

VAN AARDENNE—EHRENFEST: "On the impossibility of a just distribution." *Indag. Math.* **11** (1949) 264—269.

K. F. ROTH: "On irregularities of distribution." *Matematika* **1** (1954) 73—79.

22) Let $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ be n arbitrary integers. Denote:

$$M(a_1, \ldots, a_n) = \max_{|z|=1} \left| \prod_{i=1}^{n} (1 - z^{a_i}) \right|, \quad f(n) = \min M(a_i, \ldots a_n)$$

where the minimum is to be taken over all sequences $a_1, a_2, \ldots, a_n$. SZEKERES and I proved that

$$\text{I.22.1)} \qquad \lim f(n)^{1/n} = 1, \quad f(n) > \sqrt{2n}.$$

Recently I proved (unpublished) that for some $c_1 > 0$

$$\text{(I.22.2)} \qquad f(n) < \exp(n^{1-c_1}).$$

It is quite possible that for some c_2 $f(n) > \exp(n)^{1-c_2})$, but we were not even able to prove that $f(n) > n^k$ for every k if $n > n_0(k)$.

My proof of (I.22.2) used probabilistic arguments. Very recently ATKINSON proved $f(n) > \exp(cn^{1/2} \log n)$ in a surprisingly simple way, in fact he proved that

$$\max_{|z|=1} \left| \prod_{k=1}^{n} (1 - z^k)^{n-k+1} \right| < \exp(cn \log n) .$$

P. ERDŐS and G. SZEKERES: "On the product $\prod_{k=1}^{n} (1 - Z^{a_k})$." *Acad. Serbe des Sci.* **13** (1959). 29—34.

F. V. ATKINSON: "On a problem of Erdős and Szekeres". *Can. Math. Bull.* **4** (1961) 7—12.

23) Denote by $f(k)$ the minimum number of terms in the square of a polynomial $\sum_{i=1}^{k} a_i z^{n_i}$. Sharpening a result of RÉNYI and RÉDEI I proved that $f(k) < k^{1-c}$ for a suitable $c > 0$. RÉNYI and I conjectured that $f(k) \to \infty$ as $k \to \infty$. This seems most plausible, but we have not yet been able to prove it.

A. RÉNYI, *Hungarica Acta Math.* **1** (1947) 30–34.

P. ERDŐS: "On the number of terms of the square of a polynomial." *Nieuw Arch. Wiskunde* (1949). 63–65.

W. VERDENIUS: "On the number of terms of the square and cube of polynomials." *Indag. Math.* **11** (1949) 459–465.

24) Does there exist to every c a system of congruences

$$a_i \pmod{n_i}, \quad c < n_1 < n_2 < \ldots < n_k \qquad (k = k(c)) \tag{I.24.1}$$

so that every integer satisfies at least one of them? DEAN SWIFT and SELFRIDGE constructed such congruences for $c < 8$.

Similarly one can ask if a system (I.24.1) exists where all the n_i are >1 and odd (or not divisible by the first r primes)?

STEIN and I asked the following question: What is the maximum number of congruences $a_i \pmod{n_i}$, $n_1 < n_2 < \ldots < n_{k_x} \leqq x$ so that no integer should satisfy two of them (i. e. the arithmetic progressions $a_i + ln_i$, $1 \leqq i \leqq k_x$ should be disjoint). We proved (unpublished) $k_x > x^{1-\varepsilon}$ for every $\varepsilon > 0$ if $x > x_0(\varepsilon)$. We conjecture $k_x = o(x)$.

P. ERDŐS: "On a problem on systems of congruences". (In Hungarian) *Matematikai Lapok* **4** (1952) 122–128.

25) Let $1 < a_1 < a_2 < \ldots$ be an infinite sequence of real numbers satisfying

$$|ka_i - a_j| \geqq 1 \tag{I.25.1}$$

for every k and $i \neq j$. Is it then true that

$$\lim_{x=\infty} \frac{1}{\log x} \sum_{a_i<x} \frac{1}{a_i} = 0\,, \tag{I.25.2}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i} < \infty ? \tag{I.25.3}$$

If the a's are integers (I.25.1) means that no a divides any other, in this case (I.25.2) was proved by BEHREND and (I.25.3) by me.

F. BEHREND: „On sequences of numbers not divisible one by another." *London Math. Soc. Journal* **10** (1935) 42–45.

P. ERDŐS: „Note on sequences of integers no one of which is divisible by any other." Ibid. 126–128.

26) Let $a_1 < a_2 < \ldots$ be an infinite sequence of integers, denote by $b_1 < b_2 < \ldots$ the sequence of integers no one of which is a multiple of any of the a's. BESICOVITCH constructed a sequence a_i for which the b's do not have a density. DAVENPORT and I proved that the b's always have a logaritmic density, i.e. that

$$\lim \frac{1}{\log x} \sum_{b_i<x} \frac{1}{b_i}$$

always exists.

Make correspond to each a_i a set of residues $u_j^{(i)}$, $1 \leq j \leq i_r$. Denote now by $b_1 < b_2 < \ldots$ the integers which do not satisfy for any i

$$b \equiv u_j^{(i)} (\mathrm{mod}\, a_i), \quad 1 \leq j \leq i_r, \quad b \geq a_i . \tag{I. 26.1}$$

Is it then true that

$$\lim \frac{1}{\log x} \sum_{b_i < x} \frac{1}{b_i} \tag{I. 26.2}$$

exists? (I.26.2) if true is a generalization of (I.26.1), ($i_r = 1$, $u_1^{(i)} = 0$ for all i).

DAVENPORT and I also proved that if $a_1, a_2, \ldots$ is a sequence of positive density, we can select an nfinite subsequence $a_{i_k} (1 \leq k < \infty)$ satisfying $a_{i_k} | a_{i_{k+1}}$. It is an open problem if three distinct a's exist satisfying $[a_i, a_j] = a_l$.

A. S. BESICOVITCH: "On the density of certain sequences of integers." *Math. Annalen* **110** (1934) 336–341.

H. DAVENPORT and P. ERDŐS: "On sequences of positive integers." *Acta Arithmetica* **2** (1937) 147–151, see also *Indian Journal of Math.* **15** (1951) 19–24.

P. ERDŐS: "Density of some sequences of integers." *Bull. Amer. Math. Soc.* **64** (1948) 685–692.

27) Is it true that the density of integers having two divisors d_1 and d_2 for which $d_1 < d_2 < 2 d_1$ is 1? In my paper just quoted in 26) I prove that this density exists, but I can not show that it is 1.

Let $a_1 < a_2 < \ldots \leq n$ be any sequence of integers, $b_1 < b_2 < \ldots$ the integers no one of which is a multiple of any a. $B(x) = \sum_{b_i \leq x} 1$. Is it true that for every $m > n$

$$\frac{B(m)}{m} < \frac{2\, B(n)}{n} \; ? \tag{I. 27.1}$$

It is easy to see that in (I.27.1) 2 can not be replaced by any smaller constant, to see this let the a's consist of a_1, $n = 2 a_1 - 1$, $m = 2 a_1$.

28) BAMBAH and CHOWLA proved that for sufficiently large C the interval $(n, n + C n^{1/4})$ always contains in integer of the form $x^2 + y^2$. It has been often conjectured but never proved that this holds every C if $n > n_0(C)$. In fact it seems likely that for every $\varepsilon > 0$ the interval $(n, n + n^{\varepsilon})$ contain in integer of the form $x^2 + y^2$. I proved that for a suitable $c > 0$ and infinitely many n the interval $(n, n + c \log n/(\log \log n)^{1/2})$ does not contain any integers of the form $x^2 + y^2$.

