ERROR ESTIMATION FOR MASSAU'S METHOD OF CHARACTERISTICS

by

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The method of characteristics is one of the oldest and most frequently used numerical methods of solution of initial value problems for hyperbolic systems of quasilinear differential equations. In the present paper we shall investigate only MASSAU's original version of this method for systems in two independent and two dependent variables (the adaptation of MASSAU's method to more general equations is described in [2], methods of higher accuracy based on the same principle can be found in [1], [3] and [4]). FORSYTHE and WASOW conjectured in their recent book [1] that the error of MASSAU's method is of order $O(h)$ where h is the maximum arc length between two adjacent grid points on the initial curve (see [1], p. 65). We shall prove that this hypothesis is true under some rather trivial assumptions.

We consider systems of two quasilinear differential equations of the form

$$a_{11} u_x + a_{12} v_x + b_{11} u_y + b_{12} v_y = h_1 \tag{1a}$$

$$a_{21} u_x + a_{22} v_x + b_{21} u_y + b_{22} v_y = h_2 \tag{1b}$$

for the two unknown functions $u = u(x, y)$ and $v = v(x, y)$. The coefficients a_{ik}, b_{ik} and h_i $(i, k = 1, 2)$ are functions of x, y, u, v and have bounded third partial derivatives in a domain D of the x, y, u, v-space. We assume that the system (1a)—(1b) is of hyperbolic type in D, that is the equation

$$\begin{vmatrix} a_{11}\lambda - b_{11} & a_{12}\lambda - b_{12} \\ a_{21}\lambda - b_{21} & a_{22}\lambda - b_{22} \end{vmatrix} = 0$$

possesses two distinct real roots $\lambda_1 = \lambda_1(x, y, u, v)$ and $\lambda_2 = \lambda_2(x, y, u, v)$ at every point of D. Moreover, we assume for convenience

$$\begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix} \neq 0$$

at every point of D. The latter condition can always be satisifed by introducing new coordinates instead of x and y.

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Let $\mathcal{J}$ be a curve in the x, y-plane, on which the values of u and v are given. We assume that $\mathcal{J}$ is finite, rectifiable and has no double points. The curve $\mathcal{J}$ should be represented by the parametric equations $x = x(s)$ and $y = y(s)$ where the functions $x(s)$ and $y(s)$ have continuous third derivatives (s is the natural parameter of the curve $\mathcal{J}$). The initial values are assumed to be given parametrically by three times continuously differentiable functions $u(s)$ and $v(s)$. The points $(x(s), y(s), u(s), v(s))$ should lie in D for all possible values of s. Finally, we shall need the hypothesis that the direction of $\mathcal{J}$ is nowhere identical with one of the two "characteristic directions" determined by the vectors $[1, \lambda_1(s)]$ and $[1, \lambda_2(s)]$ respectively:

$$\big(y'(s) - \lambda_1(s)\, x'(s)\big)\big(y'(s) - \lambda_2(s)\, x'(s)\big) \neq 0$$

where $\lambda_i(s)$ is an abbreviation for $\lambda_i\big(x(s), y(s), u(s), v(s)\big)$.

We replace the system (1a)—(1b) by the so-called characteristic system

$$\lambda_1 x_\xi - y_\xi = 0\,, \tag{2a}$$

$$\lambda_2 x_\eta - y_\eta = 0\,, \tag{2b}$$

$$\begin{vmatrix} \lambda_1 a_{11} - b_{11} & h_1 x_\xi - a_{11} u_\xi^* - a_{12} v_\xi^* \\ \lambda_1 a_{21} - b_{21} & h_2 x_\xi - a_{21} u_\xi^* - a_{22} v_\xi^* \end{vmatrix} = 0\,, \tag{2c}$$

$$\begin{vmatrix} \lambda_2 a_{11} - b_{11} & h_1 x_\eta - a_{11} u_\eta^* - a_{12} v_\eta^* \\ \lambda_2 a_{21} - b_{21} & h_2 x_\eta - a_{21} u_\eta^* - a_{22} v_\eta^* \end{vmatrix} = 0\,, \tag{2d}$$

