ON COVERINGS OF GENERALIZED CHECKER BOARDS I.

by

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1. Our starting point is the following well-known problem: remove two squares at two opposite corners of a checker board and try to cover the rest with dominoes. (Each domino covers exactly two adjacent squares of the board.) The question arises: how is it possible? The answer is: it is impossible. Proof: each domino covers one light and one dark square and thus among the covered squares there are light and dark ones in equal number. However, the opposite corners are of the same colour (dark, say) and so the rest consists of thirty dark and thirty-two light squares.

This problem can easily be generalized in many ways. S. W. GOLOMB ([1] and [2]) has dealt with several generalizations. He replaced the dominoes with various "polyominoes" and examined the possibilities of different coverings of the common checker board.

In this paper we are dealing with an other kind of generalization. From the polyominoes used by GOLOMB we maintain only two types: the "straight polyominoes" (called polyominoes here) and the "monominoes" (called simply squares). But the checker board is generalized, as an arbitrary large board is considered instead of the usual 8×8 one.

After these introductory remarks let us formulate the problem more exactly.

2. Let k and n be natural numbers. Consider a "generalized checker board" (called *board* in what follows) which consists of $kn \times kn$ squares. Deleting k squares of this board (called *deleted* or *omitted squares)*, try to cover the *rest* (all the other squares) with *k-ominoes* i. e. $k \times 1$ rectangles covering exactly k consecutive squares in one row or column. The question is: what conditions must be fulfilled by the omitted squares in order to make the covering of the rest possible.

A certain position of the deleted squares on the board (related to each other) is called a *configuration*. Since this determines the localization of the remaining squares as well, we use the term "configuration" also for the whole board. (I.e., we say that a configuration is coverable if its rest is coverable, and that the conditions are fulfilled by a configuration if they are fulfilled by its deleted squares.)

The case $k = 1$ is trivial whatever n is; in the following we shall confine ourselves to cases $k > 1$. The case $k = 2$ is essentially solved by the aid of the proof mentioned in the introduction. However, we shall seek conditions valid in general and these will contain $k = 2$ as a special case.

Our first condition is also based on the solution mentioned before. Let us colour the board with k colours in such a way that k consecutive squares in one row or in one column should be all of different colours. So the k-ominoes cover always k squares of different colours. Therefore, if among the deleted squares there are two of the same colour, it is sure that the configuration cannot be covered.

This colouring, however, may be carried out not only in one way. The permutation of the colours does not give an essential change but we may alter the colouring essentially e. g. by permuting several coloured rows. Evidently, what we have stated about an arbitrary colouring, it concerns every possible colouring as well. Consequently, we may state (without the need of any proof) that the following Condition has to be necessarily fulfilled if a configuration is coverable.

(C_1) *The rest can be covered only if the deleted squares are of different colours — and this is valid for every colouring.*

In what follows we shall call some configuration "good" if it satisfies the above condition.

3. The condition (C_1) is formally very simple. However, if we want to ascertain whether a configuration may be or may not be considered good, we are confronted with great difficulties. First, we have to colour the board, then, to inspect if the deleted squares are of different colours or not. If they are, we must repeat the whole process over again till for a certain colouring there are found two deleted squares of the same colour, — or until we shall have examined every colouring. But this is almost impossible if k is great; the number of different colourings is equal to the number of k-side Latin squares.

In the following we state a (similarly necessary) condition which is much more convenient for use.

Let us number the rows and the columns of the board modulo k. This way we gave double indices to each square. We say two squares *congruent by row* if their row indices are equal. (The congruency by column is defined in the same manner.) Two rows (columns) are congruent if their squares are congruent by row (column). So, the mentioned Condition is:

(C_2) *In order that some configuration should be coverable, its deleted squares have to be pairwise incongruent by row (resp. by column) and simultaneously all of them have to be congruent by column (resp. by row).*

We assert *conditions* (C_1) *and* (C_2) *to be equivalent.*

Proof. In the proof we may restrict ourselves to the case when the omitted squares are congruent with each other by column.[1]

The properties of colouring imply that going along a row or column in the board we find the same colour again just after k steps. (Squares of the same colour may not be nearer by definition; if they would be farther, between the two equally coloured squares there must be at least k squares, the number of different colours used for these, however, is $k - 1$.) Therefore the congruent columns (and congruent rows similarly) are coloured in the same manner. So, if the deleted squares are congruent by column, we may regard them — in respect of colouring — as belonging to the same column (leaving every square

[1] This is no restriction of the generality: turning the board by 90° the possibility of covering does clearly not change.

in its original row). Now, in one column two squares are of the same colour if and only if they are congruent by row. However, deleted squares are — by (C_2) — all incongruent, consequently they are all of different colours.