Denote by $s_1, s_2, \ldots$ the squarefree integers. It is easy to prove (I do not know who did it first) that i. o.

$$s_{i+1} - s_i > \big(1 + o(1)\big) \pi^2/6 \, \log s_i / \log \log s_i . \tag{I.28.1}$$

The question if $\big(1 + o(1)\big)$ in (I.28.1) can be replaced by $1 + c$ has not yet been decided. I proved that

$$\lim \frac{1}{n} \sum_{s_i < n} (s_{i+1} - s_i)^2 \tag{I. 28.2}$$

exists. More generally one could ask the following question: Let $a_1 < a_2 < \ldots$ be any sequence of integers satisfying $a_k/k^2 \to \infty$ and denote by $b_1 < b_2 < \ldots$ the sequence of integers no one of which is a multiple of any of the a' s. Is it then true that

$$\lim \frac{1}{n} \sum_{b_i < n} (b_{i+1} - b_i)^2 \tag{I. 28.3}$$

exists and is finite (in (I.28.2) $a_i = p_i^2$)? It is easy to see that if we only require $a_k < ck^2$ then (I.28.3) does not hold in general.

R. P. Bambah and S. Chowla: "On numbers which can be expressed as a sum of two squares." *Proc. Nat. Inst. Sci. India* **13** (1947) 101–103.

P. Erdős: "Some problems and results in elementary number theory." *Publ. Math. Debrecen* **2** (1951–52) 103–109.

As far as I know the best upper bound for $s_{i+1} - s_i$ is due to Richert, who improved a previous result of K. F. Roth. Richert proved $s_{i+1} - s_i < c\, s_i^{2/9} \log s_i$.

H. E. Richert: "On the difference between consecutive squarefree numbers." *London math. Soc. Journal* **29** (1954) 16–20.

29) Denote by $A(n)$ the number of integers not exceeding n which are the product of two integers not exceeding $n^{1/2}$ I proved that for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$

$$(\log n)^{-\varepsilon} \frac{n}{\log n} (e \log 2)^{\log \log n/\log 2} < A(n) < (\log n)^{\varepsilon} \frac{n}{\log n} (e \log 2)^{\log \log n/\log 2}. \tag{I. 29.1}$$

Let $a_1 < a_2 < \ldots < a_x < \sqrt{n}$; $b_1 < b_2 < \ldots < b_y < \sqrt{n}$ be two sequences of integers so that all the products $a_i b_j$ are distinct. Is it then true that $xy < c \dfrac{n}{\log n}$? This if true is certainly best possible, to see this choose the a's to be the integers not exceeding $\frac{1}{2} n^{1/2}$ and the b's the primes in $\left(\frac{1}{2} n^{1/2}, n^{1/2}\right)$.

P. Erdős: „Об одном асимптотическом неравенстве в теории чисел." *Вестник Ленинградского* университета **3** (1960) 41—49; for a weaker result see P. Erdös: „Some remarks in number theory" (In Hebrew.) *Riveon Lematematika* (1955) 45—48.

30) Let $f(n)$ be an additive function, i. e. $f(ab) = f(a) + f(b)$ if $(a, b) = 1$. Assume that $|f(n+1) - f(n)| < c_1$. Is it true that $f(n) = c_2 \log n + g(n)$, where $|g(n)| < c_3$. I proved that if $f(n+1) - f(n) \to 0$ or if $f(n+1) \geqq f(n)$ then $f(n) = c \log n$.

P. Erdős: "On the distribution function of additive functions." *Annals of Math.* **47** (1946) 1–20. My proofs of the above theorems were unnecessarily complicated and have been simplified by various authors.

Many interesting problems and results on additive functions can be found in the following three papers:

M. Kac: "Probability methods in some problems of analysis and number theory." *Bull. Amer. Math. Soc.* **55** (1949) 641–665.

Kubiljus, *Uspehi Matem. Nauk.* **11** (1956) 31–66.

P. Erdős, Proc. International Congress of Math. Amsterdam (1954) Vol. 3, 13–19.

31) The following problem is due to W. LE VEQUE: Let $a_1 < a_2 < \dots$ be an infinite sequence tending to infinity satisfying $a_{i+1}/a_i \to 1$. Let $a_i \leqq x_n < a_{i+1}$, put $y_n = \dfrac{x_n - a_i}{a_{i+1} - a_i}$, $0 \leqq y_n < 1$. We say that the sequence x_n, $1 \leqq n < \infty$ is uniformly distributed mod $a_1, a_2, \dots$ if y_n $1 \leqq n < \infty$ is uniformly distributed. Is it true that for almost all α the sequence $n\alpha$, $1 \leqq n < \infty$ is uniformly distributed mod $a_1, a_2, \dots$? LE VEQUE proved this in some s pecial cases.

W. J. LE VEQUE: "On uniform distribution modulo a subdivision." *Pacific J. of Math.* **3** (1953) 757–771.

32) STRAUS and I conjectured that for every integer $n > 1$

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is solvable in positive integers x, y, z. SCHINZEL conjectured that for every $a > 0$ if $n > n_0(a)$ $\dfrac{a}{n} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$ is solvable in positive integers x, y, z.

SCHINZEL conjectured that there exists a k so that every sufficiently large integer can be written in the form (a_i are integers)

$$\prod_{i=1}^{k} a_i - \sum_{i=1}^{k} a_i, \quad a_i \geqq 2, \quad 1 \leqq i \leqq k .$$

33) Problem of SELFRIDGE and STRAUS. Let $Z_1, Z_2, \dots, Z_n$ be n complex numbers, $\sigma_1, \sigma_2, \dots, \sigma_{\binom{n}{k}}$, are the products of Z's taken k at a time. The authors prove that if $k = 2$, $n \neq 2^l$ and the σ's are given, there can be at most one set of Z_i, $1 \leqq i \leqq n$ which generate them. For $n = 2^l$ this is not true, here they conjecture that there can be at most two sets of Z's which generate the σ's.

If $k > 2$ they conjecture the Z's (if they exist) are determined uniquely by the σ's and they prove this in many cases, but the general problem is unsolved.

J. L. SELFRIDGE and E. *Straus:* „On the determination of numbers by their sums of a fixed order." *Pacific Journal of Math.* **8** (1958). 847–856.

34) Problem of LITTLEWOOD. Let α and β be two real numbers. Is it true that

$$\liminf n(n\alpha)\,(n\beta) = 0 \tag{I. 34.1}$$

where $(n\alpha)$ denotes the distance of $n\alpha$ from the nearest integer? (I.34.1) is trivial except if both α and β have bounded partial quotients in their continued fraction development. (I.34.1) seems very deep, even if $\alpha = \sqrt{2}$, $\beta = \sqrt{3}$ say.

Another very difficult problem in the theory of diophantine approximation is the following one: DAVENPORT and HEILBRONN proved the inequality

$$\left| \sum_{k=1}^{5} \alpha_k n_k^2 \right| < \varepsilon \tag{I. 34.2}$$

is solvable for every $\varepsilon > 0$ in positive integers n_k if not all the c_k are of the same sign and at least two of them have irrational ratios.

It is not known if for every irrational α and $\varepsilon > 0$ the inequalities

$$|(x^2 + y^2)\,\alpha - z^2| < \varepsilon \quad \text{and} \quad |x^2 + y^2 - z^2\alpha| < \varepsilon$$

are solvable in integers. The case $\alpha = \sqrt{2}$ is also undecided.

H. DAVENPORT and H. HEILBRONN: "On indefinite quadratic forms in five variables," *London Math. Soc. Journal* **21** (1946) 185—193.

Several unsolved arithmetical problems are stated in a recent paper of SIERPINSKI *L'Enseignement Mathématique* 5 (1960) 221—235, an English version appeared in *Scripta Math.* 25 (1960) 125—136.

