A GENERALIZATION OF A THEOREM OF SZEGŐ

by

BÉLA GYIRES[1]

Dedicated to Professor P. Turán
at his 50 th birthay.

1. In the present paper we are going to consider quadratic matrices of order p, the elements of which are real or complex functions defined in the interval $[-\pi, \pi]$. In the sequel such matrices will be called functional matrices. If we state a condition for a functional matrix, this must be taken in the sense that the condition must hold for all elements of the matrix. In this sense we speak of bounded, measurable, integrable, continuous etc. functional matrices. By the integral of an integrable functional matrix we understand the matrix formed from the integrals of its elements, and denote this by writing the functional matrix in question as integrated behind the integral sign. The unit matrix and the zero matrix will always be denoted by $\boldsymbol{E}$ and $\boldsymbol{O}$ respectively, and the spur of a matrix will be denoted by an "Sp" written before the sign of the matrix.

A Hermitian matrix will be called positive definite, if the corresponding Hermitian form vanishes only in the point 0, and takes in every other place positive values. If the form takes the value zero also outside the point 0, while it is nowhere negative, the corresponding matrix will be called positive semidefinite. The positive definite and the positive semidefinite Hermitian matrices will have the common name of nonnegative definite matrices. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hermitian matrices. We say that $\boldsymbol{A} \geq \boldsymbol{B}$, if $\boldsymbol{A} - \boldsymbol{B}$ is nonnegative definite, and $\boldsymbol{A} > \boldsymbol{B}$ if $\boldsymbol{A} - \boldsymbol{B}$ is positive definite.

In the sequel ν runs through the rational integers, n through the nonnegative integers, k, l through the integers from 0 to n, and α through the integers from 1 to p. $\boldsymbol{z}$ and $\boldsymbol{z}_k$ denote respectively an arbitrary row vector of the p-dimensional space with complex elements, while $\boldsymbol{x}_k$ stands for a quadratic matrix of order p built from arbitrary complex numbers.

2. Let $\boldsymbol{f}(x)$ be a Lebesgue measurable Hermitian matrix, positive definite on a set of positive measure, and nonnegative definite elsewhere. With the aid of this functional matrix we form the matrices

$$\boldsymbol{a}_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \boldsymbol{f}(x)\, e^{-i\nu x}\, dx$$

and with the aid of these the hypermatrices

$$\boldsymbol{T}_n(\boldsymbol{f}) = \begin{pmatrix} \boldsymbol{a}_0 & \boldsymbol{a}_1 & \dots & \boldsymbol{a}_n \\ \boldsymbol{a}_{-1} & \boldsymbol{a}_0 & \dots & \boldsymbol{a}_{n-1} \\ \cdot & \cdot & \dots & \cdot \\ \boldsymbol{a}_{-n} & \boldsymbol{a}_{-n+1} & \dots & \boldsymbol{a}_0 \end{pmatrix}. \tag{1}$$

[1] University of Debrecen. Department of mathematics.

$\boldsymbol{T}_n(\boldsymbol{f})$ and the determinant $D_n(\boldsymbol{f}) = \text{Det}\, \boldsymbol{T}_n(\boldsymbol{f})$ will be called the n-th Toeplitzian matrix and determinant respectively, generated by the functional matrix $\boldsymbol{f}(x)$. (For the case $p = 1$ see [2], 17, 38.)

As one easily sees, these are Hermitian matrices. It is also easy to show that they are positive definite. Indeed, we form with the vectors $\boldsymbol{z}_k$ the Hermitian form

$$\sum_{k,l=0}^{n} \boldsymbol{z}_k \boldsymbol{a}_{l-k} \boldsymbol{z}_l^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=0}^{n} \boldsymbol{z}_k e^{ikx} \right) \boldsymbol{f}(x) \left(\sum_{k=0}^{n} \boldsymbol{z}_k e^{ikx} \right)^* dx . \tag{2}$$

Since $\boldsymbol{f}(x)$ is a Hermitian functional matrix, positive definite on a set of positive measure, the function

$$\left(\sum_{k=0}^{n} \boldsymbol{z}_k e^{ikx} \right) \boldsymbol{f}(x) \left(\sum_{k=0}^{n} \boldsymbol{z}_k e^{ikx} \right)^*$$

is positive on a set of positive measure, if only $\sum_{k=1}^{n} \boldsymbol{z}_k \boldsymbol{z}_k^* > 0$, i. e. if the right hand side of (2) is positive.

