ON THE NONLINEAR EQUATION $u'' + a(t)\,u + q(t)\,f(u^2) = 0$

by

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1. This part of the paper generalizes a theorem due to Z. Opial [1]. The generalization in question is as follows :

Theorem 1. *Let $a(t)$ be continuous, non-decreasing, positive for $t \geqq 0$, the function $f(u)$ continuous, non-decreasing, positive for $u \geqq 0$, $f(0) = 0$ and*

$$\int_{u_0}^{\infty} \frac{du}{f^2(u) + u} = +\infty \qquad (u_0 > 0) \tag{1}$$

$q(t)$ continuous for $t \geqq 0$ and

$$\int_0^{\infty} \frac{|q(t)|}{\sqrt{a(t)}}\,dt < \infty\,, \tag{1'}$$

then every solution of the equation

$$u'' + a(t)\,u + q(t)\,f(u^2) = 0$$

is bounded for $t \geqq 0$, moreover the expression $A(t) = u^2(t) + \frac{u'^2(t)}{a(t)}$ tends to a finite limit as $t \to +\infty$ (e.g. the sequence of the extreme values of $|u|$ is convergent).[2]

Proof. Let $u(t)$ be an arbitrary solution of (2). Then

$$A(t) = u^2(t) + \frac{u'^2(t)}{a(t)} = u^2(0) + \frac{u'^2(0)}{a(0)} + \int_0^t d\left(u^2 + \frac{u'^2}{a}\right) =$$

$$= C + \int_0^t \left(2u\,u' + \frac{2u'\,u''}{a}\right) dt - \int_0^t \frac{u'^2}{a^2}\,da(t),$$

where $C = u^2(0) + \frac{u'^2(0)}{a(0)}$. We have from (2) $u'' = -a\,u - q\,f(u^2)$.

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[2] The proof may be modified to include the linear case too.

Therefore

$$(3) \qquad A(t) = C - \int_0^t \frac{u'^2}{a^2}\, da(t) - \int_0^t \frac{2\, u'\, f(u^2)}{a}\, q dt\,.$$

$a(t)$ being non-decreasing we have $\int_0^t \frac{u'^2}{a^2}\, d\, a(t) \geqq 0 (t \geqq 0)$. Making use of the property of the geometric and quadratic means

$$\left| \frac{2\, u'\, f(u^2)}{\sqrt{a}} \right| \leqq f^2(u^2) + \frac{u'^2}{a}\,.$$

Thus by (3)

$$(4) \quad A(t) \leqq C + \int_0^t \left| \frac{2\, u'\, f(u^2)}{\sqrt{a}} \right| \frac{|q|}{\sqrt{a}}\, dt \leqq C + \int_0^t \left[f^2(u^2) + \frac{u'^2}{a} \right] \frac{|q|}{\sqrt{a}}\, dt \leqq$$

$$\leqq C + \int_0^t \left[f^2\left(u^2 + \frac{u'^2}{a}\right) + u^2 + \frac{u'^2}{a} \right] \frac{|q|}{\sqrt{a}}\, dt\,.$$

Let us denote $\int_{u_0}^{u} \frac{ds}{f^2(s) + s}$ $(u_0 > 0)$ by $F(u)$ and its inverse function by $F^{-1}(u)$. Then by a well known lemma we obtain from (4)

$$(5) \qquad A(t) \leqq F^{-1}\left(F(C) + \int_0^t \frac{|q|}{\sqrt{a}}\, dt \right) \leqq F^{-1}\left(F(C) + \int_0^\infty \frac{|q|}{\sqrt{a}}\, dt \right)$$

and the statement concerning the boundedness of $|u|$ an $\frac{|u'|}{\sqrt{a}}$ is proved. Let us denote this boundary by K, then $|u| \leqq K$, $\frac{|u'|}{\sqrt{a}} \leqq K$. Applying this on the second integral in (3)

$$2 \int_0^t \left| f(u^2) \frac{u'}{\sqrt{a}} \frac{q}{\sqrt{a}} \right| dt \leqq 2\, f(K^2)\, K \int_0^t \frac{|q|}{\sqrt{a}}\, dt \leqq 2\, K f(K^2) \int_0^\infty \frac{|q|}{\sqrt{a}}\, dt\,.$$

This involves the convergence of the integral in question for $t = +\infty$. The first integral in (3) is — as a non-decreasing function — also convergent for $t \to +\infty$ and its limit must be finite (otherwise the right member of (3) would be negative for sufficiently large t and this is impossible). Consequently in virtue of (3)

$$A(t) = u^2(t) + \frac{u'^2(t)}{a(t)} \text{ has a finite limit.}$$

2. In the following problem the conditions imposed on $a(t)$, $q(t)$ and $f(u)$ are different from the previous ones. Let our equation be given in the form

$$u'' + a(t)\,u + q(t)\,f(u) = 0\,. \tag{6}$$

Then we have

Theorem 2. *Let $a(t)$ be a positive continuous, non-decreasing function for $t \geqq 0$, let the function $q(t)$ be continuous and satisfying together with $a(t)$ the conditions $\left|\frac{q}{a}\right| < \frac{\alpha}{t}$, $\left|\left(\frac{q}{a}\right)'\right| < \frac{\alpha}{t^2}$ $(\alpha > 0)$ for t sufficiently large, let the positive function $f(u)$ be defined for all u and $f(u) \in$ Lip (1), further let the boundedness of $\frac{|F(u)|}{u^2}$ be assumed for all u, where $F(u) = \int\limits_0^u f(s)\,ds$.*