II. Problems in combinatorial analysis and set theory

1) Let $a_1, a_2, \ldots, a_k$ be n elements. $A_1, A_2, \ldots, A_n$ are k sets formed from the a's so that no A can contain any other. SPERNER proved that

$$\max k = \binom{n}{\left[\frac{n}{2}\right]} \tag{II. 1. 1}$$

(II. 1. 1) has several applications in number theory, e. g. BEHREND's result (I.25.1) is proved by using (II.1.1).

The question has been considered that in how many ways can one select sets A_i so that no A should contain any other. Denote this number by $A(n)$. From (II.1.1) we have

$$2^{\binom{n}{\left[\frac{n}{2}\right]}} < A(n) < \binom{2^n}{T_n}, \text{ where } T_n = \binom{n}{\left[\frac{n}{2}\right]}.$$

It seems that $A(n) < \exp(c\,T_n)$, perhaps c can be chosen to be $(1 + \varepsilon)\log 2$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$.

How many sets $A_1, A_2, \ldots, A_l$ can one give so that the union of two of them never equals a third? (all three sets are supposed to be distinct i. e. $A_i \subset A_j$, $A_l \cup A_j = A_j$ is not permitted). I conjectured for a long time that $l = o(2^n)$. If I could prove this the following result in number theory would follow: Let $a_1 < a_2 < \ldots$ be an infinite sequence of positive density, then there are infinitely many triplets of distinct integers a_i, a_j, a_k satisfying $[a_i, a_j] = a_k$ (see problem I 26).

It is possible that $l < (1 + o(1))\,T_n$.

Several other problems can be asked e. g. How many sets can one give so that the union of any two of them never contains a third? How many sets

A_i can one give so that the symmetric difference of any two sets should contain at least r elements?

E. SPERNER: »Ein Satz über Untermengen einer endlichen Menge.« *Math. Zeitschrift* **27** (1928) 544—548.

2) As far as I know R. PELTESOHN and SUTHERLAND (unpublished) were the first to construct an infinite sequence formed from the symbols 0, 1, 2 where no two consecutive blocks were identical. It is easy to see that in a sequence of length four formed from the symbols 0 and 1 two consecutive blocks will be identical, I understand that EUWE proved that in an infinite sequence formed from 0 and 1 there will be arbitrarily large identical consecutive blocks, but that there do not have to be three consecutive identical blocks.

Let us now call two consecutive blocks „identical" if each symbol occurs the same number of times in both of them (i.e. we disregard order). I conjectured that in a sequence of length 2^k-1 formed from k symbols there must be two "identical" blocks. This is true for $k \leqq 3$, but for $k = 4$ de BRUIJN and I disproved it and perhaps an infinite sequence of four symbols can be formed without consecutive "identical" blocks.

3) LITTLEWOOD and OFFORD proved the following result: Let Z_i, $1 \leqq i \leqq n$ be n complex numbers. Then there exists an absolute constant c so that the number of sums

$$\sum_{i=1}^{n} \varepsilon_i Z_i, \quad \varepsilon_i = \pm 1 \tag{II. 3.1}$$

which fall into the interior of an arbitrary circle of radius 1 is less than $c\frac{2^n \log n}{n^{1/2}}$. I proved that if $Z_i \geqq 1$, $1 \leqq i \leqq n$ (i. e. the Z_i's are real) then the number of sums (II.3.1) which fall into the interior of any interval of length two is at most $\binom{n}{\left[\frac{n}{2}\right]}$ and this estimation is best possible. The proof uses the theorem of SPERNER (see problem II.1.). I do not know if this inequality remains true if the Z_i are complex numbers (my proof gives for complex z $c\,2^n/\sqrt{n}$), or more generally vectors of Hilbert space of norm $\geqq 1$. In this case I can only prove that the number of summands (II.3.1) falling into an arbitrary unit sphere is $o(2^n)$.

J. E. LITTLEWOOD and C. OFFORD, *Mat. Sbornik.* **12** (1943) 277—285.
P. ERDŐS: "On a Lemma of Littlewood and Offord." *Bull. Amer. Math. Soc.* **51** (1945) 898—902.

4) RAMSAY proved that there exists a function $f(i, k, l)$ so that if we split the i-tuples of a set of $f(i, k, l)$ elements into two classes then either there are k elements all whose i-tuplets are in the first class or l elements al whose i-tuples are in the second class. SZEKERES and I proved that

$$2^{k/2} < f(2, k, k) \leqq \binom{2k-2}{k-1}; \qquad f(2, k, l) \leqq \binom{k+l-2}{k-1}. \tag{II. 4.1}$$

The best estimation for $f(i, k, k)$, $i > 2$ is due to RADO and myself.

It would be interesting to determine $f(i, k, l)$ explicitely, this seems very difficult even for $i = 2$. I have not even be able to prove that $\lim_{i=\infty} f(2, k, k)^{1/k}$ exists. I can prove that

$$f(2, 3, k) > ck^2/(\log k)^2 \tag{II. 4.2}$$

but could not decide whether $f(2, 3, k) > c_2 k^2$ is true.

I do not wish to mention here the many problems connected with the generalisations of RAMSAY's theorem to cardinal and ordinal numbers and just state one of the simplest unsolved problems in this subject.

Let φ be a well ordered set of ordinal number ω^α, $\alpha < \Omega$. Split the pairs $a \in \varphi$, $b \in \varphi$ into two classes so that there is no triplet all whose pairs are in the first class. Does there then exist a set $\varphi' \subset \varphi$ of type ω^α all whose pairs are in the second class?

For $\alpha = 2$ this was proved by SPECKER, for $2 < \alpha < \omega$ it was disproved by him, for $\alpha \geqq \omega$ the problem is open. The most interesting unsolved case is $\alpha = \omega$.

E. P. RAMSAY: "On a problem of formal logic." Collected papers, 82—111. See also T. H. SKOLEM: »Ein kombinatorischer Satz mit Anwendung auf ein logisches Entscheidungsproblem.« *Fund. Math.* **20** (1933) 254—261.

P. ERDŐS and G. SZEKERES: "A Combinatorial Problem in geometry." *Compositio Math.* **2** (1935) 463—470.

P. ERDŐS: "Remarks on a theorem of Ramsay." *Bull. Res. Council. Israel* (1957) 21—24. See also "Graph theory and probability." *Can. Journal of Math.* I and II, **11** (1959) 34—38, **13** (1961) 346—352.

E. SPECKER: »Teilmengen von Mengen mit Relationen.« *Comm. Math. Helv.* **31** (1956—57) 302—314.

P. ERDŐS and R. RADO: „A partition calculus in set theory." *Bull. Amer. Math. Soc.* **62** (1956) 427—489. (See also the forthcoming triple paper of ERDŐS—HAJNAL—RADO.)

5) Let $a_1, a_2, \ldots, a_n$ be n elements $A_1, A_2, \ldots, A_k$, $k > 1$ sets whose elements are the a's. Assume that each pair (a_i, a_j) is contained in one and only one A. Then $k \geqq n$. This is a result of de BRUIJN and myself (also proved by SZEKERES and HANANI). We can not determine the smallest l so that there should exist sets $A_1, A_2, \ldots, A_l$, $l > 1$ so that every triplet (a_i, a_j, a_r) is contained in one and only one A.

N. G. DE BRUIJN and P. ERDŐS: "On a combinatorial problem." *Ind. Math.* (1948) 421—423.

C. STEINER conjectured that if $n = 6k + 1$ or $6k + 3$ there exists a system of triplets of n elements so that every pair is contained in one and only one triplet (if n is not of the above form it is easy to see that such a system can not exist). STEINER's conjecture was first proved by REISS and later independently by MOORE.

Let now $2 \leqq r < s$ be any two integers. For which n is there a system of combinations taken s at a time formed from n elements so that all r tuples should be contained in one and only one s tiuple. The case $r = 2$, $s = 3$ is STEINER's. The only other case which has been settled is $r = 3$, $s = 4$ H. HANANI recently proved that such a system exists if and only if $n \equiv 2$ or 4(mod 6). (Very recently HANANI settled the cases $r = 2$, $s = 4$ and $r = 2$, $s = 5$).

