UNSOLVED PROBLEMS

Problems for this section as well as comments on published problems should be sent to G. ALEXITS, editor of the section, to the adress of the redaction of the journal (Budapest, V. Reáltanoda u. 13—15).

НЕРАЗРЕШЕННЫЕ ППОБЛЕМЫ

Проблемы предназначные для этого раздела, а также замечания, связанные с сообщаемыми проблемами просим направить по адрессу редакции журнала (Budapest, V. Reáltanoda u. 13—15) для редактора раздела G. ALEXITS.

RESEARCH PROBLEMS

by

PAUL TURÁN

(1) If $1 \geqq x_1 > x_2 \ldots > x_n \geqq -1$ and $l_\nu(x)$ $(\nu = 1, 2, \ldots, n)$ are the fundamental functions of the Lagrange interpolation on these x_j's, then what is the minimum of

$$\int_{-1}^{1} \left(\sum_{\nu=1}^{n} |l_\nu(x)| \right) dx$$

and what are the extremal $(x_1, x_2, \ldots, x_j)$-systems? (Probably this is $\sim c \log n$, giving a far reaching generalization of G. FABER's classical theorem.)

(2) Let the x_j 's and $l_\nu(x)$'s of the previous problem be such that the numbers

$$\lambda_\nu \stackrel{\text{def}}{=} \int_{-1}^{1} l_\nu(x)\, dx \geqq 0 \qquad \nu = 1, 2, \ldots, n$$

are all nonnegative. If $1 \leqq j \leqq n$ is fixed, what is the exact range of x_j? (If $x_1, x_2, \ldots, x_n$ form a so-called strongly normal sequence introduced by FEJÉR, then the corresponding problem has been solved by P. ERDŐS. (*Proc. of the Nat. Acad. of USA* (1940) p. 294—297).

(3) If $2 \leqq l < k \leqq n$, then what is the minimal number μ of combinations $C_1, C_2, \ldots, C_\mu$ taken l at a time out of $1, 2, \ldots, n$ with the property that each combination taken k at a time out of $1, 2, \ldots, n$ contains at least one C_j? (For $l = 2$ the question is settled with exhibiting the *only* minimal C-system in my paper "Egy gráfelméleti szélsőértékfeladatról", *Mat. és Fiz. Lapok* (1941) 436—451, in Hungarian with German abstract.)

(4) If

$$\max_{j=1,\ldots,n} |z_j| = 1,$$

what is the minimum of

$$\max_{\nu=1,2,\ldots,n} |z_1^\nu + z_2^\nu + \ldots + z_n^\nu|$$

and what are the extremal-systems? (The solution of this longstanding problem can be applied in the theory of approximative solution of algebraic equations. See my paper "Remark on the preceding paper of J. W.S. CASSELS "*Acta*

Math. Hung. **7** (1956) p. 291—294. Recently F. V. ATKINSON made a large step towards the solution showing that this minimum is at least $\frac{1}{6}$. See his paper „On sums of powers of complex numbers." *Acta Math. Hung.* **12** (1961) p. 185—188.

(5) Does it follow from the truth of Riemann's conjecture that for all sufficiently large n's the Dirichlet-polynomials

$$u_n(s) = \sum_{\nu \leq n} \nu^{-s} \qquad (s = \sigma + it)$$

do not vanish for $\sigma > 1$, $|t| < e^n$? (In my paper „Nachtrag . . ." *Acta Math. Hung.* **10** (1959) 277—298, I proved it for

$$\sigma > 1, \quad c_1 \leq |t| \leq e^{e^{c_2 \sqrt{\log n \log \log n}}}$$

with a positive numerical c_1, c_2.)

(6) In our paper "On a problem . . ." *Indag. Math.* **10** (1948) 1146—1154. resp. 406—413, with P. ERDŐS we proved for real $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $\lambda \leq 1$ that if

$$\max_{1 \leq k \leq n^{\frac{1}{\lambda+1}}} \left| \sum_{j=1}^{n} e^{ki\varphi_j} \right| k^{-\lambda} \leq 1,$$

then for $0 \leq \alpha < \beta \leq 2\pi$ we have

$$\left| \sum_{\substack{\alpha \leq \varphi_\nu \leq \beta \bmod 1 \\ \nu \leq n}} 1 - \frac{\beta - \alpha}{2\pi} n \right| < C n^{\frac{\lambda}{\lambda+1}}$$

with a positive numerical C. Is the order $n^{\frac{\lambda}{\lambda+1}}$ the bestpossible?