where $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, $u^* = u^*(\xi, \eta)$ and $v^* = v^*(\xi, \eta)$ are unknown functions of the new coordinates ξ and η. Expanding the determinants on the left side of (2c) and (2d) the characteristic system can be brought into the form

$$\lambda_1 x_\xi - y_\xi = 0, \tag{3a}$$

$$\lambda_2 x_\eta - y_\eta = 0, \tag{3b}$$

$$a_{11}^* u_\xi^* + a_{12}^* v_\xi^* + h_1^* x_\xi = 0\,, \tag{3c}$$

$$a_{21}^* u_\eta^* + a_{22}^* v_\eta^* + h_2^* x_\eta = 0\,. \tag{3d}$$

It is easy to show (see, for example [2], p. 75) that

$$\begin{vmatrix} \lambda_1 & -1 & 0 & 0 \\ \lambda_2 & -1 & 0 & 0 \\ h_1^* & 0 & a_{11}^* & a_{12}^* \\ h_2^* & 0 & a_{21}^* & a_{22}^* \end{vmatrix} \neq 0$$

at every point of D.

By the so-called equivalence theorem (see, for example [2], p. 76) our initial value problem for the system (1a)—(1b) is equivalent to the following initial value problem for the characteristic system: determine solutions $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, $u^* = u^*(\xi, \eta)$ and $v^* = v^*(\xi, \eta)$ of the characteristic system (3a)—(3d) such that these solutions on the line $\xi + \eta = 0$ satisfy the following initial conditions:

$$x\ \big(\xi(s), \eta(s)\big) = x(s)\,,$$
$$y\ \big(\xi(s), \eta(s)\big) = y(s)\,,$$
$$u^*\big(\xi(s), \eta(s)\big) = u(s)\,,$$
$$v^*\big(\xi(s), \eta(s)\big) = v(s)\,,$$

where s is now the natural parameter of the line $\xi + \eta = 0$. It is a well-known result of FRIEDRICHS and LEWY (see [2], pp. 79 ff.) that the initial value problem for the characteristic system has uniquely determined solutions $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, $u^* = u^*(\xi, \eta)$ and $v^* = v^*(\xi, \eta)$ inside a trapezoid[1] T bounded by the initial segment AB of the line $\xi + \eta = 0$, the line $\eta = $ const. through A, the line $\xi =$ const. through B and a line parallel to the line $\xi + \eta = 0$; moreover the functions x, y, u^* and v^* have continuous second partial derivatives inside T. Then by the equivalence theorem the transformation

$$x = x(\xi, \mu) \qquad y = y(\xi, \eta)$$

has an inverse

$$\xi = \xi(x, y) \qquad \eta = \eta(x, y)$$

and the functions

$$u(x, y) = u^*\big(\xi(x, y), \eta(x, y)\big) \quad v(x, y) = v^*\big(\xi(x, y), \eta(x, y)\big)$$

are the (unique) solutions of our original initial value problem in a trapezoid-like region S of the x, y-plane, bounded by the curve $\mathcal{J}$, the curves $\eta =$ const. and $\xi =$ const. through the end points A^* and B^* of the curve $\mathcal{J}$, and a curve parallel to $\mathcal{J}$; moreover the functions $u(x,y)$ and $v(x,y)$ have continuous second partial derivatives in S.

MASSAU's method can now be described as a process in the ξ, η-plane. We choose a sequence of (not necessarily equally spaced) grid points on the segment AB of the line $\xi + \eta = 0$. These grid points will be called the grid points at the 0-th level. If P_1 and P_2 are two adjacent grid points at the $j - 1$-th level and the η coordinate of P_1 is greater than that of P_2, then the point of intersection Q of the line $\eta =$ const. through P_1 with the line $\xi =$ const. through P_2 will be, by definition, a grid point at the j-th level. By successive application of this construction we get a system of grid points in the triangle ABC (see Fig. 1). It should be noted that at each level there is one point less than at the preceding one.