The purpose of the present paper is to demonstrate the following

Theorem. *If $\boldsymbol{f}(x)$ is a Lebesgue measurable, nonnegative definite Hermitian functional matrix, then*

$$\frac{D_n(\boldsymbol{f})}{D_{n-1}(\boldsymbol{f})} \downarrow \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \text{Det}\, \boldsymbol{f}(x)\, dx \right\} \tag{3}$$

and this limit is to be taken equal to zero, if

$$\int_{-\pi}^{\pi} \log \text{Det}\, \boldsymbol{f}(x)\, dx = -\infty . \tag{4}$$

The author has investigated matrices of the form (1) in several papers ([3], [4], [5]). In all three of these he has derived the theorem just formulated under more special conditions. So, in his paper [4], the author has obtained the limit (3) under the assumption that $\boldsymbol{f}(x)$ is a continuous, real, symmetric functional matrix, and

$$\inf \boldsymbol{z}\boldsymbol{f}(x)\boldsymbol{z}^* = m > 0\,, \quad x \in [-\pi, \pi]\,, \quad \boldsymbol{z}\boldsymbol{z}^* = 1\,. \tag{5}$$

In papers [3] and [5] the expression (3), and its equivalent form based on the limit relation

$$\lim_{n\to\infty} \frac{D_n(\boldsymbol{f})}{D_{n-1}(\boldsymbol{f})} = \lim_{n\to\infty} \sqrt[n+1]{D_n(\boldsymbol{f})} \tag{6}$$

has occurred as a special case of more general theorems. Indeed, this was the case in [3] for continuous Hermitian functional matrices satisfying condition (5), while in [5] the condition of continuity was replaced by boundedness and measurability.

As is well known, eigenvalues of Toeplitzian matrices in the case $p = 1$, i.e. when the generating functional matrix is replaced by a generating function, have first been investigated by G. Szegő ([8], [9], [10]). He was also the first to obtain for the case $p = 1$ the result expressed in our theorem ([8], [9], [10]).

In the proof of our theorem we make use of the following theorem of HELSON and LOWDENSLAGER ([6], 186):

If $\boldsymbol{f}(x)$ is a Lebesgue measurable nonnegative definite Hermitian matrix and $d\boldsymbol{M}$ an arbitrary measure taking as values nonnegative Hermitian matrices and having the absolute continuous part $\frac{1}{2\pi}\boldsymbol{f}(x)\,dx$, then

$$\inf_{\boldsymbol{x}_0, \boldsymbol{P}} \frac{1}{p}\int_{-\pi}^{\pi} \mathrm{Sp}\,[(\boldsymbol{x}_0 + \boldsymbol{P})(\boldsymbol{x}_0 + \boldsymbol{P})^*\,d\boldsymbol{M}] = \exp\left\{\frac{1}{2\pi p}\int_{-\pi}^{\pi} \mathrm{Sp}\log \boldsymbol{f}(x)\,dx\right\}, \tag{7}$$

where $\boldsymbol{x}_0$ runs through all unimodular matrices and $\boldsymbol{P}$ through the trigonometric polynomial matrices

$$\boldsymbol{P}(e^{ix}) = \sum_{k>0} \boldsymbol{x}_k\, e^{ikx}. \tag{8}$$

(7) *must be taken equal to zero, if*

$$\int_{-\pi}^{\pi} \mathrm{Sp}\,\log \boldsymbol{f}(x)\,dx = -\infty\,.$$

It is also known that the theorem just quoted of HELSON and LOWDENSLAGER has been proved for the case $p = 1$ by G. SZEGŐ under the assumption that $\boldsymbol{f}(x)$ is absolutely continuous, and later without this assumption by KOLMOGOROV ([7]).