(7) If p is an odd prime, we denote by $N_p(n)$ the number of the solutions of the Fermat-equation $x^p + y^p = z^p$ with

$$1 \leq x, y, z \leq n, \qquad (x, y) = (x, z) = (y, z) = 1$$

Is it true that the inequality

$$N_p(n) < c(p)\, n^{\frac{1}{p}}$$

holds, where $c(p)$ depends only upon p ? In our paper "A second note on Fermat's conjecture" with P. DÉNES in *Publ. Math.* **4** (1955) 28—32, we proved only

$$N_p(n) < c(\mathrm{p}) \frac{n^{\frac{2}{p}}}{\log^{2-\frac{2}{p}} n}.$$

(8) As a conversion of the theorems of Descartes sign-rule-type I proved (*Bull. of Amer. Math. Soc.* **55** (1949) 797—800.) that writing an arbitrary polynomial $f(x)$ of degree n with real coefficients in the form

$$f(x) = \sum_{\nu=0}^{n} b_\nu L_\nu(x)$$

where $L_\nu(x)$ stands for the ν^{th} Laguerre-polynomial

$$L_\nu(x) = \frac{e^x}{\nu!}\frac{d^\nu}{dx^\nu}(e^{-x}x^\nu),$$

then the number of positive zeros of $f(x)$ is *not less* than the number of changes of sign in the sequence

$$b_0, (b_0 - b_1), \quad (b_0 - 2b_1 + b_2), \ldots, \left(b_0 - \binom{n}{1}b_1 + \binom{n}{2}b_2 - \ldots + (-1)^n\binom{n}{n}b_n\right).$$

For this theorem an integral-free proof is wanted.

(9) If the ν^{th} Hermite-polynomial $H_\nu(z)$ is defined as usual by

$$H_\nu(z) = (-1)^\nu e^{z^2}\frac{d^n}{dz^n}(e^{-z^2}),$$

does there follow from

$$0 < a_0 < a_1 < \ldots < a_n$$

the reality of all zeros of the polynomial

$$\sum_{\nu=0}^{n}\frac{(-1)^\nu a_\nu}{2^{2\nu}(2\nu)!}H_{2\nu}(x) \text{ ?}$$

(For the background of this problem see my paper "Sur l'algèbre fonctionnelle", *Comptes Rendus du Prem. Congr. des Math. Hongr.* 1950.)

(10) Is it true that if

$$f(z) = \sum_{\nu=0}^{n} b_\nu H_\nu(z)$$

($H_\nu(z)$ again the ν^{th} Hermite-polynomial) then fixing the positive integers p and h with $p + h \leq n$ and the coefficients $b_1, b_1, \ldots, b_{p-1}$ and b_{p+h} *the imaginary parts* of p zeros of $f(z)$ remain absolutely bounded when the other b_j's and n vary? (The analogous theory for the "Vieta-expansion" $\sum_{\nu=0}^{n} c_\nu z^\nu$ of $f(z)$ is due to LANDAU, FEJÉR, BIERNACKI, FEKETE, MONTEL and others.) For indications see my previously quoted paper.

(11) If with arbitrary complex a_ν's and $z = x + iy$

$$V(z) = \sum_{\nu=0}^{n} a_\nu z^\nu, \qquad H(z) = \sum_{\nu=0}^{n} a_\nu H_\nu(z),$$

is it true that for all positive D the strip $|y| \leq D$ can contain at most as many from the zeros of $V(z)$ as the strip $|y| \leq \frac{D}{2}$ from the zeros of $H(z)$ (both counted with multiplicity)? I could prove this theorem only in special cases, see my previous quoted paper "Sur l'algèbre fonctionnelle".

(12) Denoting by $k_j(n)$ the minimal number of 0 's in an $n \times n$ -matrix with elements 0, 1 which ensures the existence of a $j \times j$ submatrix consisting exclusively from 0 's, is it true that

$$k_j(n) > c(j)\, n^{2-\frac{1}{j}},$$

where $c(j)$ depends only upon j ? The inequalities

$$k_j(n) < 1 + jn + \left[(j-1)^{\frac{1}{j}} n^{2-\frac{1}{j}}\right]$$

and

$$\lim_{n\to\infty} \frac{k_2(n)}{n^{3/2}} = 1$$

are proved in our paper with T. KŐVÁRI and Vera T. Sós entitled "On a problem of K. Zarankiewicz" in *Coll. Math. Vol.* **3** (1954) 50—57.

(13) What is the "exact domain" of Lipschitz-classes for the "fine" quadrature-convergence theory in the sense of our paper with P. ERDŐS entitled "On the role of the Lebesgue functions in the theory of Lagrange-interpolation", *Acta Math. Hung.* **6** (1955) 1—2, p. 47—66 in particular p. 50?