Thus (C_2) implies (C_1). Let us now prove the implication in the opposite direction.

First we establish that in a good configuration (i.e. satisfying (C_1)) it is impossible for two squares to be congruent by row and by column simultaneously. For, in such a case, they would have the same indices and hence they ought to be of the same colour in any colouring — contradicting (C_1).

Similarly, in a good configuration there are not two squares which are incongruent by row and by column simultaneously. Suppose — opposing to our assertion — that two such squares can be found. Let these be called a and b. Consider a colouring. Let a be, say, red, and b green. (They may not be of the same colour because of (C_1).) Let the square in the row of b and in the column of a be coloured blue. (This one may not be red or green: in the first case a and b would be congruent by row, and in the other case they would be congruent by column.) Starting from this blue square and going towards b let us seek the first red and first green square. The green will be the j^{th} and the red will be the l^{th} one. Evidently $j \neq l,\ j,l < k$ (the k^{th} square is blue again). Starting from the column of a we exchange the j^{th} and l^{th} column. So we get an other colouring where in the row of b the j^{th} square will be red. Extending this new colouring to the whole board in the row of b every green square becomes red (therefore also b itself), while a remains red unchanged. But this is impossible because these squares belong to a good configuration.

So we are ready essentially. Namely, choosing two deleted squares from a good configuration these are congruent either by row or by column (and by no means by both). Let these be, say, congruent by column. Now, choose a third square. If this is congruent by row with one of the preceding squares (and therefore incongruent with the other), it must be incongruent by column with them according to our first establishment. But in this case the third is incongruent simultaneously by row and by column with one of the preceding squares — in contradiction to our second conclusion. Therefore, if the third square belongs to a good configuration, it must be incongruent by row with the preceding ones — and necessarily congruent by column with them. The assertion may be successively seen for the other deleted squares in the same way.

So we have proved the equivalency of (C_1) and (C_2). In what follows we shall always refer to the condition (C_2) because the verification of its fulfilment is incomparably simpler than in the case of (C_1). We maintain the expression "good configuration" concerning (C_2) too.

4. Condition (C_2), as we have seen, is necessary for the possibility of constructing a covering. We cannot prove (C_2) to be sufficient too. Indeed, (C_2) is not a sufficient condition in general. Moreover, n and k must be very special in order to make (C_2) sufficient.

In the case $n = 1$, whatever k may be, (C_2) is sufficient too. Then it is easy to see the omitted squares of a good configuration may be placed only in one row or in one column; the other rows (columns) we can easily cover with k-ominoes which cover an entire row (column) here.

In the case $k = 2$ (n is arbitrary) (C_2) is a necessary and sufficient condition. Although it would be very easy to prove this, we shall obtain this result as a special conclusion from the general sufficient condition to be worded later.

The above mentioned cases are the only ones where (C_2) is sufficient and necessary. Already in the simplest case $n = 2$ $k = 3$ we can show a good configuration which is not yet coverable[2] (Figure 1).

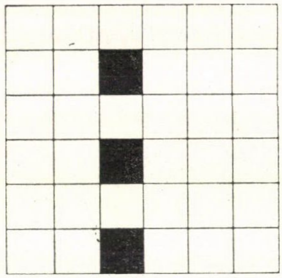

Fig. 1.

We note that this is not the only similarly uncoverable configuration. Such a configuration can not be covered being just at any place of this board. Moreover, there exist some another good yet not coverable configurations.

5. Our next purpose is to find a sufficient condition as strong as possible for the general case.

First we shall show that if we can cover a complete rectangle which contains every omitted square and one side of which is a multiple of k then the board is simply coverable. Suppose this rectangle consists of m rows and l columns where $m = d \cdot k$ $(d = 1, 2, \ldots, n)$ and $1 \leq l \leq n \cdot k$. (The role of the rows and columns may be evidently exchanged.) Thus beside the rectangle we can put d k-ominoes in one column. So we may extend the covered rectangle to an $m \times nk$ one. The $nk - m$ empty rows can be easily covered by k-ominoes: each row is coverable with n k-ominoes.