In order to establish a connection between our theorem and that of HELSON and LOWDENSLAGER, we shall need the matricial generalization of the Lagrange transformation relative to positive definite Hermitian hypermatrices built from matrices of order p. This generalization will be given in **3.** The theorem itself will be proved under **4.** There it will also be pointed out that from a special case proved earlier [5] of the theorem of the author formulated in this paper the special case of the theorem of HELSON and LOWDENSLAGER can easily be derived.

3. Let the positive definite Hermitian matrix of order $(m + 1)p$

$$\boldsymbol{A}_m = \begin{pmatrix} \boldsymbol{a}_{00} & \boldsymbol{a}_{01} & \cdots & \boldsymbol{a}_{0m} \\ \boldsymbol{a}_{10} & \boldsymbol{a}_{11} & \cdots & \boldsymbol{a}_{1m} \\ \cdot & \cdot & \cdots & \cdot \\ \boldsymbol{a}_{m0} & \boldsymbol{a}_{m1} & \cdots & \boldsymbol{a}_{mm} \end{pmatrix} \tag{9}$$

be given, which is built from matrices $\boldsymbol{a}_{kl}$ of order p. By the form belonging to the matrix (9) we understand the matrix

$$\sum_{k,l=0}^{m} \boldsymbol{x}_k\, \boldsymbol{a}_{kl}\, \boldsymbol{x}_l^* = \boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)\,. \tag{10}$$

Now we prove the following lemma, constituting a generalization of the well known Lagrange transformation:

Lemma 1. *By the transformation*

$$(\xi_0, \xi_1, \ldots, \xi_m) = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)\,\boldsymbol{B}$$

defined with the aid of the hypermatrix

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{o} & \boldsymbol{o} & \ldots & \boldsymbol{o} \\ \beta_{01} & \boldsymbol{E} & \boldsymbol{o} & \ldots & \boldsymbol{o} \\ \beta_{02} & \beta_{12} & \boldsymbol{E} & \ldots & \boldsymbol{o} \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \beta_{0m} & \beta_{1m} & \beta_{2m} & \ldots & \boldsymbol{E} \end{pmatrix}$$

built from matrices of order p which depend only on the elements of the matrix (9)' *the positive definite Hermitian form* (10) *can be given in the form*

$$\boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) = \xi_0\,\gamma_0\,\xi_0^* + \xi_1\,\gamma_1\,\xi_1^* + \ldots + \xi_m\,\gamma_m\,\xi_m^*,$$

where $\gamma_0, \gamma_1, \ldots, \gamma_m$ are positive definite Hermitian matrices of order p.

For $p = 1$ our theorem gives the Lagrange transformation for positive definite Hermitian forms.

Proof. Let

$$\boldsymbol{a}_{j0}\,\boldsymbol{a}_{00}^{-1} = \beta_{0j} \qquad (j = 1, \ldots, m)$$

and

$$\xi_0 = \boldsymbol{x}_0 + \boldsymbol{x}_1\,\beta_{01} + \ldots + \boldsymbol{x}_m\,\beta_{0m}.$$

Since

$$\xi_0\,\boldsymbol{a}_{00}\,\xi_0^* = \boldsymbol{x}_0\,\boldsymbol{a}_{00}\,\boldsymbol{x}_0^* + \boldsymbol{x}_1\,\boldsymbol{a}_{10}\,\boldsymbol{x}_0^* + \ldots + \boldsymbol{x}_m\,\boldsymbol{a}_{m0}\,\boldsymbol{x}_0^* + \boldsymbol{x}_0\,\boldsymbol{a}_{01}\,\boldsymbol{x}_1^* +$$

$+ \boldsymbol{x}_0\,\boldsymbol{a}_{02}\,\boldsymbol{x}_2^* + \ldots + \boldsymbol{x}_0\,\boldsymbol{a}_{0m}\,\boldsymbol{x}_m^* +$ are members which do not contain $\boldsymbol{x}_0$, using the notation $\boldsymbol{a}_{00} = \gamma_0$ we obtain

$$\boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) = \xi_0\,\gamma_0\,\xi_0^* + \boldsymbol{H}_{m-1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m).$$