(14) what is the lim inf of those A-numbers for which the inequality

$$\max_{\nu=m+1,\ldots,m+n} |z_1^\nu + \ldots + z_n^\nu| \geqq \left(\frac{n}{A(m+n)}\right)^n$$

holds whenever for the variable z_j 's

$$\max_{j=1,\ldots,n} |z_j| = 1,$$

m and n *are fixed* positive integers. That $A^* \leq 8e$, is contained in our paper with Vera T. Sós entitled "On some new theorems etc. . ." *Acta Math. Hung.* **6** (1955) 241—256; that $A^* > 1{,}473$, was proved by E. MAKAI in his paper "An estimation in the theory of diophantine approximations" *Acta Math. Hung.* **9** (1958) 299—307.

(15) Let $f(z) = \sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$ an entire-function with the Fabry-condition $\lim_{n\to\infty} \frac{n}{\lambda_n} = 0$, further we denote

$$\max_{|z|=r} |f(z)| = M(r), \quad \max_{|z|=r} |f(z)| = M(r, \alpha, \beta),$$

$$\alpha \leqq \operatorname{arc} z \leqq \beta$$

as usual. Is it true that for an arbitrarily small but fixed positive ε and δ the measure $R_{\delta,\varepsilon}(\omega)$ of r-values not exceeding ω with the property

$$M(r)^{1-\varepsilon} \leq \max_{\alpha} M(r, \alpha, \alpha + \delta) \leq M(r)$$

has the property

$$\lim_{\omega\to\infty} \frac{1}{\omega} R_{\delta,\varepsilon}(\omega) = 1 .$$

Results in this direction in my paper „Über lakunären Potenzreihen", *Revue de Math. pures et appl. Romania* **1** (1956) 27—32. and T. KŐVÁRI's paper entitled "On the gap-theorems of G. Polya and P. Turán." *Journal d'Analyse.* (1958) 323—332.

(16) For fixed $A > 0$ and $c > 1$ we denote by $S(N, A, c)$ the set of those α 's in $0 < \alpha < 1$ for which with an integer $N \geq 2$, x and $N \leq y \leq cN$ the system

$$\left|\alpha - \frac{x}{y}\right| \leq \frac{A}{y^2}, \qquad (x, y) = 1 , \quad y > 1$$

is solvable. Denoting by $|S(N, A, c)|$ the measure of $S(N, A, c)$ does

$$\lim_{N\to\infty} |S(N, A, c)| = f(A, c)$$

exist? Results in this direction are in our paper with ERDŐS and SZÜSZ in *Coll. Math.* **6** (1958) 119—125 entitled „Remarks on the theory of diophantine approximation" and ERDŐS's paper entitled "Some results on diophantine approximation" *Acta Arith.* **5** (1959) 359—369.

(17) Denoting the set of those α 's in $0 < \alpha < 1$ for which with an integer $N \geq 2$ and $c > 1$ the interval $N \leq y \leq cN$ contains at least one denominator q_ν of the regular continued fraction of α, by $R(N, c)$, does

$$\lim_{N\to\infty} |R(N, c)| = \varphi(c)$$

exist? (See again our above-quoted paper with Erdős and Szüsz.)

(18) If $N_n(V, x)$ stands for the number of integers $k \leq n$, for which

$$\frac{V(k) - \log\log n}{\sqrt{\log\log n}} \leq x,$$

($V(k)$ the number of *all* prime-factors of k) then we proved with A. RÉNYI in our paper "On a theorem of Erdős-Kac", *Acta Arith.* **4** (1958) 71—84, that for fixed x and $n \to \infty$

$$N_n(V, x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du + O\left(\frac{n}{\sqrt{\log\log n}}\right)$$

uniformly in x. How to modify the proof to obtain the corresponding theorem of ERDŐS—KAC *with remainder-term* for general additive functions

$$f(mn) = f(m) + f(n) \quad \text{for} \quad (m, n) = 1$$

$$B_m = \sum_{p\leq m} \frac{f(p)^2}{p} \to \infty \quad \text{for} \quad m \to \infty$$

$$f(p) = O\left(\sqrt{B_p}\right), \quad p \quad \text{prime}$$

in the form that if $N_n(f, x)$ stands for the number of integers $k \leqq n$ for which

$$\frac{f(k) - \sum\limits_{p \leqq n} \frac{f(p)}{p}}{B_n^{1/2}} \leqq x ,$$

then for fixed x and $n \to \infty$

$$N_n(f, x) = \frac{n}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{u^2}{2}} du + O\left(\frac{n}{\sqrt{B_n}}\right)$$

uniformly in x ? [ERDŐS—KAC published their theorem in the paper "The Gaussian Law of Errors in the theory of additive numbertheoretic functions", *Amer. J. of Math.* **62** (1940).] How to modify our proof for the corresponding situation concerning $V(g(k))$ where $g(x)$ is an irreducible polynomial with integer coefficients (for the first result in this direction in my paper „Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan", *Journ. of Lond. Math. Soc.* **11** (1936) 125—133) or more generally to $f(g(k))$ with the above $f(x)$ (see the paper of H. HALBERSTAMM "On the distribution of additive numbertheoretic functions II." *Journ. of Lond. Math. Soc.* **31** (1956) 1—14)?