In the following we need some definitions.

We look at the board always so that the the omitted squares should be congruent by column.[3] Columns which are congruent with ones containing omitted squares we call *significant columns*. Since the row index is characteristic for each deleted square, row indices may be regarded as "names" of different deleted squares. Starting from the first row of the board and going on row by row we can establish a natural order of the deleted squares; thus their "names" form a permutation of the numbers $1, 2, \ldots, k$. We call this one *the permutation of the configuration*. In what follows, the *circular permutations* will have an important role; we call a permutation circular if putting it around a circle it is indistinguishable from the identical one.

A *minimal rectangle* is the smallest rectangle which contains all the omitted squares. A *period* consists of k consecutive rows or columns of the rectangle eventually in question (let it be just the minimal rectangle as well as the whole board).

[2] We don't prove the impossibility of covering; anybody can easily verify it by trying.

[3] In this section we regard good configurations exclusively.

We shall prove the following simple sufficient condition:

(S) *A good configuration is coverable if its permutation is circular.*

Proof. Since in this case the number of rows of the minimal rectangle is evidently divisible by k in order to prove (S) it is enough to show the minimal rectangle to be coverable. (We note that the number of the columns of any minimal rectangle is congruent with 1 modulo k, because its first and last columns are both significant ones and so they are congruent.)

In the minimal rectangle first we cover the rows which contain omitted squares. (There are k such rows.) Such a row is cut by any significant column (the possible places of the omitted squares) into two parts the length of which is a multiple of k (eventually zero). So we can cover these rows separately with k-ominoes.

As the permutation of the configuration is circular the rows covered before either are neighbouring ones or between two covered rows there are one or more empty periods. Thus these rows are coverable by periods setting k-ominoes along columns.

So we have obtained a covering of the minimal rectangle. Thus (S) is proved.

6. In this section we want to give some sharpening of (S).

(S*) *Some good configuration is coverable if from its permutation we can make a circular one with the following discrete changes.*[4] *Two or three consecutive elements may be changed among themselves if the corresponding deleted squares are in different columns. Before the row of the first deleted square there must be as many rows as the number of those elements of the obtained circular permutation*[5] *which precede the first element of the original permutation. Similarly, after the row of the last deleted square there must be as many rows as the number of those elements of the obtained circular permutation which follow the last element of the original permutation.*

If we add other conditions, some of the requirements listed in (S*) may be cancelled. Indeed, covering is possible even in some cases not satisfying (S*). However, as we have said, we state *some* sharpening of (S) and not the *strongest* one.

Proof. We regard the permutation of the given configuration as a "perturbed" circular one. If the first and last element of the respective circular order are also the first and last element of the given permutation then the number of the rows of the minimal rectangle is divisible by k. In the opposite case, if the outside elements are changed ones we must augment the minimal rectangle by so many rows as the number of elements of the circular order preceding the first (resp. following the last) element of the original permutation; thus this *augmented rectangle* has rows of a number divisible by k. Therefore it is enough to show the covering of the minimal resp. the augmented rectangle.

[4] We don't permit repeated changes at any element since then no restriction would be done for the permutations.

[5] It is easy to see, in the cases $k < 6$ we can often get different circular permutations from a given one: e. g. from 32154 both 12345 and 23451 are obtainable. In such a case we may choose the most convenient order.

These rectangles are coverable as in the case of the circular permutation (see the proof of (S)) except the parts where the circular order is "perturbed". Take out such a part from the permutation and consider the corresponding deleted squares. These (two or three) squares also determine their own minimal rectangle. We augment this rectangle by one or two rows above and by one or two rows below so that the index of the first row of the rectangle so obtained should be equal to the smallest of the indices of the omitted squares of this rectangle and the index of its last row should be equal to the greatest of the same indices.[6] The rectangle so obtained we call *elementary rectangle.*

The first and the last column of any elementary rectangle are significant. Therefore, beside these rectangles the minimal rectangle has entire periods of columns and so these parts are also coverable.