Let $\boldsymbol{B}_m$ denote the matrix which arises from $\boldsymbol{B}$, if we replace by zeros all elements of $\boldsymbol{B}$ with the exception of those in the first column and in the diagonal, and let $\boldsymbol{A}_{m-1}$ denote the matrix of the form $\boldsymbol{H}_{m-1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$, then from the equality

$$\boldsymbol{B}_m \begin{pmatrix} \gamma_0 & (\boldsymbol{o}) \\ (\boldsymbol{o}) & \boldsymbol{A}_{m-1} \end{pmatrix} \boldsymbol{B}_m^* = \boldsymbol{A}_m$$

it is clear that $\boldsymbol{A}_{m-1}$ is a positive definite Hermitian matrix of order np. But then, by the foregoing

$$\boldsymbol{H}_{m-1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) = \xi_1\,\gamma_1\,\xi_1^* + \boldsymbol{H}_{m-2}(\boldsymbol{x}_2, \ldots, \boldsymbol{x}_m).$$

Continuing this process, we obtain the theorem to be proved.

From our theorem there follows

$$\boldsymbol{A}_m = \boldsymbol{B} \begin{pmatrix} \gamma_0 & & & \\ & \gamma_1 & & (\boldsymbol{o}) \\ & & \ddots & \\ (\boldsymbol{o}) & & & \gamma_m \end{pmatrix} \boldsymbol{B}^*$$

and from this

$$\operatorname{Det} \boldsymbol{A}_m = D_m = \operatorname{Det} \boldsymbol{\gamma}_0 \operatorname{Det} \boldsymbol{\gamma}_1 \dots \operatorname{Det} \boldsymbol{\gamma}_m .$$

In view of

$$\boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_s, \boldsymbol{o}, \dots, \boldsymbol{o}) = \sum_{i,j=0}^{s}{}' \boldsymbol{x}_i \boldsymbol{a}_{ij} \boldsymbol{x}_j^* =$$

$$= (\boldsymbol{x}_0 + \boldsymbol{x}_1 \beta_{01} + \dots + \boldsymbol{x}_s \beta_{0s}) \boldsymbol{\gamma}_0 (\boldsymbol{x}_0 + \boldsymbol{x}_1 \beta_{01} + \dots + \boldsymbol{x}_s \beta_{0s})^* +$$
$$+ (\boldsymbol{x}_1 + \boldsymbol{x}_2 \beta_{12} + \dots + \boldsymbol{x}_s \beta_{1s}) \boldsymbol{\gamma}_1 (\boldsymbol{x}_1 + \boldsymbol{x}_2 \beta_{12} + \dots + \boldsymbol{x}_s \beta_{1s})^* +$$
$$+ \dots + \boldsymbol{x}_s \boldsymbol{\gamma}_s \boldsymbol{x}_s^* ,$$

if we denote by D_s the determinant obtained from Det $\boldsymbol{A}_m$ by leaving only the first $(s+1)p$ rows and columns while cancelling everything else, the relation

$$D_s = \operatorname{Det} \gamma_0 \operatorname{Det} \gamma_1 \dots \operatorname{Det} \gamma_s \tag{11}$$

follows.

Lemma 2. *If the eigenvalues of γ_m are $\lambda_\alpha^{(m)}$, and $\boldsymbol{x}_m$ runs through all unitary matrices, then*

$$\inf \frac{1}{p} \operatorname{Sp} \boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m) = \frac{1}{p} [\lambda_1^{(m)} + \dots + \lambda_p^{(m)}]. \tag{12}$$

If, moreover, $\boldsymbol{x}_m$ runs through the matrices, whose determinants have the absolute value 1, then

$$\inf \frac{1}{p} \operatorname{Sp} \boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m) = \sqrt[p]{\lambda_1^{(m)} \dots \lambda_p^{(m)}} = \sqrt[p]{\frac{D_m}{D_{m-1}}} .$$

Proof. By Lemma 1 Sp $\xi_j \gamma_j \xi_j^* \geqq 0$, and therefore

$$\operatorname{Sp} \boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \geqq \operatorname{Sp} \boldsymbol{x}_m \boldsymbol{\gamma}_m \boldsymbol{x}_m^* . \tag{13}$$

Let γ_m have the Jordan normal form

$$\boldsymbol{\gamma}_m \boldsymbol{U} \boldsymbol{\Lambda} \boldsymbol{U}^*, \quad \boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{E} .$$

