EXTENSION OF A THEOREM OF ARMELLINI—TONELLI—SANSONE TO THE NONLINEAR EQUATION $u'' + a(t)f(u) = 0$

by

I. BIHARI

1. According to a well-known theorem the integrals of the *linear* equation $u'' + a(t)u = 0$ are for positive continuous non-decreasing $a(t)$, with $\lim_{t=\infty} a(t) = \infty$ all bounded and there is at least one solution tending to zero as $t \to \infty$. It is an interesting problem how further conditions must be imposed on $a(t)$ so that every solution may behave similarly. The theorem mentioned in the title gives reply on this point. In order to be able to quote this theorem we need the knowledge of two notions: one from set-theory, another from function-theory.

a) Density of an interval-sequence: Let $\{(\alpha_n, \beta_n)\}$ $(n = 1, 2, \ldots)$ be an interval-sequence on the half line $t \geqq 0$ having no point in common. It is said of density ε $(\varepsilon > 0)$ on $(0, \infty)$ provided that

$$\overline{\lim_{n\to\infty}} \frac{\sum_{i=1}^{n} (\beta_i - \alpha_i)}{\beta_n} = \varepsilon .$$

b) Function tending to infinity "quasi jumping" respectively "varying regularly": The function $F(t)$ being positive continuous non-decreasing, tending to infinity as $t \to \infty$ is said "tending to infinity quasi jumping", if to every $\varepsilon > 0$ number there is an interval-sequence $\{(\alpha_n, \beta_n)\}$ of density less than ε so that on its complementary set — on the set $(0, \infty) - \sum_{n=1}^{\infty} (\alpha_n, \beta_n)$ — the increase of $F(t)$ is finite. In the opposite case $F(t)$ is called tending to infinity "regularly".

The theorem of Armellini—Tonelli—Sansone (s. [1] p. 60.) is as follows.

If in the equation $u'' + a(t) = 0$ $a(t)$ is positive non-decreasing continuous together with its derivative, $\lim_{t=\infty} a(t) = \infty$ and $\log a(t)$ tends to infinity regularly, then every integral of the equation tends to zero as $t \to \infty$.

Z. Opial generalized this theorem for more complicated $a(t)$ and showed that the differentiability of $a(t)$ need not be supposed (s. [2]).

2. The purpose of the present paper is to find an extension of the A.—T.—S.-theorem to the *nonlinear* equation

$$u'' + a(t)\, f(u) = 0 . \qquad (1)$$

This is intended by the following

Theorem. *If in equation* (1)

1. $a(t) > 0$, *is continuous non-decreasing*, $\lim_{t=\infty} a(t) = \infty$, $\log a(t)$ *tends to infinity regularly*,
2. $f(u)$ *is a continuous non-decreasing odd function*,
3. $\frac{f(u)}{u} = O(1)$ $(u \to 0)$ *and* $\frac{f(u)}{u}$ *is non-increasing for* $u > 0$,
4. $|f(u_1) - f(u_2)| \leqq \omega(|u_1 - u_2|)$, *where* $\omega(z)$ *is positive non-decreasing and* $\int^{u_0} \frac{dz}{\omega(z)} = \infty$ $(u_0 > 0)$, *then every solution of* (1) *tends to zero as* $t \to \infty$.

Proof. As is well-known these hypotheses assure the existence and uniqueness of an oscillatory solution (for $t \geqq 0$) with given initial conditions (s. [3] and [4]). Let us suppose there exists a solution $u(t)$ contradicting this theorem. Its "amplitude" defined by $A(t) = \sqrt{\frac{u'^2}{a(t)} + 2\,F(u)}$ is non-increasing (s. [4]) ($F(u)$ means here $\int_0^u f(z)dz$). Let its limit be denoted by A. By our assumptions $A > 0$. Taken (1) into account

$$A^2(t) = 2\,F(u) + \frac{u'^2}{a(t)} = A^2(0) + \int_0^t dA^2(t) = A^2(0) + \int_0^t \left(2 f(u)\, u' + \frac{2\,u'\,u''}{a}\right) dt -$$

$$- \int_0^t \frac{u'^2}{a^2(t)}\, da(t) = A^2(0) - \int_0^t \frac{u'^2}{a^2(t)}\, da(t) = A^2(0) - \int_0^t \frac{u'^2}{a(t)}\, \frac{da(t)}{a(t)}\,.$$