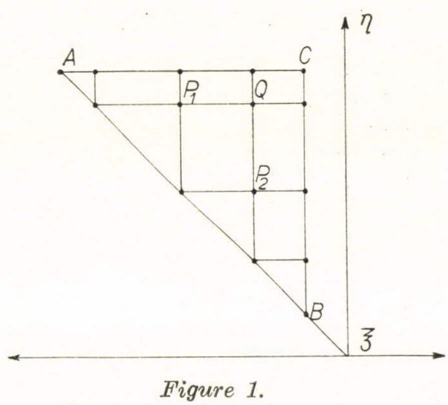

Figure 1.

[1] In what follows we shall regard the half-plane above the line $\xi + \eta = 0$ only.

We replace the characteristic system (3a)—(3d) by the finite-difference equations

$$\text{(4a)} \qquad \bar{y}(Q) - \bar{y}(P_1) = \bar{\lambda}_1(P_1)\,[\bar{x}(Q) - \bar{x}(P_1)],$$

$$\text{(4b)} \qquad \bar{y}(Q) - \bar{x}(P_2) = \bar{\lambda}_2(P_1)\,[\bar{x}(Q) - \bar{x}(P_2)],$$

$$\text{(4c)} \qquad \bar{a}^*_{11}(P_1)\,[\bar{u}^*(Q) - \bar{u}^*(P_1)] + \bar{a}^*_{12}(P_1)\,[\bar{v}^*(Q) - \bar{v}^*(P_1)] + \\ + \bar{h}^*_1(P_1)\,[\bar{y}(Q) - \bar{x}(P_1)] = 0,$$

$$\text{(4d)} \qquad \bar{a}^*_{21}(P_1)\,[\bar{u}^*(Q) - \bar{u}^*(P_2)] + \bar{a}^*_{22}(P_1)\,[\bar{v}^*(Q) - \bar{v}^*(P_2)] + \\ + \bar{h}^*_2(P_1)\,[\bar{x}(Q) - \bar{x}(P_2] = 0,$$

where $\bar{\lambda}_i(P_1)$ is an abbreviation for $\lambda_i(\bar{x}(P_1), \bar{y}(P_1), \bar{u}^*(P_1), \bar{v}^*(P_1))$ and $\bar{a}^*_{ik}(P_1)$ is an abbreviation for $a^*_{ik}(\bar{x}(P_1), \bar{y}(P_1), \bar{u}^*(P_1), \bar{v}^*(P_1))$. The initial conditions for the system (4a)—(4d) are

$$\text{(5)} \qquad \bar{x}(R_0) = x(s_0)\,; \quad \bar{y}(R_0) = y(s_0)\,; \quad \bar{u}^*(R_0) = u(s_0)\,; \quad \bar{v}^*(R_0) = v(s_0)$$

where R_0 is an arbitrary grid point at the 0-th level and s_0 is the corresponding value of the parameter s. If the values of $\bar{x}(P_i)$, $\bar{y}(P_i)$, $\bar{u}^*(P_i)$ and $\bar{v}^*(P_i)$ are already known ($i = 1, 2$) and

$$\bar{\Delta}(P_1) = \begin{vmatrix} \bar{\lambda}_1(P_1) & -1 & 0 & 0 \\ \bar{\lambda}_2(P_1) & -1 & 0 & 0 \\ \bar{h}^*_1(P_1) & 0 & \bar{a}^*_{11}(P_1) & \bar{a}^*_{12}(P_1) \\ \bar{h}^*_2(P_1) & 0 & \bar{a}^*_{21}(P_1) & \bar{a}^*_{22}(P_1) \end{vmatrix} \neq 0,$$

then we can determine $\bar{x}(Q)$, $\bar{y}(Q)$, $\bar{u}^*(Q)$ and $\bar{v}^*(Q)$ from the linear equations (4a)—(4d).