If the sum of the squares of the absolute values of the elements standing in the α-th row of the matrix $\boldsymbol{U}^* \boldsymbol{x}_m^*$ is denoted by s_α^2, then

$$\frac{1}{p} \operatorname{Sp} \boldsymbol{x}_m \boldsymbol{\gamma}_m \boldsymbol{x}_m^* = \frac{1}{p} \operatorname{Sp} \boldsymbol{\Lambda} (\boldsymbol{u}^* \boldsymbol{x}_m^* \boldsymbol{x}_m \boldsymbol{u}) = \frac{1}{p} [s_1^2 \lambda_1^{(m)} + \dots + s_p^2 \lambda_p^{(m)}] . \tag{14}$$

If $\boldsymbol{x}_m$ is unitary, in which case $\boldsymbol{U}^* \boldsymbol{x}_m^*$ is also unitary, we have $s_\alpha^2 = 1$, and so (14), together with (13), already proves (12), the relation expressing the first half of our theorem.

From (14) one gets

$$\frac{1}{p} \operatorname{Sp} \boldsymbol{x}_m \boldsymbol{\gamma}_m \boldsymbol{x}_m^* \geqq \sqrt[p]{(s_1 \dots s_p)^2 \lambda_1^{(m)} \dots \lambda_p^{(m)}} , \tag{15}$$

and in view of the well known determinant theorem of Hadamard, by which

$$(s_1 \dots s_p)^2 \geqq \operatorname{Det} \boldsymbol{x}_m \boldsymbol{x}_m^*$$

and by the equality

$$\lambda_1^{(m)} \dots \lambda_p^{(m)} = \mathrm{Det}\, \gamma_m = \frac{D_m}{D_{m-1}}$$

resulting from (11), we get the inequality

$$(16) \qquad \frac{1}{p} \mathrm{Sp}\, \boldsymbol{x}_m \gamma_m \boldsymbol{x}_m^* \geqq \sqrt[p]{\mathrm{Det}\, \boldsymbol{x}_m \boldsymbol{x}_m^* \lambda_1^{(m)} \dots \lambda_p^{(m)}} = \sqrt[p]{\mathrm{Det}\, \boldsymbol{x}_m \boldsymbol{x}_m^* \frac{D_m}{D_{m-1}}}.$$

Let $|\,\mathrm{Det}\, \boldsymbol{x}_m\,| = 1$. From inequality (13) we get through inequality (16) the relation

$$(17) \qquad \frac{1}{p} \mathrm{Sp}\, \boldsymbol{H}_m(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \geqq \sqrt[p]{\lambda_1^{(m)} \dots \lambda_p^{(m)}} = \sqrt[p]{\frac{D_m}{D_{m-1}}}.$$

Now it remains only to be shown that the left hand side of (17) actually attains the lower bound given in (17). Indeed, if

$$\boldsymbol{x}_m = (\beta_1, \dots, \beta_p), \quad |\beta_\alpha|^2 = \frac{1}{\lambda_\alpha} \sqrt[p]{\lambda_1^{(m)} \dots \lambda_p^{(m)}},$$

then $|\,\mathrm{Det}\, \boldsymbol{x}_m\,| = 1$ and

$$\frac{1}{p} \mathrm{Sp}\, \boldsymbol{x}_m \gamma_m \boldsymbol{x}_m^* = \frac{1}{p} [\lambda_1^{(m)} |\beta_1|^2 + \dots + \lambda_p^{(m)} |\beta_p|^2] = \sqrt[p]{\lambda_1^{(m)} \dots \lambda_p^{(m)}}.$$

From the proof of Lemma 2 it is clear that the second assertion valid also for $|\,\mathrm{Det}\, \boldsymbol{x}_m\,| \geqq 1$.

By the proof of Lemma 2 and by Lemma 1 it is clear that there holds the following

Corollary. *The minimizing matrices $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ are given in both cases of Lemma 2 by*

$$(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m) = (\boldsymbol{o}, \boldsymbol{o}, \dots, \boldsymbol{x}_m) \boldsymbol{B}^{-1},$$

if $\boldsymbol{x}_m$ denotes an arbitrary unitary matrix, resp. a matrix satisfying the condition $|\,\mathrm{Det}\, \boldsymbol{x}_m\,| = 1$ and minimizing the expression $\mathrm{Sp}\, \boldsymbol{x}_m \gamma_m \boldsymbol{x}_m^*$.