It has been known for a long time that if $n = p^{2l} + p^l + 1$ (p prime), $r = 2$, $s = p^l + 1$, then there exist n $(p^l + 1)$ — tuples so that every pair is contained in one and only one $(p^l + 1)$-tuplet. If $n = k^2 + k + 1$, $k \neq p^\alpha$

it has been conjectured that such a system of $(k+1)$-tuplets does not exist. Special cases of this conjecture have been proved by BRUCK and RYSER, the first unsettled case is $k = 10$ (see I.10).

Connected with this problem is the following conjecture of SYLVESTER: For every $n \equiv 0 \pmod 4$ there exists an orthogonal matrix of order n all whose elements are ± 1 (it is easy to see that if $n \not\equiv 0 \pmod 4$ such a matrix does not exist.) If $n = 2^k$ SYLVESTER showed that such a matrix exists, if $p \equiv -1 \pmod 4$. PALEY proved that such a matrix exists for $p+1$, the general case is still unsolved.

Denote by M_n the maximum value of an n by n determinant whose elements are ± 1. From HADAMARD's theorem it follows that $M_n \leqq n^{n/2}$ and if a SYLVESTER matrix exists $M_n = n^{n/2}$. It follows easily from the prime number theorem for arithmetic progressions that for every $n > n_0(\varepsilon)$

$$M_n > (1-\varepsilon)^n\, n^{n/2}. \tag{II.6.1}$$

COLUCCI and BARBA proved that if $n \not\equiv 0 \pmod 4$ then

$$M_n < (2n-1)^{1/2}\,(n-1)^{(n-1)/2} = \left(1+o(1)\right)\left(\frac{2}{e}\right)^{\frac{1}{2}} n^{n/2}. \tag{II.6.2.}$$

M. REISS: »Über eine Steinersche kombinatorische Aufgabe.« *J. reine und angewandte Math.* **56** (1859) 326–344.

E. H. MOORE: "Concerning triple systems." *Math. Annalen* **43** (1893) 271–285. See also „Practical memoranda." *Amer. J. Math.* **18** (1896) 264–303.

H. HANANI: "On quadruple systems." *Can. J. Math.* **12** (1960) 145–157.

J. H. SYLVESTER: "Thoughts on inverse orthogonal matrices." *Phil. Mag.* (4) **24** (1867) 461–475.

R. E. A. C. PALEY: "On orthogonal matrices." *Journal of Math. and Phys* (1933) 311–320.

COLUCCI: «Sui valori massimi dei determinanti ad elementi ± 1.» *Gior. di Matem. di Battaglini* 54. See also G. BARBA ibid. 71.

See also G. SZEKERES and P. TURÁN: „An extremal problem in the theory of determinants." (In Hungarian, German summary) *Sitzungsber. III. Klasse Ung. Akad.* **54** (1937) 796–806.

7) Problem of VAN der WAERDEN: Let $|a_{i,k}|$ be an n by n doubly stochastic matrix $\left(\text{i. e. } a_{ik} \geqq 0 \text{ and } \sum_{i=1}^{n} a_{i,k} = \sum_{k=1}^{n} a_{ik} = 1 \text{ for every } i \text{ and } k\right)$. Then the value of the permanent is $\geqq \frac{n!}{n^n}$, equality only for $a_{i,k} = \frac{1}{n}$. The permanent (a terminology of SYLVESTER) is the sum of the expansion terms of the determinant. The fact that the permanent of a doubly stochastic matrix can not be 0 is a theorem of FROBENIUS—KÖNIG. VAN der WAERDEN's problem seems to be difficult.

I made the following two weaker conjectures: The value of at least one term of the permanent is $\geqq \frac{1}{n^n}$, and the still weaker one: There is at least one non-zero expansion term of the permanent where the sum the factors is $\geqq 1$. This was proved by R. REE and S. MARCUS $\left(\text{in fact they prove that the sum is } \geqq \frac{1}{n} \sum_{1 \leqq i,k \leqq n} a_{i,k}^2\right)$.

R. REE and S. MARCUS: „Diagonals of doubly stochastic matrices." *Quarterly Journal of Math.* **10** (1959) 296–301.

8) A special case of a theorem of TURÁN states that if in a graph of n vertices the number of edges is greater than $\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)$, then the graph always contains a triangle. He points out that the following analogous problem is unsolved: Let there be given n elements what is the smallest number $f(n)$ so that to every system φ of $f(n)$ triplets formed from the n elements there are always four elements all four triplets of which occur in φ.

P. TURÁN: "On the theory of graphs." *Coll. Math.* **3** (1954) 19–30.
D. KŐNIG: *Theorie der endlichen und unendlichen Graphen.*

9) HAJNAL and I proved the following theorem: To every real x make correspond a bounded set of real numbers $f(x)$ whose outer measure is less than 1. Then for every finite k there exists an independent set of k elements (i. e. a set $x_1, x_2, \ldots, x_k$ so that for every $1 \leqq i, j \leqq k$, $i \neq j$, $x_i \notin f(x_j)$). We can not prove that there always exists an infinite independent set (not even if we also assume that the sets $f(x)$ are compact.)

If we assume that the sets $f(x)$ are closed and of measure < 1, we can not even prove that there are two independent points. (Recently GLADYSZ proved in a very ingenious way the existence of two independent points. The existence of an independent triplet is open).

P. ERDŐS and A. HAJNAL: „Some remarks on set theory, VIII." *Michigan Math Journal* **7** (1960) 187—191, for further problems in this direction see P. ERDŐS and A. HAJNAL: "On the structure of set mappings. *Acta Math. Hung.* **9** (1958) 111–131. and P. ERDŐS: "Some remarks on set theory." **3** (1953) 51–57.

III. Problems in elementary geometry

1) Let $x_1, x_2, \ldots, x_n$ be n points in the plane. Denote by $M_n(x_1, x_2, \ldots, x_n)$ the number of distinct distances between any two of the points. Put

$$f(n) = \min M_n(x_1, x_2, \ldots, x_n),$$

where $x_1, x_2, \ldots, x_n$ ranges over all sets of n distinct points of the plane. It seems to be difficult to get a good estimate for $f(n)$, the best results (due to MOSER and myself are)

$$c_1 n^{2/3} < f(n) < c_2 n/\sqrt{\log n}. \tag{III. 1.}$$

I would guess that the upper bound is the right one and perhaps even the following result holds: There is one point x_i so that amongst the distances (x_j, x_i) there are at least $c_3 n/\sqrt{\log n}$ distinct ones.

If the set $x_1, x_2, \ldots, x_n$ is convex it seems that $f(n) = \left[\frac{n}{2}\right]$; despite its seeming simplicity I have not been able to prove this. A somewhat stronger conjecture is: In every convex polygon there is a vertex which has no three vertices equidistant from it.

How often can the same distance occur between n points of the plane? Denote this maximum by $g(n)$. I proved

$$n^{1+c_4/\log\log n} < g(n) < n^{3/2}.$$

I believe that the lower bound is close to being the correct one.

COXETER asked me how many points does one have to have in n-dimensional space so that one should be sure to have more than two distinct distances between them. I stated that for c_5 sufficiently large n^{c_5} points suffice, but my proof was wrong and if corrected it only gave $\exp(n^{1-\varepsilon})$.

One can show that from 7 points in the plane one can always find three of them which do not determine an isosceles triangle, it is easy to see that this is false for 6 points. How many points does one have to have in n-dimensional space to be sure that one can find three of them which do not determine an isosceles triangle? This is not even known for $n = 3$.

P. ERDŐS: "On sets of distances of n points." *Amer. Math. Monthly.* **54** (1946) 248–250.

L. MOSER: "On the different distances determined by n points." Ibid. **59** (1952) 85–91.

P. ERDŐS: "On some problems in geometry." (In Hungarian) *Mat. Lapok* (1954) 86–92. Many further problems are stated in this paper.