(19) To determine all polynomial solutions $\psi(z)$ of degree n of the differential-functional equation

$$\lambda\psi'(z) - \bar{\lambda}\bar{\psi}'\left(\frac{1}{z}\right) = \frac{z^n}{1-z}\psi(z)\,\bar{\psi}\left(\frac{1}{z}\right)$$

with $\psi(1) = 0$; λ is a suitable constant. (For the background see my paper "Über die Potenzsummen komplexer Zahlen" *Archiv der Math.* **9** (1958) 59—64.)

(20) Does there exist an $f(z) = \sum\limits_{\nu=1}^{\infty} a_\nu z^\nu$ regular in $|z| < 1$ and continuous for $|z| \leqq 1$ such that $\sum\limits_{\nu} a_\nu$ converges and putting

$$f\left(\frac{2z-1}{2-z}\right) = \sum_{\nu=0}^{\infty} b_\nu z^\nu$$

the series $\Sigma\, b_\nu$ diverges? (for the fact that without the continuity-requirement this is possible and for general background see my paper "A remark concerning the behaviour of a power-series . . ." *Publ. de l'Inst. Math. de l'Acad. Serbe des Sc. Beograd* **12** (1958) 19—26.)

(21) With the notations of the problem 20 does it follow from the convergence of $\sum\limits_{\nu} |a_\nu|$ that of $\sum\limits_{\nu} |b_\nu|$? (This problem was in the meantime solved by L. ALPÁR in the negative. See his paper in *Matematikai Lapok* **11** (1960) 4, p. 312—322 in Hungarian with Russian and French summaries).

(22) Let $\varphi_0, \varphi_1, \ldots$ be a normed orthogonal system with respect to $[-\pi, \pi]$ say, where the φ_j 's are here continuous and $\varphi_j(\pi) = \varphi_j(-\pi)$ ($j =$

$= 0, 1, \ldots$). We say that this system has the Haar-property if the expansion

$$f = \sum_{\nu}^{\infty} a_\nu \varphi_\nu$$

converges uniformly to f in $[-\pi, \pi]$, whenever f is continuous in $[-\pi, \pi]$ and $f(\pi) = f(-\pi)$. Does there exist a φ-system having the Haar-property so that the system of conjugate functions (in the sense of the trigonometrical series) which form automatically a normed orthogonal system too, has the Haar-property too? For background see my paper "On the infinite product representation of functions", *Bull. de l'Acad. Polon. des Sc.* **7** (1959) 481—486.

(23) If k is an arbitrary positive integer then I showed that in whatever way we split the integers $k, k+1, \ldots, 5k+3$ into two classes, the equation $x + y = z$ is solvable in at least one class in *different* integers and for $k, k+1, \ldots, 5k+2$ this is not true. What is the corresponding theorem in the case of three classes? As I. Schur proved in whatever way we split the integers $1, 2, \ldots, ([en!]+1)$ into n classes, the equation $x + y = z$ is in at least one class solvable (not necessarily in different integers). See his paper "Über die Kongruenz $x^m + y^m \equiv z^m \bmod p$", *Jahresber. der deutsch. Math. Ver.* **25** (1916) 114—117.

(24) If $2 = p_1 < p_2 < \ldots < p_n$ are the n first primes, $\beta_1, \beta_2 \ldots, \beta_n$ real and ω integer and $\geqq 4$, is it true that the system

$$\|t \log p_v - \beta_v\| \leqq \frac{1}{\omega}, \qquad v = 1, 2, \ldots, n$$

is solvable in

$$0 \leqq t \leqq e^{c_1 n}$$

with a suitable positive numerical c_1? Here $\|x\|$ stands as usual for the distance of x from the *next* integer. My paper "A theorem on diophantine approximation with application to Riemann zeta function" to appear in *Acta Szeged* will contain the proof of the corresponding theorem for the interval $0 \leqq t \leqq e^{n^{\frac{3}{2}}}$.

(25) The classical theorem of Dirichlet (for positive α_j's and integer $q \geqq 3$ there is an integer $\nu \leqq q^n$ so that $\|\nu \alpha_j\| \leqq \frac{1}{q}$ $(j = 1, \ldots, n)$) is to be proved on the analyticl way I postulated in my book „*Eine neue Methode in der Analysis und deren Anwendungen*" (p. 19-20). (Essential progress was made in this direction by J. W. S. Cassels in his paper „On the sums of powers of complex numbers." *Acta Math. Hung.* **7**(1956) 3-4, p. 283-289).