In order to complete the proof we show how the different kinds of elementary rectangles can be covered. Ways of covering may be seen in Figure 2.[7]

It is easy to see, that all the possible elementary rectangles may be obtained from these types by the simplest geometrical transformations.

7. In this section we seek answer to the question: among all possible configurations how many ones can be covered? More exactly: if in a $kn \times kn$ board we choose the k omitted squares at random what is the probability that the configuration so obtained will be coverable?

We cannot compute this probability as a sufficient and necessary condition is lacking. However, conditions (C_2) and (S) supply lower and upper bounds for this probability.

Let $P_n(k)$ denote the probability that some chosen configuration is "good" (in the sense of the preceding sections), and $p_n(k)$ the probability that some configuration satisfies (S). ($P_n(k)$ is an upper, $p_n(k)$ is a lower boundary of the sought probability.)

It is easy to see[8] that

$$P_n(k) = \frac{2\,kn^{2k}}{\binom{n^2\,k^2}{k}}. \tag{1}$$

In order to formulate $p_n(k)$ too, let us at first compute the number of configurations satisfying (S) which have omitted squares only in a preassigned column.

In this column, let the first square be an omitted one. The other $k-1$ deleted squares can be divided into the n periods in $\binom{n+k-2}{k-1}$ ways. The permutation must always be the identical one: this determines the place of the omitted squares within the periods. Cutting this column at the end of any period and exchanging the two sections we get circular permutations. In such a way n circular permutations are got from each former permutation. The

[6] The smallest and greatest indices are to be taken in "circular sense". Therefore, if the indices in question are e. g. k, 1 and 2 then the smallest is k and the greatest is 2.

[7] In order to make the illustration easier, in figures we have used 5-ominoes. Furthermore, we have represented the deleted squares in position as near as possible; if they are farther from each other, the intermediate entire periods are easily coverable.

[8] Now the board is considered in fixed position.