2) BLUMENTHAL's problem. Let there be given n points in the plane, denote by $A(x_1, x_2, \ldots, x_n)$ the largest angle ($\leq \pi$) determined by the n points, and define

$$\alpha_n = \inf A(x_1, x_2, \ldots, x_n)$$

where the minimum is taken over all sets of n points. SZEKERES proved that $\alpha_{2^n+1} > \pi\left(1 - \frac{1}{n} + \frac{1}{n(2^n+1)^2}\right)$ and that for every $\varepsilon > 0, 2^n$ points can be given with $A(x_1, x_2, \ldots, x_{2^n}) > \pi\left(1 - \frac{1}{n} + \varepsilon\right)$ $\left(\text{this implies } \alpha_{2^n} \leq \pi\left(1 - \frac{1}{n}\right)\right.$.

SZEKERES and I recently proved that $\alpha_{2^n} = \pi\left(1 - \frac{1}{n}\right)$ and in fact for every 2^n points $A(x_1, x_2, \ldots, x_{2^n}) > \pi\left(1 - \frac{1}{n}\right)$, we also showed $\alpha_{2^n-1} = \pi\left(1 - \frac{1}{n}\right)$.

Let there be given $2^n + 1$ points in n dimensional space. I conjectured that there are always three of them which determine an angle $> \frac{\pi}{2}$. This is trivial for $n = 2$, for $n = 3$ it was proved by (unpublished) N. H. KAIPER and A. H. BOERDIJK. For $n > 3$ the problem is open. (Recently this conjecture was proved by L. DANZER and G. GRÜNBAUM in a simple and ingenious way.)

G. SZEKERES: "On an extremum problem in the plane." *Amer. Journal of Math.* **53** (1941) 208–210. Our paper with SZEKERES will appear in the *Annales of the Univ. of Budapest* **3** (1961).

3) BORSUK's problem. Is it true that every set of diameter one in n dimensional space is the union of $n + 1$ sets of diameter < 1? This is trivial for $n = 1$, easy for $n = 2$. For $n = 3$ it was first proved by Eggleston and later simultaneously and independently GRÜNBAUM and HEPPES found a considerably simpler proof. The problem is open for $n > 3$.

BORSUK and ULAM proved that the n dimensional sphere is not the union of n sets of smaller diameter.

K. BORSUK: »Drei Sätze über die n-dimensionale euklidische Sphäre.« *Fundamenta Math.* **20** (1933) 177—190.

H. G. EGGLESTON: "Covering a three dimensional set with sets of smaller diameter." *Journal of Lond. Math. Soc.* **30** (1955) 11—24.

B. GRÜNBAUM: „A simple proof of Borsuk's conjecture in three dimensions." *Proc. of Cambridge Phil. Soc.* **53** (1957) 776—778.

A. HEPPES—P. RÉVÉSZ: »Zum Borsukschen Zerteilungsproblem." *Acta Math. Acad. Sci. Hung.* **7** (1956) 159—162.

A. HEPPES: „Térbeli ponthalmazok felosztása kisebb ármérőjű részhalmazok összegére." *MTA III. Oszt. Közl.* **7** (1957) 413—416.

H. HADWIGER: »Über die Zerstückung eines Eikörpers.« *Math. Zeitschr.* **51** (1949) 161—165.

H. LENZ: "Zur Zerlegung von Punktmengen in solche kleineren Durchmessers.« *Arch. Math.* **6** (1955) 413—416.

4) SYLVESTER conjectured and GALLAI first proved that if we have n points, not all on a line then there is at least one line which goes through exactly two of the points. Denote by G_n the minimum number of such lines, de BRUIJN and I conjectured that $G_n \to \infty$ as $n \to \infty$. This was proved by MOTZKIN (his paper contains many more problems and results in this direction). MOSER and KELLY proved that $G_n \geqq \left[\frac{3n}{7}\right]$ and this is best possible for $n = 7$. For $n > n_0$ perhaps $G_n = n - 1$. For large n perhaps there always is a triangle all whose lines goes through only two of our points (except if n—1 of them are on a line).

SYLVESTER asked: Let there be given n points no four on a line. What is the maximum number of lines which goes through three of them? He proved that this maximum is greater than $\frac{1}{3}\binom{n}{2} - cn$ on the other hand the maximum is $< \frac{1}{3}\binom{n}{2}$.

Let there be given n points not all on a line I observed that it easily follows from GALLAI's result that these points determine at least n lines (see also II.5). G. DIRAC conjectured that there always exists a point which is connected with the other points by more than cn lines.

Let there be given n points not all on a circle. What is the minimum number of circles these points determine? This problem is unsolved (see also II.5).

T. H. MOTZKIN: "The lines and planes connecting the points of a finite set." *Trans. Amer. Math. Soc.* **70** (1951) 451—464. This paper contains many more problems and results and also the history of this problem and many references to the literature.

5) Miss KLEIN asked: Does there exist for every n an $f(n)$ so that if $f(n)$ points in the plane are given no three on a line then there always exist n of them which are the vertices of a convex polygon. She proved $f(4) = 5$ and MAKAI and TURÁN proved that $f(5) = 9$. SZEKERES conjectured $f(n) = 2^{n-2} + 1$. He and I proved

$$2^{n-2} \leqq f(n) \leqq \binom{2n-4}{n-2}. \tag{III. 5.1.}$$

P. ERDŐS and G. SZEKERES: "A combinatorial problem in geometry", *Compositio Math.* 2 (1935) 463—470. The proof of the lower bound in (III. 5. 1) will appear in the *Annales of the Univ. of Budapest.*

6) HEILBRONN's problem. Let there be given n points in the unit circle. Denote by $A(x_1, x_2, \ldots, x_n)$ the smallest area of all the triangles determined by the x_i. Estimate $\max A(x_1, \ldots, x_n)$ where the maximum is taken over all the x_i in the unit circle. $A_n < c_1/n$ is trivial. ROTH proved $A_n = o\left(\frac{1}{n}\right)$, more precisely

$$A_n < c_2/n(\log\log n)^{1/2},$$

and I observed that $A_n > c_3/n^2$. It seems to be a difficult and interesting problem to improve these inequalities for A_n.

K. F. ROTH: "On a problem of Heilbronn." *London Math. Soc. Journal* **26** (1951 198—209.

7) Recently it was asked if the plane can be split into four sets φ_i, $1 \leq i \leq 4$ so that no φ_i should contain two points whose distance is 1. Several mathematicians observed that this certainly can not be done with three sets. (I can not trace the origin of this problem.)

8) ANNING and I proved that if in an infinite set of points in the plane all the distances between the points are integers then the points all are on a line. On the other hand it is known that one can give an infinite set of points, not all on a line so that all the distances should be rational. ULAM asked: Does there exist a set φ dense in the plane so that all the distances between points of φ are rational? I think the answer is no, but the question seems very difficult. SCHOENBERG asked if to every polygon and every ε there exists a polygon whose vertices are at distance $< \varepsilon$ from the corresponding vertices of the original polygon and all whose sides and diagonals have rational length. Clearly if ULAM's problem has an affirmative answer, then the same holds for SCHOENBERG's problem. BESICOVITCH dealt with some special cases of this problem.

P. ERDŐS and A. ANNING: "Integral distances." *Bull. Amer. Math. Soc.* (1945) 598—600 and 996.

A. S. BESICOVITCH: "Rational polygons." *Mathematika* **6** (1959). 98.

Further literature on these similar problems and results: L. FEJES TÓTH: *Lagerungen in der Ebene auf der Kugel und im Raum.* Berlin, 1953 and H. HADWIGER and H. DEBRUNNER: »Kombinatorische Geometrie in der Ebene.« *L'Enseignement Math.* **1** (1955) 56—67. The paper also appeared in French a more detailed version of this paper recently appeared in book form Monographies de L'Enseignement Mathématique No 2. See also a forthcoming book of HADWIGER on these subjects.