This may be written in the form

$$A^2(t) = A^2(0) - \int_0^t [A^2(t) - 2\,F(u)]\, \frac{da(t)}{a(t)} \tag{2}$$

too. Tending $\log a(t)$ to infinity regularly there is such a number $\varepsilon_0 > 0$, that the growth of $\log a(t)$ on the complementary set of all sequences $\{(\alpha_n, \beta_n)\}$ $(\alpha_n < \beta_n < \alpha_{n+1})$ of intervals of density less than ε_0 is infinite, i.e. the series

$$\sum_{i=1}^{\infty} [\log a(\alpha_{i+1}) - \log a(\beta_i)] = \sum_{i=1}^{\infty} \log \frac{a(\alpha_{i+1})}{a(\beta_i)}$$

is a divergent one. It will be proved later, that one can choose (to ε_0) such a number $\eta > 0$, that the sequence of all the intervals (α_n, β_n), where the inequality $A^2(t) - 2F(u) \leqq \eta$ is satisfied, have a density less than ε_0. On account of (2)

$$A^2(\alpha_n) \leqq A^2(0) - \sum_{i=1}^{n-1} \int_{\beta_i}^{\alpha_{i+1}} [A^2(t) - 2\,F(u)]\, \frac{da(t)}{a(t)}\,.$$

But on the intervals (β_i, α_{i+1}) we have $A^2(t) - 2F(u) \geqq \eta$, therefore,

$$A^2(\alpha_n) \leqq A^2(0) - \eta \sum_{i=1}^{n-1} \int_{\beta_i}^{\alpha_{i+1}} \frac{da(t)}{a(t)} = A^2(0) - \eta \sum_{i=1}^{n-1} \log \frac{a(\alpha_{i+1})}{a(\beta_i)} .$$

However this results in $A^2(\alpha_n) < 0$ for n large enough, what is impossible. — Hence we have still to show, that a choice of η like above is possible.

3. The inequality $A^2(t) - 2F(u) \leqq \eta$ implies

$$A(t) - \sqrt{2\,F(u)} \leqq \frac{\eta}{A(t) + \sqrt{2\,F(u)}} \leqq \frac{\eta}{A(t)} \leqq \frac{\eta}{\varrho} ,$$

where ϱ denotes the greatest lower bound of $A(t)$ for $t \geqq 0$. This is a positive number accordingly to our assumption on $u(t)$. Therefore it is sufficient to show that a number $\eta^* > 0$ can be chosen so that the sequence of the intervals, where

$$A(t) - \sqrt{2\,F(u)} \leqq \eta^* , \tag{3}$$

is of a density less than ε_0. Namely multiplying (3) by $A(t) + \sqrt{2F(u)} \leqq K$ ($A(t)$ and consequently $u(t)$ are bounded) we have $A^2(t) - 2F(u) \leqq \eta^* K = \overline{\eta}$ and $\overline{\eta}$ satisfies our requirements. Being $\lim\limits_{t=\infty} A(t) = A$ there is a place $t_1 \geqq 0$ so that the inequality $A + \eta^* \geqq A(t) \geqq A - \eta^*$ is fulfilled for $t \geqq t_1$. In other words: every number $t \geqq t_1$ satisfying (3) satisfies the inequality

$$A - \sqrt{2\,F(u)} \leqq 2\,\eta^* \quad \text{or} \quad A \left(1 - \frac{2\,\eta^*}{A}\right) \leqq \sqrt{2\,F(u)}$$

