ON SOME PROBLEMS OF P. VERMES

by

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1. Introduction

At the end of a recent paper [1], P. VERMES lists some problems which arose in his work and we are now in a position to answer four of his questions, all in the negative. We first list these questions.

(a) Is the convolution of two sequences in $\mathcal{D}'$ always in $\mathcal{D}'$? In other words if the power series of $u(z)$ and $v(z)$ are convergent at the rational points on the unit circle, is the same true for the power series of $u(z)\,v(z)$?

(b) if u is in $\mathcal{D}'$, i. e. if $u(z) = \Sigma u_k z^{k-1}$ is convergent at every rational point on the unit circle, is the set of values of $u(z)$ at such points bounded?

(c) is $\mathcal{S}^+$ a ring?

(d) if A, B are in $\mathcal{S}$, x in $\mathcal{D}'$ is always $A(Bx) = (AB)x$?

2. Solutions

In the first place, as VERMES points out, if (d) were true it would imply (c) and if (c) were true it would imply (a). Hence to show that the answer to all four questions is no it is sufficient to show that the answers to both (a) and (b) are no.

Let $u_k = \binom{\alpha}{k-1} e^{i(k-1)} = e^{i(k-1)} \dfrac{\Gamma(\alpha+1)}{(k-1)!\,\Gamma(\alpha-k+2)}$. Then, the series $\sum\limits_1^\infty u_k z^{k-1}$ is $\sum\limits_{k=1}^{\infty} \dfrac{\Gamma(\alpha+1)}{(k-1)!\,\Gamma(\alpha-k+2)} (e^i z)^{k-1}$, or the binomial expansion of $(1+ze^i)^\alpha$. If $0 > \alpha > -1$ the series is convergent for $|ze^i| \leq 1$ except if $ze^i = -1$, that is $z = e^{i(\pi-1)}$, which is not a rational point. Hence the series is convergent at all rational points on the unit circle. However, the value of $u(z)$ is $(1+ze^i)^\alpha$ and since $0 > \alpha > -1$, if $\{t_n\}$ is a sequence of rational numbers with limit point $\frac{1}{2}(1-1/\pi)$, then the sequence $\{u(e^{2it_n\pi})\}$ is unbounded. This answers (b).

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Now consider two such series, and let

$$u_k = v_k = e^{i(k-1)} \frac{\Gamma(\alpha + 1)}{(k-1)!\,\Gamma(\alpha - k + 2)}$$

where now $-\dfrac{1}{2} > a > -1$.

Then $u(z) = v(z) = (1 + ze^i)^a$, and both the series are convergent at every rational point on $|z| = 1$. But $u(z)\,v(z) = (1 + ze^i)^{2a}$, and since $-1 > 2\alpha$ this binomial expansion is *not* convergent at any point on $|ze^i| = 1$, that is $|z| = 1$. Hence the answer to (a) is also no.

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REFERENCE

[1] P. Vermes: "Transformations of Periodic Sequences." This journal **5** (1960) 153—163.