IV. Problems in analysis

1) Let $z^n + \ldots$ be a polynomial of degree n. H. CARTAN proved that the set $|f(z)| \leq 1$ (which we will call $E_f^{(n)}$ can be covered by a set of circles the sum of whose radii is $< 2e$. It seems likely that $2e$ can be replaced by 2 (which if true is known to be best possible). If $E_f^{(n)}$ is connected this was proved by POMMERENKE, and in the general case he recently proved this with 2.59 instead of 2.

Assume that $E_j^{(n)}$ is connected. Is it true that

$$\max_{z \subset E_j^{(n)}} |f'(z)| < \frac{n^2}{2} ? \tag{IV. 1.1}$$

POMMERENKE proved this with $\frac{e}{2} n^2$. (IV. 1.1) if true is best possible as is shown by the n-th TCHEBICHEFF polynomial $T_n(z)$.

Is it true that to every $c > 0$ there exists an $A(c)$ independent of n so that $E_f^{(n)}$ can have at most $A(c)$ components of diameter $> 1 + c^2$,

Is it true that the length of the curve $|f_n(z)| = 1$ is maximal for $f_n(z) = z^n - 1$.

Let $|z_i| \leqq 1$. Estimate from below the area of $E_f^{(n)}$. HERZOG, PIRANIAN and I prove that for every ε there exists an n_0 so that for $n > n_0(\varepsilon)$ the area of $E_f^{(n)}$ can be made to be $< \varepsilon$, but we have not succeeded in getting an a good estimate of the area from below.

Let $-1 \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_n \leqq 1$. Is it true that the measure of the set on the real line for which $|f(x)| \leqq 1$ is $\leqq 2\sqrt{2}$? (We can prove that the diameter of this set is less than 3). Most of these problems are discussed in our paper with *Herzog* and *Piranian*.

P. ERDŐS, F. HERZOG and G. PIRANIAN: "Metric properties of polynomials." *Journal d'Analyse Math.* **6** (1958) 125–148.

CHRISTIAN POMMERENKE: "On some problems of Erdős, Herzog and Piranian." *Michigan Math. Journal* **6** (1959) 221–225; „On the derivative of a polynomial." *Ibid.* 373–375; "On some metric properties of polynomials with real zeros." Ibid. 377–380; »Einige Sätze über die Kapazität ebener Mengen.« *Math. Annalen* **141** (1960) 143–152.

2) Littlewood conjectured that for every sequence of integers $n_1 < n_2 < \ldots < n_k$

$$\int_0^{2\pi} \left| \sum_{i=1}^{k} \cos n_i x \right| dx > c \log k \tag{IV. 2. 1}$$

$n_i = i$ shows that if true this is best possible. It was not even known that the integral (IV.2.1) tends to infinity with k. Recently P. COHEN proved (IV.2.1) with $c\,(\log k/\log\log k)^{1/8}$ and DAVENPORT improved this to $c\,(\log k/\log\log k)^{1/4}$.

ANKENY and CHOWLA conjectured that to every $c > 0$ there exists a k so that for

$$\min_{0 \leqq x < 2\pi} \sum_{i=1}^{k} \cos n_i x < -c\,. \tag{IV. 2.2}$$

(IV.2.2) immediately follows from the result of COHEN.

CHOWLA observed that if $n_1 < n_2 < \ldots < n_k$ is a sequence for which the sums $n_i \pm n_j$ are all distinct then $\left(\text{i. e.} \left(\sum_{i=1}^{k} \cos n_i x\right)^2\right)$

$$\sum_{1 \leqq i < j \leqq k} \cos (n_i \pm n_j)\, x$$

gives a trigonometric polynomial of $k^2 - k$ terms whose minimum is $\geqq -\sqrt{k}$. He then asked: is it true that the minimum of (IV.2.2) is less than $-c\sqrt{k}$ for a suitable absolute constant $c > 0$?

PAUL COHEN: "On a conjecture of Littlewood and idempotent measures." *Amer. Journal of Math.* **82** (1960) 190—212.
H. DAVENPORT: „On a theorem of P. J. Cohen." *Mathematika* **7** (1960) 93—97.

3) Let $f_n(\Theta)$ be a trigonometric polynomial of degree n all whose roots are real. Is it true that

$$\int_0^{2\pi} |f_n(\theta)| \leqq 4 . \tag{IV. 3.1}$$

$f_n(\Theta) = \cos n\Theta$ shows that if (IV.3.1) is true, it certainly is best possible

For similar problems see P. ERDŐS: "Note on some elementary properties of polynomials." *Bull. Amer. Math. Soc.* **46** (1940) 954—958.

4) It is known that there exists a polynomial $\sum_{k=1}^{n} \varepsilon_k z^k$, $\varepsilon_k = \pm 1$ for which

$$\max_{|z|=1} \left| \sum_{k=1}^{n} \varepsilon_k z^k \right| < c_1 \sqrt{n} . \tag{IV. 4.1}$$

As far as I know it is not known if there exists a polynomial of the above form which besides (IV.4.1) also satisfies

$$\min_{|z|=1} \left| \sum_{k=1}^{n} \varepsilon_k z^k \right| > c_2 \sqrt{n} . \tag{IV. 4.2}$$

In fact it is (as far as I know) not known if a polynomial satisfying (IV.4.2) exists.

Does there exist an absolute constant $c > 0$ so that

$$\max_{|z|=1} \left| \sum_{k=1}^{n} \varepsilon_k z^k \right| > (1 + c) \sqrt{n} \; ? \tag{IV. 4.3}$$

(IV.4.3) is trivial for $c = 0$ (PARSEVAL's inequality).

I can prove (my paper will appear in Annales Polonici Math.) the analogous inequality for trigonometric polynomials i. e.

$$\max_{0 \leqq \theta < 2\pi} \left| \sum_{k=1}^{n} \varepsilon_k \cos k\theta \right| > \frac{1 + c}{\sqrt{2}} \sqrt{n} . \tag{IV. 4.4}$$

A generalisation of (IV.4.3) would be

$$\max_{|z|=1} \left| \sum_{k=1}^{n} \varepsilon_k z^{n_k} \right| > (1 + c) \sqrt{n} . \tag{IV. 4.5}$$

Here I can not even prove

$$\max_{0 \leqq \theta < 2\pi} \left| \sum_{k=1}^{n} \varepsilon_k \cos n_k \theta \right| > \frac{1 + c}{\sqrt{2}} \sqrt{n} . \tag{IV. 4.6}$$

J. CLUNIE: "The minimum modulus of a polynomial on the unit circle." *Quarterly Journal of Math.* **10** (1959) 95—98.

5) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function

$$M(r) = \max_{|z|=r} |f(z)|, \qquad m(r) = \max |a_n r^n|.$$

Is it true that if $\lim_{r=\infty} m(r)/M(r)$ exists it must be 0? CLUNIE (unpublished) proved this if $a_n \geqq 0$. Determine

$$\max_f \lim m(r)/M(r) = c.$$

$\frac{1}{2} \leqq c < 1$ is trivial. KŐVÁRI observed $c > \frac{1}{2}$, but the exact value of c is not known.

S. M. SHAH: „The behavior of entire and a conjecture of Erdős" *Amer. Math. Monthly* **68** (1961) 419—425.

6) Let $f(z)$ be an entire function. I conjectured and BOAS proved (unpublished) that there exists a path L so that for every n

$$\lim |f(z)/z^n| \to \infty \tag{IV. 6.1}$$

where $z \to \infty$ on L. Can one estimate the length of this path in terms of $M(r)$? Does there exist a path along of which $|f(z)|$ tends to ∞ faster than a fixed function of $M(r)$ e. g. $M(r)^{\varepsilon}$?