too. Since the density of a sequence of intervals does not change by omission of a finite number of intervals (preceding t_1), it is enough to prove the existence of such a number $0 < \sigma < 1$, that the density of the interval-sequence S_σ, where $\sigma A \leqq \sqrt{2F(u)}$, is less than ε_0. Viz., conversely, the previous inequality implies $A - \sqrt{2F(u)} \leqq 2\,\eta^*$ with a certain η^* $\left(\eta^* = \frac{1}{2} A(1 - \sigma)\right)$ and this the further inequality $A(t) - \eta^* - \sqrt{2F(u)} \leqq A - \sqrt{2F(u)} \leqq 2\eta^*$, i.e. $A(t) - \sqrt{2F(u)} \leqq 3\eta^* = \overline{\eta}$. By the notation $\mu = F^{-1}\left(\frac{\sigma^2 A^2}{2}\right)$ the condition $\sigma A \leqq \sqrt{2F(u)}$ takes the form $|\,u\,| \geqq \mu$. — Let us estimate the density of the sequence S_σ. Denoting the successive intervals of S_σ by (α_n', β_n') $(n = 1, 2, \ldots)$ we have $|\,u\,(\alpha_n')\,| = |\,u\,(\beta_n')\,| = \mu$. Regard simultaneously with (1) the auxiliary comparison equation

$$v'' + a(\alpha_n') f(v) = 0 \tag{4}$$

too and take its solution $v_n(t)$ satisfying the initial conditions $v_n(\alpha_n') = |\,u(\alpha_n')\,|$, $v_n'(\alpha_n') = |\,u'(\alpha_n')\,|$. Let the first root of the equation $v_n(t) = \mu$ lying to the right of α_n' be denoted by β_n'', resp. the first root of the equation

$v_n(t) = 0$ situated left from α'_n by γ'_n. Applying a comparison theorem of the Sturmian type (s. [4]) on $u(t)$ resp. $v_n(t)$ (solutions of (1) resp. (4))

$$\beta'_n - \alpha'_n \leqq \beta''_n - \alpha'_n \quad \text{and} \quad \alpha'_n - \gamma'_n \leqq \alpha'_n - \beta'_{n-1}\,.$$

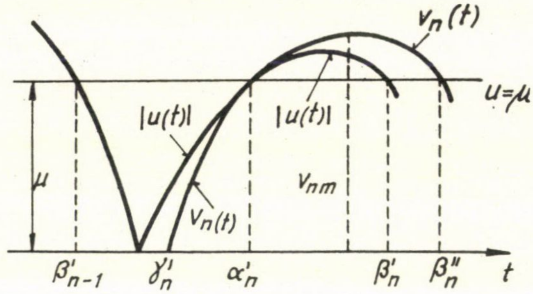

Figure.

Evaluate the lengths $\beta''_n - \alpha'_n$ and $\alpha'_n - \gamma'_n$. — Equation (4) may be solved. Obtaining

$$(5) \qquad 2\,v'\,v'' + 2\,a(\alpha'_n)\,f(v)\,v' = 0 \quad \text{resp.} \quad v'^2 + 2\,a(\alpha'_n)\,F(v) = K\,,$$

where

$$F(v) = \int_0^v f(z)\,dz \quad \text{and} \quad K = v'^2(\alpha'_n) + 2a(\alpha'_n)\,F\big(v_n(\alpha'_n)\big) = u'^2(\alpha'_n) + $$
$$+\,2a(\alpha'_n)\,F\big(u(\alpha'_n)\big) = a(\alpha'_n)\,A^2(\alpha'_n)\,.$$

Thus (5) gives

$$\frac{dv}{dt} = \sqrt{a(\alpha'_n)\,A^2(\alpha'_n) - 2\,a(\alpha'_n)\,F(v)} \quad \text{resp.} \quad dt = \frac{1}{\sqrt{a(\alpha'_n)}}\,\frac{dv}{\sqrt{A^2(\alpha'_n) - 2\,F(v)}}\,.$$

Hence

$$\beta''_n - \alpha'_n = \frac{2}{\sqrt{a(\alpha'_n)}}\int_\mu^{v_{nm}}\frac{dz}{\sqrt{A^2(\alpha'_n) - 2\,F(z)}} = \frac{2}{\sqrt{a(\alpha'_n)}}\int_{\frac{1}{2}\sigma^2A^2}^{F(v_{nm})}\frac{d\lambda}{f(z)\sqrt{A^2(\alpha'_n) - 2\,\lambda}}\,,$$
$$\big(\lambda = F(z)\big)$$

where $v_{nm} = \max\limits_{(\alpha'_n,\,\beta''_n)} v_n(t)$ $\big(v_n(t)$ is symmetrical on its maxima$\big)$ and

$$\alpha'_n - \gamma'_n = \frac{1}{\sqrt{a(\alpha'_n)}}\int_0^\mu\frac{dz}{\sqrt{A^2(\alpha'_n) - 2\,F(z)}} \geqq \frac{1}{\sqrt{a(\alpha'_n)}}\int_0^\mu\frac{dz}{\sqrt{A^2(t_1) - 2\,F(z)}}\,.$$

