ON A PROBLEM OF INFORMATION THEORY

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Abstract

A typical example of the situation, a mathematical model of which is discussed in this paper, is the following: one has to find that part of a complicated network which got out of order. In such a situation if the number of places of the defect is very large, a possible method to find the defect is to try whether certain sub-networks do work or not. If a sub-network (consisting of a subset of the parts of the total network) does not work, then it contains the defective part (it is supposed that only a single part is defective), while if it works, then the defective part is contained in the complementary subset. If the network consists of n parts and if $k > \log_2 n$ suitably chosen sub-networks are tested in this way, the defective part can be determined. The question arises, how many sub-networks have to be tested, if the sub-networks are chosen completely at random. The answer to this question may be useful if one wants to construct defect-searching automata. The corresponding mathematical problem is solved under the supposition that the results of the tests of the sub-networks are correct with probability β only ($1/2 < \beta \leq 1$) while with probability $1-\beta$ they are false and thus misleading.

Let us put

$$I(\beta) = \beta \log_2 \frac{1}{\beta} + (1-\beta) \log_2 \frac{1}{1-\beta}. \tag{1}$$

It is shown that if the total number of parts any one of which may be defective is equal to n, and $k = k(n)$ randomly and independently chosen sub-networks are tested (so that the probability of choosing any particular sub-network is the same) where

$$k(n) = \frac{\log_2 n + y\sqrt{\log_2 n} + o(\sqrt{\log_2 n})}{1 - I(\beta)} \tag{2}$$

where y is a fixed real number, then if P_{nk} denotes the probability that by evaluating the tests in a suitable way, the defective part can be identified, one has

$$\lim_{n \to +\infty} P_{n,k(n)} = \Phi\left(\frac{y}{\sigma}\right) \tag{3}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{4}$$

and

$$\sigma = \sqrt{\frac{\beta(1-\beta)}{I(\beta)}} \log_2 \frac{\beta}{1-\beta} \tag{5}$$

It is mentioned that the mathematical problem solved above may also be considered as a highly simplified model of other processes too, e. g. the process

applied by a physician trying to make a diagnose, or the work of a chemist who wants to analyse some material of an unknown composition, or even of a judge trying to find out the truth in some criminal case. The special case when all tests lead to reliable results (i. e. $\beta = 1$) has been considered in previous papers ([1], [2], [3]) of the author. It is pointed out that from the point of view of statistics the problem is one of discrimination, while from the point of view of information theory the problem can be characterized as follows: a discrete noisy memoryless channel is given and the same symbol is transmitted several times; one has to determine this transmitted symbol from the received symbols. The special channel corresponding to the problem solved in the paper is such that the number of symbols which can be sent is n, that of symbols which can be received is 2^n while the matrix of transition probabilities (the channel probability function) contains only two sorts of elements, so that in each row exactly one half of the elements belong to the first, and one half to the second sort and all columns are different.