4. In this section we are going to prove our theorem.

Since $\boldsymbol{T}_n(\boldsymbol{f})$ is a positive definite Hermitian matrix, we can employ the generalized Lagrange transformation defined in our Lemma 1. By giving closer attention to the procedure which led us to this transformation, we remark that the matrix γ_k arising in the diagonal in the k-th step depends only on the matrices $\boldsymbol{a}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_k$. From this it follows that continuing this procedure for all values of n, we make correspond to the matrix (1) a sequence $\{\gamma_n\}$ of positive definite Hermitian matrices of order p.

Lemma 3. *The sequence $\{\gamma_n\}$ is monotonically nonincreasing.*

In order to prove this we start with the form

$$\boldsymbol{H}_n(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n) = \xi_0 \gamma_0 \xi_0^* + \xi_1 \gamma_1 \xi_1^* + \dots + \xi_n \gamma_n \xi_n^*$$

gained by use of the generalized Lagrange transformation of the form belonging to the matrix $\boldsymbol{T}_n(\boldsymbol{f})$. Indeed, hence we get

$$\gamma_n \leqq \boldsymbol{H}_n(\boldsymbol{o}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-1}, \boldsymbol{E}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=1}^{n} \boldsymbol{x}_k e^{ikx} \right) \boldsymbol{f}(x) \left(\sum_{k=1}^{n} \boldsymbol{x}_k e^{ikx} \right)^* dx =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=1}^{n} \boldsymbol{x}_k e^{i(k-1)x} \right) \boldsymbol{f}(x) \left(\sum_{k=1}^{n} \boldsymbol{x}_k e^{i(k-1)x} \right)^* dx = \boldsymbol{H}_{n-1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-1}, \boldsymbol{E}).$$

Since this inequality holds for all matrices $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-1}$, the relation $\gamma_n \leqq \gamma_{n-1}$ follows.

All elements of the sequence $\{\gamma_n\}$ are positive definite Hermitian matrices. So we have the following

Corollary 1. *The limit* $\lim\limits_{n\to\infty} \gamma_n = \gamma$ *exists.*

Indeed, by Lemma 3, the sequence $\{\boldsymbol{z}\, \gamma_n\, \boldsymbol{z}^*\}$ is monotonically nonincreasing and bounded for any fixed vector $\boldsymbol{z}$, and so it has a limit. With the aid of suitably chosen vectors $\boldsymbol{z}$ it is easy to show that the sequence $\{\gamma_n\}$ converges elementwise.

Corollary 2. *If the eigenvalues of* γ_n *resp. of* γ *are in monotonically nondecreasing order* $\lambda_1^{(n)}, \ldots, \lambda_p^{(n)}$ *and* $\lambda_1, \ldots, \lambda_p$ *respectively, then* $\lambda_a^{(n)} \downarrow \lambda_a$.

For a proof see e. g. [1], 298, Satz 15.

Corollary 3. *The sequence* $\left\{ \dfrac{D_n(\boldsymbol{f})}{D_{n-1}(\boldsymbol{f})} \right\}$ *is monotonically nonincreasing.*

On the basis of (11) and of Corollary 2 we indeed have

$$\frac{D_n(\boldsymbol{f})}{D_{n-1}(\boldsymbol{f})} = \operatorname{Det} \gamma_n = \lambda_1^{(n)} \ldots \lambda_p^{(n)} \leqq \lambda_1^{(n-1)} \ldots \lambda_p^{(n-1)} = \operatorname{Det} \gamma_{n-1} = \frac{D_{n-1}(\boldsymbol{f})}{D_{n-2}(\boldsymbol{f})}.$$

Corollary 3 gives the first assertion of our theorem.