Since the coefficients of the equations (4a)—(4d) do not contain explicitly the ξ, η coordinates, MASSAU's method can also be formulated as a process in the x, y-plane. The grid points on the line $\xi + \eta = 0$ correspond to grid points on the initial curve $\mathcal{J}$ such that the distance between two adjacent grid points on the line $\xi + \eta = 0$ is equal to the arc length between the corresponding grid points on the curve $\mathcal{J}$; the coordinates of these grid points and the corresponding values of $\bar{u}$ and $\bar{v}$ can be determined from the initial conditions (5). By successive application of the equations (4a)—(4d) we can find the coordinates $\bar{x}$ and $\bar{y}$ of a system of irregularly spaced grid points in the x, y-plane (see Fig. 2) and the corresponding values $\bar{u}(\bar{x}, \bar{y})$ and $\bar{v}(\bar{x}, \bar{y})$ which may be taken as approximate values of $u(\bar{x}, \bar{y})$ and $v(\bar{x}, \bar{y})$ respectively.

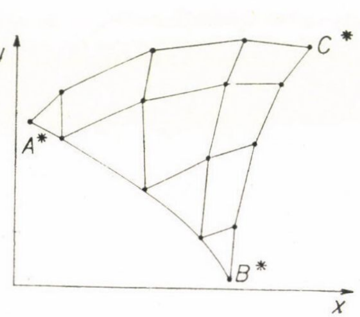

Figure 2.

Let h denote the maximum arc length between two adjacent grid points of $\mathcal{J}$ (that is the maximum of the distance between two adjacent grid points of the line $\xi + \eta = 0$), and r denote the number of grid points on the curve $\mathcal{J}$. We shall prove the following theorem:

If the „stability condition" $rh = O(1)$ is satisfied[2] *then as long as the point (x, y) lies in the region S, there exists a grid point $(\bar{x}, \bar{y})$ such that the inequalities*

$$x - \bar{x} = O(h); \quad y - \bar{y} = O(h); \quad u(x, y) - \bar{u}(\bar{x}, \bar{y}) = O(h);$$
$$v(x, y) - \bar{v},(\bar{x}, \bar{y}) = O(h)$$

hold.

In order to prove our theorem we shall return to the formulation of MASSAU's method as a process in the ξ, η-plane. Instead of the characteristic equations (3a)—(3d) we shall first consider the simpler characteristic system

$$\alpha_{11} f_\xi + \alpha_{12} g_\xi = 0\,, \tag{6a}$$

$$\alpha_{21} f_\eta + \alpha_{22} g_\eta = 0 \tag{6b}$$

for the two unknown functions $f = f(\xi, \eta)$ and $g = g(\xi, \eta)$. We assume that the coefficients $a_{ik} = a_{ik}(f, g)$ have bounded third partial derivatives in a domain D_2 of-the f, g-plane and the inequality

$$|\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}| \geqq \delta > 0 \tag{7}$$

holds in D_2. The functions f and g should satisfy on the line $\xi + \eta = 0$ the initial conditions

$$f(\xi(s), \eta(s)) = f(s), \quad g(\xi(s), \eta(s)) = g(s)\,,$$

where $f(s)$ and $g(s)$ are three times continuously differentiable functions of the natural parameter s, and the points $(f(s), g(s))$ are in the domain D_2.

The finite-difference equations corresponding to (6a)—(6b) are

$$\bar{\alpha}_{11}(P_1)[\bar{f}(Q) - \bar{f}(P_1)] + \bar{\alpha}_{12}(P_1)[\bar{g}(Q) - \bar{g}(P_1)] = 0\,, \tag{8a}$$

$$\bar{\alpha}_{21}(P_1)[\bar{f}(Q) - \bar{f}(P_2)] + \bar{\alpha}_{22}(P_1)[\bar{g}(Q) - \bar{g}(P_2)] = 0\,, \tag{8b}$$

where $\bar{\alpha}_{11}(P_1)$ is an abbreviation for $\alpha_{11}(\bar{f}(P_1), \bar{g}(P_1))$.