HUBER proved the following theorem: Let $f(z)$ be an entire function, not a polynomial. Then to every $\lambda > 0$ there exists a locally rectifiable path C_λ tending to infinity, such that

$$\int_{C_\lambda} |f(z)|^{-\lambda} |dz| < \infty. \tag{IV. 6.2}$$

Does there exist a path C independent of λ so that for every $\lambda > 0$

$$\int_{C} |f(z)|^{-\lambda} |dz| < \infty ? \tag{IV. (6.3)}$$

A. HUBER: "On subharmonic functions and differential geometry in the large." *Comment. Math. Helv.* **32** (1957) 13—72.

7) Pólya's problem. Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ be an entire function of finite order. Assume that $\lim_{k \to +\infty} n_k/k = \infty$. Does it then follow that

$$\overline{\lim} \log m(r)/\log M(r) = 1 ? \tag{IV.7.1}$$

PÓLYA remarks that WIMAN's results (*Acta Math.* **37** (1914) 305—326, and **41** (1916)1—28) imply that if

$$\log (n_{k+1} - n_k)/\log n_k > 1/2, \tag{IV. 7.2}$$

then

$$\overline{\lim}\, m(r)/M(r) = 1 \tag{IV. 7.3}$$

holds (also for functions of inifinite order). MACINTYRE and I proved that if $\sum_{k=2}^{\infty} \frac{1}{n_{k+1} - n_k} < \infty$ then (IV.7.3) holds and that if $\sum_{k=2}^{\infty} \frac{1}{n_{k+1} - n_k} = \infty$ there always exists an entire function $\sum_{k=1}^{\infty} a_k z^{n_k}$ for which

$$\underline{\lim}\, m(r)/M(r) = 0 .$$

$$\left(\sum_{k=2}^{\infty} \frac{1}{n_{k+1} - n_k} < \infty \text{ implies (IV. 7.2)} \right).$$

G. PÓLYA: "Lücken und Singularitäten der Potenzreihen." *Math. Zeitschrift* **29** (1929) 549–640.

P. ERDŐS and A. J. MACINTYRE: "Integral functions with gap power series." *Edinburgh Math. Proc. Ser.* 2. **10** (1954) 62–70.

8) FEJÉR proved that if $\sum_{k=1}^{\infty} 1/n_k < \infty$ then the entire function $\sum_{k=1}^{\infty} a_k z^{n_k}$ assumes every value at least once and BIERNACKI proved that it assumes every value infinitely often. FEJÉR and PÓLYA conjectured that if $n_k/k \to \infty$ then $\sum_{k=1}^{\infty} a_k z^{n_k}$ assumes every value infinitely often.

L. FEJÉR: "Über die Wurzel vom kleinsten absoluten Betrage einer algebraischen Gleichung." *Math. Annalen* **65** (1908) 413–423.

M. BIERNACKI: "Sur les equations algèbriques contenant des parametres arbitraires." *Thèse*, Paris, 1928.

9) Let φ_k, $1 \leqq k < \infty$ be a set of complex numbers which has no limit point in the finite part of the plane. Does there exist an entire function $f(z)$ and a sequence $n_1 < n_2 < \ldots$ so that for every $z \in \varphi_k$, $f^{(n_k)}(z) = 0$, $1 \leqq k < \infty$ (i. e. the set of zeros of $f^{(n_k)}(z)$ contains φ_k?)

10) HANANI and I proved (unpublished) that if $|a_n| > c > 0$, $\lim |a_n|/\sqrt{n} = 0$, a_n real, then to every real α there exists a sequence $\varepsilon_n = \pm 1$ so that the series $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is C_1-summable to α. It is easy to see that $|a_n|/\sqrt{n} \to 0$ can not be replaced by $|a_n| < \varepsilon \sqrt{n}$. But we conjectured that if $|a_n| > c > 0$ and the series $\sum_{n=1}^{\infty} a_n$ is C_1-summable to a finite number then the conclusion of our result remains true. We were unable to prove this, even if we assume $|a_n| < \varepsilon \sqrt{n}$.

Let a_n, $1 \leqq n < \infty$ be a sequence of real numbers. Assume that $\sum_{n=1}^{\infty} a_n$ is C_1-summable. Denote by φ the set of values to which some rearrangement of $\sum_{n=1}^{\infty} a_n$ is C-summable. BAGEMIHL and I proved that φ either consists of a single number, or is the whole real axis or is the set of all numbers

$\alpha + \nu\beta$, $\nu = 0, \pm 1, \pm 2, \ldots$ It would be interesting to extend this for C_k summability and for series with complex terms.

P. ERDŐS and F. BAGEMIHL, "Rearrangements of C_1-summable Series." *Acta Math.* **92** (1954) 35—53. The problem has been considered previously by S. MAZUR, *Soc. Sav. Sci. Lett. Lwow* **4** (1929) 411—424. See also K. ZELLER and G. G. LORENTZ: "Series rearrangements and analytic sets." *Acta Math.* **100** (1958) 144—169.

11) TURÁN's problem. Let $z_1 = 1, z_2, \ldots, z_n$ be any complex numbers Put $s_k = \sum_{i=1}^{n} z_i^k$. TURÁN conjectured that there exists an absolute constant c so that

$$\max_{1 \leq k \leq n} |s_k| > c . \tag{IV. 11.1}$$

About 20 years ago TURÁN proved $\max_{1 \leq k \leq n} |s_k| > \frac{c_1}{n}$, this was improved by me to $\frac{1}{2 \log n}$, by TURÁN to $\log 2/\log n$ and by de BRUIJN and UCHIJAMA to c_2 loglog $n/\log n$. Very recently ATKINSON proved TURÁN's conjecture with $c = \frac{1}{6}$. The best value of c is unknown.

For problems of this type and their application see P. TURÁN's book: *Eine neue Methode in der Analysis und deren Anwendungen*. The book also appeared in Hungarian and there is a Chinese edition which contains new material. A new American edition of the book will appear soon. I would like to mention just one problem I proved (see TURÁN's book) that one can find n complex numbers $z_1 = 1, |z_i| \leq 1, 2 \leq i \leq n$ for which

$$\max_{2 \leq k \leq n+1} |s_k| < (1 + c_3)^{-n} \tag{IV. 11.2}$$

where $c_3 > 0$ is an absolute constant. Can one find n complex numbers z_i satisfying (IV.11.2) and $|z_i| \geq 1, 1 \leq i \leq n$?

F. V. ATKINSON: On sums of powers of complex numbers." *Acta Math. Hung.* **12** (1961) 185—188.

V. Problems on probability

1) Let $r_n(t)$ be the sequence of Rademacher functions i. e. $r_n(t) = \pm 1$ with probability $\frac{1}{2}$ and the $r_n(t)$ are independent functions. The well known law of the iterated logarithm states that for almost all t

$$\limsup_{n=\infty} \sum_{k=1}^{n} r_k(t)/\sqrt{2\, n \log \log n} = 1 . \tag{V. 1.1}$$

Assume now that $\varphi_p(t)$ (p prime) is a sequence of independent functions $\varphi_p(t) = \pm 1$ with probability $\frac{1}{2}$. Further assume that for $n = a \cdot b$, $\varphi_n(t) = \varphi_a(t)\, \varphi_b(t)$. Thus if the φ's are defined for all primes they are defined for

all integers. WINTNER proved that for all ε and almost all t

$$\lim_{n\to\infty} \sum_{k=1}^{n} \varphi_k(t)/n^{1/2+\varepsilon} = 0\,, \tag{V.1.2}$$

and I improved this (unpublished) to

$$\lim_{n\to\infty} \sum_{k=1}^{n} \varphi_k(t)/n^{1/2} (\log n)^c = 0\,. \tag{V.1.3}$$

It would be interesting to prove a result analogous to (V.1.1). I can not even prove that

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \varphi_k(t)/n^{1/2} = \infty\,. \tag{V.1.4}$$

I was unable to locate the paper of WINTNER.