Then every solution of (6) *is bounded if $t \to \infty$.*

Proof. Let t_0 be a positive constant, to be determined later. Then similarly to (3) we have (denoting $A(t_0)$ by C)

$$A(t) = C - 2\int\limits_{t_0}^{t} \frac{dF(u)}{dt}\,\frac{q}{a}\,dt - \int\limits_{t_0}^{t} \frac{u'^2}{a^2}\,da(t) \leqq C + 2\,F(u(t_0))\,\frac{q(t_0)}{a(t_0)} -$$

$$- 2\,F(u(t))\,\frac{q(t)}{a(t)} + 2\int\limits_{t_0}^{t} F(u)\left(\frac{q}{a}\right)' dt\,,$$

whence

$$A(t) \leqq K + 2\,|F(u)|\left|\frac{q}{a}\right| + 2\int\limits_{t_0}^{t} |F(u)|\left|\left(\frac{q}{a}\right)'\right| dt \tag{7}$$

where $K = C + 2\,|F(u(t_0))|\left|\frac{q(t_0)}{a(t)}\right|$. Let the maximum of $|u|$ in the interval $[t_0, t]$ be M, and let $|u|$ take on the value M at τ $(t_0 \leqq \tau \leqq t)$. Then by (7)

$$M^2 \leqq A(\tau) \leqq K + 2\frac{\alpha}{\tau}|F(M)| + 2\,|F(M)|\,\alpha\left(\frac{1}{t_0} - \frac{1}{\tau}\right) = K + 2\,|F(M)|\,\frac{\alpha}{t_0}\,. \tag{8}$$

Hence

$$M^2 - 2\,|F(M)|\,\frac{\alpha}{t_0} = M^2\left(1 - \frac{|F(M)|}{M^2}\,\frac{2\,\alpha}{t_0}\right) \leqq K\,. \tag{9}$$

Let us choose for t_0 so much a large value that the inequality $\frac{|F(M)|}{M^2}\,\frac{2\,\alpha}{t_0} < \frac{1}{2}$ be satisfied. Then by (9) $M^2 \leqq 2\,K$, what was to be proved.

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REFERENCES

[1] OPIAL, Z.: "Nouvelles remarques sur l'équation différentielle $u'' + a(t)\,u = 0$." *Annales Polonici Mathematici* **6** (1959) 75–81.

ЗАМЕЧАНИЯ ОТНОСИТЕЛЬНО НЕЛИНЕЙНОГО УРАВНЕНИЯ $u'' + a(t)\,u + q(t)\,f(u^2) = 0$

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Резюме

Теорема 1. (Обобщение одной теоремы OPIAL). *Пусть $u(t)$ непрерывная неубывающая положительная функция от $t \geqq 0$, $f(u)$ непрерывная неубывающая функция от u, которая положительна для $u \geqq 0$,*

$$f(0) = 0 \quad и \quad \int_{u_0}^{\infty} \frac{du}{f^2(u) + u} = +\infty \qquad (u_0 > 0)$$

далее, $q(t)$ непрерывная функция от $t \geqq 0$ и $\int_0^{\infty} \frac{|q(t)|}{\sqrt{a(t)}}\,dt < \infty$, тогда любое решение уравнения

$$u'' + a(t)\,u + q(t)\,f(u^2) = 0$$

ограничено для $t \geqq 0$ и выражение

$$A(t) = u^2(t) + \frac{u'^2(t)}{a(t)}$$

имеет конечный предел при $t \to \infty$.

Теорема 2. *Пусть 1) функция $a(t) > 0$ непрерывна и неубывающая, функция $q(t)$ непрерывна для $t \geqq 0$ и пусть будет выполнено условие*

$$\left|\frac{q}{a}\right| < \frac{\alpha}{t}, \quad \left|\left(\frac{q}{a}\right)'\right| < \frac{\alpha}{t^2} \qquad (\alpha > 0)$$

для достаточно больших $t > 0$,

2) *Функция $f(u)$ определена для всех значений u и $f(u) \in \mathrm{Lip}\,(\alpha)$,*

3) $\frac{|F(u)|}{u}$ *ограничена* $\left(F(u) = \int_0^u f(s)\,ds\right)$ *и $F(u)$ неубывающая функция для $u > 0$, тогда все решения уравнения*

$$u'' + a(t)\,u + q(t)\,f(u^2) = 0$$

ограничены при $t \to \infty$.