By the existence theorem of FRIEDRICHS and LEWY the initial value problem for the equations (6a)—(6b) has uniquely determined solutions $f(\xi, \eta)$ and $g(\xi, \eta)$ with continuous second partial derivatives in a trapezoid T_2 whose sides are the initial segment of the line $\xi + \eta = 0$, the lines $\xi =$ const. and $\eta =$ const. through the end points of this segment, and a line parallel o th e line $\xi + \eta = 0$. It is clear from their proof that the points $(f(R), g(R))$ and $(\bar{f}(R), \bar{g}(R))$ lie in D_2 for all grid points R in the trapezoid T_2.

From the construction of the grid points it follows that

$$\Delta\xi = \overline{P_1Q} < h\,; \quad \Delta\eta = \overline{P_2Q} < h\,, \quad \overline{P_1P_2} \leqq h\,.$$

[2] Here and in what follows $A = O(B)$ means that for all sufficiently small values of h $|A| \leqq c\,|B|$ where c is a positive constant whose numerical value may depend on bounds for derivatives of the coefficients and the solutions but not on h and the coordinates of the grid points.

Let us now assume that the points P_1, P_2 and Q lie in the trapezoid T_2. Then since f and g have continuous second partial derivatives in T_2

$$\alpha_{11}(P_1)[f(Q)-f(P_1)]+\alpha_{12}(P_1)[g(Q)-g(P_1)]=$$

$$=\alpha_{11}(P_1)f_\xi(P_1)\Delta\xi+\alpha_{12}(P_1)g_\xi(P_1)\Delta\xi+O(h^2) \tag{9a}$$

and since $f_\eta(P_2)=f_\eta(P_1)+O(h)$; $g_\eta(P_2)=g_\eta(P_1)+O(h)$

$$\alpha_{21}(P_1)[f(Q)-f(P_2)]+\alpha_{22}(P_1)[g(Q)-g(P_2)]=$$

$$=\alpha_{21}(P_1)f_\eta(P_1)\Delta\eta+\alpha_{22}(P_1)g_\eta(P_1)\Delta\eta+O(h_2). \tag{9b}$$

But the functions f and g satisfy the equations (6a)—(6b) thus from (9a)—(9b) we get

$$\alpha_{11}(P_1)[f(Q)-f(P_1)]+\alpha_{12}(P_1)[g(Q)-g(P_1)]=O(h^2) \tag{10a}$$

$$\alpha_{21}(P_1)[f(Q)-f(P_2)]+\alpha_{22}(P_1)[g(Q)-g(P_2)]=O(h^2). \tag{10b}$$

Let us now put

$$w(R)=f(R)-\bar{f}(R)\,;\quad z(R)=g(R)-\bar{g}(R)$$

where R is an arbitrary grid point in T_2. Then because of the continuous differentiability of the coefficients α_{ik} in the domain D_2, we have

$$\alpha_{ik}(P_1)-\bar{\alpha}_{ik}(P_1)=O\big(w(P_1)\big)+O\big(z(P_1)\big).$$

Substitution of these inequalities into (8a)—(8b) yields for $i=1, 2$

$$\alpha_{i1}(P_1)[\bar{f}(Q)-\bar{f}(P_i)]+\alpha_{i2}(P_1)[\bar{g}(Q)-\bar{g}(P_i)]=$$

$$=O\big(w(P_1)[\bar{f}(Q)-\bar{f}(P_i)]\big)+O\big(z(P_1)[\bar{f}(Q)-\bar{f}(P_i)]\big)+ \tag{11}$$

$$+O\big(w(P_1)[\bar{g}(Q)-\bar{g}(P_i)]\big)+O\big(z(P_1)[\bar{g}(Q)-\bar{g}(P_i)]\big).$$