Consider now the Hermitian forms (2) belonging to the matrices $\boldsymbol{T}_n(\boldsymbol{f})$. By Lemma 2 and by Corollary 3 to Lemma 3 we have the following

Lemma 4. *If* $\boldsymbol{f}(x)$ *is a Hermitian functional matrix, Lebesgue measurable, nonnegative definite and positive definite on a set of positive measure, then*

$$(18) \quad \inf_{\boldsymbol{x}_0, \boldsymbol{P}} \frac{1}{p \cdot 2\pi} \int_{-\pi}^{\pi} \operatorname{Sp}(\boldsymbol{x}_0 + \boldsymbol{P})(\boldsymbol{x}_0 + \boldsymbol{P})^* \boldsymbol{f}(x)\, dx = \lim_{n\to\infty} \sqrt[p]{\frac{D_n(\boldsymbol{f})}{D_{n-1}(\boldsymbol{f})}} = \sqrt[p]{\operatorname{Det} \gamma},$$

where $\boldsymbol{x}_0$ *runs through all matrices satisfying* $|\operatorname{Det} \boldsymbol{x}_0| = 1$ *and* $\boldsymbol{P}$ *through the trigonometric polynomial matrices* (8).

It is clear that (18) vanishes if and only if $\lambda_1 = 0$.

In case $\boldsymbol{f}(x)$ is positive semidefinite almost everywhere, we have $D_n(\boldsymbol{f}) = 0$, and so the quotients $\dfrac{D_n(\boldsymbol{f})}{D_{n-1}(\boldsymbol{f})}$ are senseless. Then however the infimum (18) can be taken equal to zero by (6), which is true for nonnegative Hermitian functional matrices $\boldsymbol{f}(x)$, positive definite on a set of positive measure.

Using now the theorem of HELSON and LOWDENSLAGER in (18), we obtain the limit (3), proving at the same time our theorem.

By the preceding considerations it is necessary and sufficient for the validity of (4) that $\lambda_1 = 0$ be valid, resp. that the Lebesgue measurable functional matrix $\boldsymbol{f}(x)$ be almost everywhere positive semidefinite.

As we have already mentioned in the introduction, the author has obtained the theorem just proved also independently of the theorem of HELSON and LOWDENSLAGER for the case when $\boldsymbol{f}(x)$ is a Lebesgue measurable, bounded, positive definite functional matrix satisfying the condition (5). Of course our Lemma 4 can also be used in order to prove with its help the theorem of HELSON and LOWDENSLAGER for a functional matrix $\boldsymbol{f}(x)$ having these properties. Indeed, it even follows from our remark on Lemma 2 that (7) remains also for $|\operatorname{Det} \boldsymbol{x}_0| \geqq 1$.

In his paper [11] T. BALOGH has investigated the infimum of $\operatorname{Sp} \int_{-\pi}^{\pi} \boldsymbol{P}(x)\, \boldsymbol{P}^*(x)\, \boldsymbol{f}(x)\, dx$, where $\boldsymbol{P}(x)$ runs through all matrix polynomials of grade n and of order p, which also satisfy the condition $\boldsymbol{P}(\alpha) = \boldsymbol{E}$, where α is an arbitrary fixed complex number. Formula (12) of Lemma 2 also answers this question by a method and in a form different from those of T. BALOGH in the special case, when $\alpha = 0$, and in the more general case, when $\boldsymbol{P}(0)$ is an arbitrary unitary matrix. The result of T. BALOGH is a generalization of a theorem of SZEGŐ ([2], 38).

Finally it is worth mentioning, that a new possibility would arise for generalizing the theorem of SZEGŐ mentioned in the introduction and to be obtained from our theorem by putting $p = 1$, by determining the limit γ of the sequence $\{\gamma_n\}$, which exists in view of Corollary 1 to Lemma 3. This would perhaps give a new matricial generalization of the theorem of SZEGŐ. Namely a matricial generalization of this theorem has already been given in the paper [5] by the author.

(Received November 11, 1960.)