The next few questions deal with random polynomials and power series.

2) Let $\varepsilon_k = \pm 1$. Completing previous results of LITTLEWOOD, OFFORD and KAC, OFFORD and I proved that if we neglect $o\left(\frac{2^n}{(\log\log n)^{1/2}}\right)$ polynomials $\sum_{k=0}^{n} \varepsilon_k z^k$ the number of real roots of the remaining polynomials is of the form $\frac{2}{\pi}\log n + o\big((\log n)^{2/3}\log\log n\big)$.

Our result was not quite strong enough to prove the following conjecture (which as far as I know is still open): Put $0 < t < 1$, let the binary expansion of t be $t = \sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{2^k}$. Denote by $R_n(t)$ the number of real roots of the n-th partial sum of $\sum_{k=0}^{\infty} \varepsilon_k(t)\, z^k$. Then for almost all t

$$\lim_{n=\infty} R_n(t)/\frac{2}{\pi}\log n = 1\,. \tag{V.2.1}$$

Denote by $R_n'(t)$ the number of roots in the unit circle of the n-th partial sum of $\sum_{k=0}^{\infty} \varepsilon_k(t)\, z^k$. Is it true that

$$R_n'(t)/n \to \frac{1}{2} \tag{V.2.2}$$

for almost all t? Here I can not even prove that for all but $o(2^n)$ polynomials $\sum_{k=0}^{n} \varepsilon_k z^k$ the number of roots in $|z| < 1$ is $\frac{n}{2} + o(n)$.

J. E. LITTLEWOOD and C. OFFORD, *Proc. Cambridge Phil. Soc.* **35** (1939) 133—148.

M. KAC, *Bull. Amer. Math. Soc.* **49** (1943) 314—320 and 938, see also *Proc. London Math. Soc.* **50** (1948) 390—408.

P. ERDŐS and C. OFFORD: "On the number of real roots of a random algebraic equation." *Proc. London Math. Soc.* 139—160.

3) SALEM and ZYGMUND proved the following theorem: For almost all t and $n > n_0(t)$

$$c_1(n \log n)^{1/2} < \max_{|z|=1} \left| \sum_{k=0}^{n} \varepsilon_k(t)\, z^k \right| < c_2(n \log n)^{1/2}. \tag{V.3.1}$$

The proof of the upper bound in (V.3.1) is easy, the difficult part was the proof of the lower bound. One would except that for almost all t

$$\lim_{n=\infty} \max_{|z|=1} \frac{\left| \sum_{k=0}^{n} \varepsilon_k(t)\, z^k \right|}{n^{1/2} (\log n)^{1/2}} = C \tag{V.3.2}$$

where C does not depend on t. The following weaker statement has also not been proved so far: For every ε if we neglect $o(2^n)$ polynomials $\sum_{k=0}^{n} \varepsilon_k z^k$ we have

$$(C - \varepsilon)(n \log n)^{1/2} < \max_{|z|=1} \left| \sum_{k=0}^{n} \varepsilon_k z^k \right| < (C + \varepsilon)(n \log n)^{1/2}. \tag{V.3.3}$$

Denote

$$M_n(t) = \max_{1 \leqq x \leqq 1} \left| \sum_{k=0}^{n} \varepsilon_k(t)\, x^k \right|.$$

The upper bound for $M_n(t)$ is given by the law of the iterated logarithm, but the lower bound is much more difficult. I proved (unpublished) that for almost all t and every $\varepsilon > 0$

$$\lim_{n=\infty} M_n(t)/n^{1/2-\varepsilon} = \infty. \tag{V.3.4}$$

A theorem of CHUNG implies that for almost all t there are infinitely many n for which

$$M_n(t) < c \left(\frac{n}{\log \log n} \right)^{1/2}. \tag{V.3.5}$$

The exact lower bound for $M_n(t)$ seems very difficult (the problem is due to SALEM and ZYGMUND).

Is it true that for all but $o(2^n)$ polynomials $\sum_{k=0}^{\infty} \varepsilon_k z^k$.

$$\min_{|z|=1} \left| \sum_{k=0}^{n} \varepsilon_k z^k \right| < 1 \; ? \tag{V.3.6}$$

or more precisely how can one estimate the minimum (V.3.6) as accurately as possible.

R. SALEM and A. ZYGMUND: "Some properties of trigonometric series whose terms have random signs." *Acta Math* **91** (1954) 245–301.

4) DVORETZKY's problem. Let

$$a_n \geqq 0, \; a_n \to 0, \; \sum_{n=1}^{\infty} a_n = \infty. \tag{V.4.1}$$

Place on the circle of circumference 1 at random arcs of length a_n. It is easy to see that if (V.4.1) is satisfied then with probability one almost all

points of the unit circle are covered by the arcs. DVORETZKY showed that for suitable choice of a_n *all* points of the unit circle are covered for almost all choices of the arcs of length a_n (satisfying (V.4.1)], and that for suitable choice of the a_n for almost all choices of the arcs there are points not covered by them. The first case we shall call the case of covering, the second of not covering. $a_n = \frac{1+c}{n}$ where $c>0$ was shown by KAHANE to be in the case of covering. I proved (unpublished) that $a_n = \frac{1}{n}$ is in the case of covering but $a_n = \frac{1-c}{n}$ in the case of not covering. At present no necessary and sufficient condition for the case of covering is known.

Let $\sum_{n=1}^{\infty} |b_n|^2 = \infty$. It is well known that for almost all choices of $\varepsilon_n = \pm 1$, $\sum_{k=1}^{\infty} \varepsilon_n b_n z^n$ diverges for almost all points of the unit circle. Sharpening previous results of DVORETZKY, he and I proved that if $|b_n| > \frac{c}{\sqrt{n}}$, then for almost all choices of $\varepsilon_n = \pm 1$, $\sum_{n=1}^{\infty} \varepsilon_n b_n z^n$ diverges for *all* points $|z| = 1$. We have an example of a series $\sum_{n=1}^{\infty} |b_n|^2 = \infty$, $|b_{n+1}| \leqq |b_n|$ so that for almost all choices of $\varepsilon_n = \pm 1$ there exists a z_0, $|z_0| = 1$, (z_0 depends on the sequence ε_n) so that $\sum_{n=1}^{\infty} \varepsilon_n b_n z_0^n$ converges. Perhaps every series satisfying $n^{1/2} |b_n| \to 0$ has this property.

A. DVORETZKY: "On the covering of the circle by randomly placed arcs." *Proc. Nat. Acad. Sci. USA* **42** (1956) 199–203.

J. P. KAHANE: "Sur le recouvrement d'un cercle par des arcs disposés au hasard." *Comptes rendus* **248** (1959) 184–186.

A. DVORETZKY and P. ERDŐS: "Divergence of random power series." *Michigan Math. Journal* **6** (1959) 343–347.

5) Denote by $f(n, k)$ the number of random walk paths of n steps in k dimensional space where we assume that the path does not intersect itself. It has been obeserved that $\lim_{n=\infty} f(n, k)^{1/n} = C_k$ exists, but no sharper inequalities are known for $f(n, k)$ even for $k = 2$.

The expected position and distribution of the point after n steps has also not been determined. It has often been conjectured that for $k = 2$ the expected distances from the origin divided by $n^{1/2}$ tends to ∞ and divided by n tends to 0, for $k \geq 3$ the expected distance was supposed to be $O(n^{1/2})$.

I do not know the origin of these problems (probably applications in polymer chemistry, I first heard of them in 1949). See the forthcoming paper of B. Rennie in the Publications of the Mathematical Institute of the Hungarian Academy of Sciences, Series A.

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