The inequalities

$$\bar{f}(Q)-\bar{f}(P_i)=O(h)\,;\quad \bar{g}(Q)-\bar{g}(P_i)=O(h)$$

can easily be derived as supplementary results from the existence proof of Friedrichs and Lewy (see [2], pp. 82—83) thus from (11) we get

$$\alpha_{11}(P_1)[\bar{f}(Q)-\bar{f}(P_1)]+\alpha_{12}(P_1)[\bar{g}(Q)-\bar{g}(P_1)]=$$

$$=O\big(w(P_1)\,h\big)+O\big(z(P_1)\,h\big)$$

$$\alpha_{21}(P_1)[\bar{f}(Q)-\bar{f}(P)]+\alpha_{22}(P_1)[\bar{g}(Q)-\bar{g}(P_2)]=$$

$$=O\big(w(P_1)\,h\big)+O\big(z(P_1)\,h\big).$$

Subtraction of these inequalities from (10a)—(10b) then yields

$$\alpha_{11}(P_1)[w(Q)-w(P_1)]+\alpha_{12}(P_1)[z(Q)-z(P_1)]= \tag{12a}$$

$$=O\big(w(P_1)\,h\big)+O\big(z(P_1)\,k\big)+O(h^2)$$

$$\alpha_{21}(P_1)\,[w(Q) - w(P_2)] + \alpha_{22}(P_1)\,[z(Q) - z(P_2)] = \tag{12b}$$
$$= O\big(w(P_1)\,h\big) + O\big(z(P_1)\,h\big) + O(h^2)\,.$$

Let now M_j denote the maximum of $|\,\alpha_{i1}(R)\,w(R) + \alpha_{i2}(R)\,z(R)\,|$ for $i = 1,2$ and for all grid points at the j-th level, and N_j denote the maximum of max $[\,|\,w(R)\,|, \,|\,z(R)\,|\,]$ for all grid points at the j-th level. Then[3]

$$M_j \leqq c_1 N_j\,;\;\; N_j \leqq c_2 M_j\,. \tag{13}$$

The first of these inequalities immediately follows from the boundedness of the coefficients α_{ik}, the second is a consequence of (7).

Because of the continuous differentiability of the coefficients α_{ik} and the solutions f and g we have

$$\alpha_{ik}(Q) - \alpha_{ik}(P_1) = O(h)$$

whence for $i = 1, 2$

$$\alpha_{i1}(P_1)\,w(Q) + \alpha_{i2}(P_1)\,z(Q) = \alpha_{i1}(Q)\,w(Q) + \alpha_{i2}(Q)\,z(Q) + O(M_j\,h)\,. \tag{14}$$

From (12a) and (13) we obtain

$$\alpha_{11}(P_1)\,w(Q) + \alpha_{12}(P_1)\,z(Q) = \tag{15a}$$
$$= \alpha_{11}(P_1)\,w(P_1) + \alpha_{12}(P_1)\,z(P_1) + O(M_{j-1}\,h) + O(h^2)\,.$$

Similarly from (12b) using (13) and the continuous differentiability of the coefficients we get

$$\alpha_{21}(P_1)\,w(Q) + \alpha_{22}(P_1)\,z(Q) =$$
$$= \alpha_{21}(P_1)\,w(P_2) + \alpha_{22}(P_1)\,z(P_2) + O(M_{j-1}\,h) + O(h^2) = \tag{15b}$$
$$= \alpha_{21}(P_2)\,w(P_2) + \alpha_{22}(P_2)\,z(P_2) + O(M_{j-1}\,h) + O(h^2)\,.$$

(14), (15a) and (15b) together yield

$$(1 - c_3\,h)\,M_j \leqq M_{j-1} + c_4(M_{j-1}h + h^2)$$
$$M_j \leqq M_{j-1} + c_5(M_{j-1}\,h + h^2)\,.$$

Since $M_0 = 0$ it is evident that if F_j satisfies for $j \geqq 1$ the linear difference equation

$$F_j = (1 + c_5\,h)\,F_{j-1} + c_5\,h^2$$

and the initial condition $F_0 = 0$, then $M_j \leqq F_j$. The solution of the latter difference equation problem is

$$F_j = c_5\,h^2(1 + c_5\,h)^{j-1} + h\,[(1 + c_5\,h)^{j-1} - 1]$$

thus

$$F_j < (c_5\,h^2 + h)\;e^{c_5 h(j-1)}$$

[3] c_1, c_2, c_3, c_4 and c_5 are positive constants whose numerical value is independent of h and the coordinates of the grid points (but may depend on bounds for derivatives of the coefficients and the solutions).

whence by the "stability condition" $rh = O(1)$ we have

$$F_j = O(h); \quad M_j = O(h); \quad N_j = O(h)$$

so that

$$f(R) - \bar{f}(R) = O(h); \quad g(R) - \bar{g}(R) = O(h)$$

for all grid points in the trapezoid T_2.