REFERENCES

[1] GANTMACHER, F. R.: *Matrizenrechnung* I., Berlin, 1958.
[2] GRENANDER, U.—SZEGŐ, G.: *Toeplitz forms and their applications*. Berkeley—Los Angeles, 1958.
[3] GYIRES, B.: „Eigenwerte verallgemeinerter Toeplitzscher Matrizen". *Publ. Math., Debrecen* **4** (1956), 171—179.
[4] GYIRES, B.: „Valós függvénymatrixhoz tartozó Toeplitz-féle determinánsokról." *Acta Univ. Debreceniensis* **5** (1958), 145—158.
[5] GYIRES, B.: „Spuren der verallgemeinerten Toeplitzschen Matrizen." *Publ. Math., Debrecen*, **8** (1961) 93—116.
[6] HELSON, H.—LOWDENSLAGER, D.: „Prediction Theory and Fourier Series in several Variables". *Acta Math.* **99** (1958), 165—202.
[7] KOLMOGOROV, A.: „Stationary Sequences in Hilbert Space". *Bull. Math. Univ. Moscou* 2. No. 6 (1941), 40 pp. (English translation by Natasa Artin.)
[8] SZEGŐ, G.: „Ein Grenzwertsatz über die Toeplitzschen Determinanten einer reellen positiven Funktion." *Math. Ann.* **76** (1915), 490—503.
[9] SZEGŐ, G.: „A Toeplitz-féle formákról". *Mat. Termtud. Értesítő* **35** (1917), 185—222.
[10] SZEGŐ, G.: „Beiträge zur Theorie der Toeplitzschen Formen". *Math. Zeitschr.* **6** (1920), 167—202, **9** (1921), 167—190.
[11] BALOGH, T.: „Solution of a minimum problem". *Publ. Math, Debrecen*, **8** (1961) 131—142.

ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ SZEGŐ

B. GYIRES

Резюме

В настоящей работе автором доказывается следующая теорема:

Пусть $\boldsymbol{f}(x)$ определенная в интервале $[-\pi, \pi]$ измеримая по Лебегу функция с матричными значениями, типа $p \times p$, неотрицательно определенная и ермитова. Если

$$D_n(\boldsymbol{f}) = \operatorname{Det}(\boldsymbol{a}_{\mu-\nu})_{\mu,\nu=0}^{n},$$

где

$$\boldsymbol{a}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \boldsymbol{f}(x)\, e^{-ikx}\, dx,$$

то

$$\frac{D_n(\boldsymbol{f})}{D_{n-1}(\boldsymbol{f})} \downarrow \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \operatorname{Det} \boldsymbol{f}(x)\, dx\right\}$$

и этот предел равен 0, если

$$\int_{-\infty}^{+\infty} \log \operatorname{Det} \boldsymbol{f}(x)\, dx = -\infty.$$

При доказательстве автор установит связь со следующим результатом Helson и Lowdenslager [6], стр. 186):

Если $d\boldsymbol{M}$ данная мера состоящая из неотрицательных самосопряженных матриц типа $p \times p$, $\frac{1}{2\pi} \boldsymbol{f}(x)$ абсолютно непрерывная часть меры $d\boldsymbol{M}$, $\boldsymbol{P}(x)$ бежит по тригонометрическим полиномам формы

$$\sum_{k=0}^{n} \boldsymbol{A}_k\, e^{-ikt}$$

коэффициенты которых являются матрицами типа $p \times p$, подчиненные условию $\operatorname{Det} \boldsymbol{A}_0 = 1$, то

$$\inf_{P} \frac{1}{p} \int_{-\pi}^{\pi} \operatorname{Sp}[\boldsymbol{P}(x)\, d\boldsymbol{M}(x)\, \boldsymbol{P}^*(x)] = \exp \frac{1}{2\pi p} \int_{-\pi}^{\pi} \operatorname{Sp}[\log \boldsymbol{f}(x)]\, dx.$$

Автор получил свою вышеформулированную теорему и независимо от теоремы Хелсона—Ловденслагера для случая, когда $\boldsymbol{f}(x)$ измеримая по Лебегу функция с матричными значениями, ермитова и типа $p \times p$, обладающая положительной нижней и конечной верхней гранью. ([5]). Исходя из этой теоремы можно доказать методом автора теорему Хелсона и Ловденслагера для случая, когда $d\boldsymbol{M}$ абсолютно непрерывна, т. е. $d\boldsymbol{M} = \frac{1}{2\pi} \boldsymbol{f}(x)\, dx$ и $\boldsymbol{f}(x)$ удовлетворит условиям вышеприведенной теоремы.