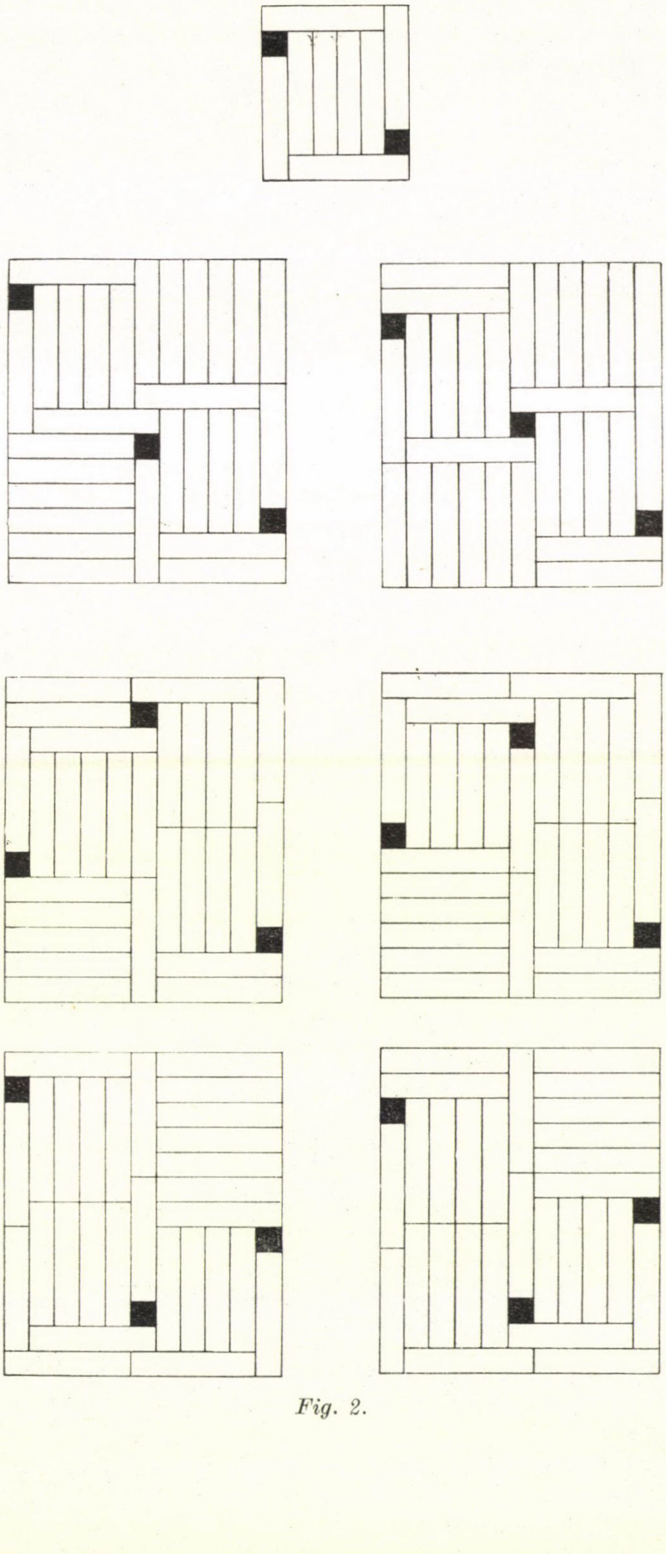

Fig. 2.

choice of some significant row or column and the division of the deleted squares into the n significant columns (rows) is likely performed as in computing $P_n(k)$. Thus we have

$$(2)\qquad p_n(k) = \frac{2k\binom{n+k-2}{k-1} n^{k+1}}{\binom{n^2k^2}{k}}.$$

(All the formulas are valid only in the cases $k \geqq 2$.) Computing the quotient of (1) and (2) we have

$$(3)\qquad \frac{P_n(k)}{p_n(k)} = \frac{n^{k-1}}{\binom{n+k-2}{k-1}} = (k-1)!\,\frac{1}{\left(1+\frac{k-2}{n}\right)\left(1+\frac{k-3}{n}\right)\cdots\left(1+\frac{1}{n}\right)}.$$

If n tends to infinity (and k is fixed) this quotient tends to $(k-1)!$ from below.

We are interested in the behaviour of these probabilities as functions of k. If n is great enough, the dependence on n of the probabilities may be neglected. Namely, for any fixed k we have:

$$(4)\qquad P_\infty(k) \equiv \lim_{n\to\infty} P_n(k) = \lim_{n\to\infty} \frac{2kn^{2k}}{\binom{n^2k^2}{k}} = \frac{2k\cdot k!}{k^{2k}}.$$

In addition, from (3) and (4):

$$(5)\qquad p_\infty(k) = \frac{P_\infty(k)}{(k-1)!} = \frac{2k^2}{k^{2k}}.$$

Applying very rough estimations we see that $p_\infty(k)$ may not be smaller than the square of $P_\infty(k)$:

$$(6)\qquad P_\infty(k) \leqq 2k^{-k}$$

and

$$(7)\qquad p_\infty(k) > 4k^{-2k}.$$

The cause of this great difference is the weakness of (S). Our sharpening given in Section 6. is not suitable for computation, on the other hand, it would not change the situation essentially. Nevertheless, if we examine the probability whether a configuration is coverable if the omitted squares are chosen at random, we shall see this probability to be very small. Namely, the function $P_\infty(k)$ which majorizes the preceding probability will be small already for relatively not great values of k. Applying the Stirling-formula, for $P_\infty(k)$ we obtain an estimation better than (6):

$$(8)\qquad P_\infty(k) \sim \frac{2\sqrt{2\pi}}{e^k k^{k-3/2}}.$$

(Received March 10, 1961.)

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ПРОБЛЕМЫ ПОКРЫТИЯ ОБОБЩЕННОЙ ШАХМАТНОЙ ДОСКИ

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Резюме

В статье автор занимается следующей проблемой: Дана «обобщенная шахматная доска» содержащая $kn \times kn$ квадраты и даны также «обобщенные дощечки домино», которые могут покрыть точно k смежных квадратов доски. Если выбросить k квадратов доски, то спрашивается, можем ли мы покрыть оставшиеся квадраты? Возможность покрытия зависит, конечно, от расположения выброшенных квадратов. Автор дает, сначала, два необходимых условия (эквивалентных между собой) для возможности покрытия (легко доказуемое условие C_1 и хорошо применяемое условие C_2), а потом дает еще одно довольно слабое достаточное условие S и его уточнение S*. Наконец, он исследует вопрос, какая вероятность возможности покрытия полученного расположения, если выброшенные квадраты выбрать случайно.