The considerations illustrated here are applicable without essential modification to the general characteristic equation in n unknown functions

$$\begin{aligned} &\alpha_{11} \quad f^1_\xi + \dots + \alpha_{1n} f^n_\xi = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\alpha_{m1} \quad f^1_\xi + \dots + \alpha_{mn} f^n_\xi = 0, \\ &\alpha_{m+1,1} f^1_\eta + \dots + \alpha_{m+1,n} f^n_\eta = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\alpha_{n1} \quad f^1_\eta + \dots + \alpha_{nn} f^n_\eta = 0, \end{aligned}$$

where $0 < m < n$ and $\alpha_{ik} = \alpha_{ik}(f^1, f^2, \dots, f^n)$. In the case of the characteristic equations (3a)—(3d) we get

$$(16) \quad x(R) - \bar{x}(R) = O(h); \quad y(R) - \bar{y}(R) = O(h); \quad u^*(R) - \bar{u}^*(R) = O(h), \\ v^*(R) - \bar{v}^*(R) = O(h)$$

for all grid points in the trapezoid T.

Now let (x, y) be an arbitrary point in the region S and (ξ, η) the point in the ξ, η-plane which corresponds to (x, y) by the transformation

$$\xi = \xi(x, y) \quad \eta = \eta(x, y).$$

From the construction of the grid points in the ξ, η-plane it follows that there exists a grid point R such that

$$|\xi - \xi(R)| < h, \quad |\eta - \eta(R)| < h.$$

Then because of the continuous differentiability of the functions $x(\xi, \eta)$, $y(\xi, \eta)$, $u(x, y)$ and $v(x, y)$ we have

$$(17) \quad \begin{aligned} &x - x(R) = O(h); \quad y - y(R) = O(h), \\ &u(x, y) - u\big(x(R), y(R)\big) = O(h); \quad v(x, y) - v\big(x(R), y(R)\big) = O(h). \end{aligned}$$

The inequalities (16) may be rewritten as

$$(18) \quad \begin{aligned} &x(R) - \bar{x} = O(h); \quad y(R) - \bar{y} = O(h), \\ &u\big(x(R), y(R)\big) - u(\bar{x}, \bar{y}) = O(h); v\big(x(R), y(R)\big) - \bar{v}(\bar{x}, \bar{y}) = O(h). \end{aligned}$$

From (17) and (18) immediately follows the assertion of our theorem.

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ОЦЕНКА ПОГРЕШНОСТИ МЕТОДА ХАРАКТЕРИСТИК (МЕТОДА МАССО)

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Резюме

В связи с так называемым методом характеристик (методом Массо) доказывается следующая теорема:

Пусть h — максимальная длина дуги между двумя соседними точками сетки на начальной кривой $\mathcal{J}$ и r — число точек сетки на кривой $\mathcal{J}$. Пусть, далее, выполняется «условие устойчивости» $rh = O(1)$. Тогда к каждой точке (x, y) в некоторой области S ограниченной кривой $\mathcal{J}$, характеристиками, проходящими через концы $\mathcal{J}$ и кривой $\mathcal{J}'$ параллельной к $\mathcal{J}$, можно найти точку сетки $(\bar{x}, \bar{y})$ так что выполняются неравенства

$$x - \bar{x} = O(h);\; y - \bar{y} = O(h);\; u(x, y) - \bar{u}(\bar{x}, \bar{y}) = O(h)\,;\; v(x, y) - \bar{v}(\bar{x}, \bar{y}) = O(h)$$

где u и v — точные решения задачи Коши для системы уравнений (1a)—(1b) а $\bar{u}$ и $\bar{v}$ — приближенные значения по методу Массо.