By the last relation

$$(6) \qquad \beta'_n - \beta'_{n-1} > \alpha'_n - \beta'_{n-1} \geqq \alpha'_n - \gamma'_n \geqq \frac{1}{\sqrt{a(\alpha'_n)}}\int_0^\mu\frac{dz}{\sqrt{A^2(t_1) - 2\,F(z)}}\,.$$

The amplitude $B_n^2(t) = \dfrac{v_n'^2(t)}{a(\alpha_n')} + 2F(v_n(t))$ of $v_n(t)$ is constant (s. [4]). For this reason

$$B_n^2(v_{nm}) = 2\,F(v_{nm}) = B_n^2(\alpha_n') = A^2(\alpha_n'),$$

consequently

$$\beta_n'' - \alpha_n' = \frac{2}{\sqrt{a(\alpha_n')}} \int_{\frac{1}{2}\sigma^2 A^2}^{\frac{1}{2}A^2(\alpha_n')} \frac{d\lambda}{f(z)\sqrt{A^2(\alpha_n') - 2\lambda}} \leqq \frac{2}{f(\mu)\sqrt{a(\alpha_n')}} \int_{\frac{1}{2}\sigma^2 A^2}^{\frac{1}{2}A^2(\alpha')} \frac{d\lambda}{\sqrt{A^2(\alpha_n') - 2\lambda}} =$$

$$= \frac{2\sqrt{A^2(\alpha_n') - \sigma^2 A^2}}{f(\mu)\sqrt{a(\alpha_n')}} = \frac{2\sqrt{A^2 + \nu_n - \sigma^2 A^2}}{f(\mu)\sqrt{a(\alpha_n')}} = \frac{2\sqrt{A^2(1-\sigma^2) + \nu_n}}{f(\mu)\sqrt{a(\alpha_n')}},$$

where $\nu_n \geqq 0$ and $\lim_{n=\infty} \nu_n = 0$. Therefore $\nu_n \leqq \nu$ $(\nu > 0)$ fore some $n \geqq n_0$ and

$$\beta_n' - \alpha_n' \leqq \beta_n'' - \alpha_n' \leqq \frac{2\sqrt{A^2(1-\sigma^2) + \nu}}{f(\mu)\sqrt{a(\alpha_n')}} \qquad (n \geqq n_0).$$

By means of this and (6)

$$\frac{\beta_n' - \alpha_n'}{\beta_n' - \beta_{n-1}'} \leqq \frac{2\sqrt{A^2(1-\sigma^2) + \nu}}{f(\mu) \displaystyle\int_0^{\mu} \frac{dz}{\sqrt{A^2(t_1) - 2\,F(z)}}} = G(\sigma, \nu), \tag{7}$$

and finally

$$\frac{\sum_{n=n_0+1}^{N} (\beta_n' - \alpha_n')}{\beta_N' - \alpha_{n_0}'} \leqq \frac{\sum_{n=n_0+1}^{N} (\beta_n' - \alpha_n')}{\sum_{n=n_0+1}^{N} (\beta_n' - \beta_{n-1}')} \leqq G(\sigma, \nu) \qquad (N > n_0). \tag{8}$$

The number μ is increasing with σ, therefore the denominator of (7) too. Once having chosen the number ν, i.e. n_0 so that $G(1, \nu) < \varepsilon_0$ is, we have also $G(\sigma, \nu) < \varepsilon_0$ provided that σ is near enough to 1. Therefore the left hand side of (8) may for arbitrary $N > n_0$ be made less than ε_0 by taking σ near enough to 1 and just this was to be proved.

(Received April 28, 1961.)

REFERENCES

[1] SANSONE, L.: *Equazioni differenziali nel campo reale II*, second edition, 1949.
[2] OPIAL, Z.: "Sur l'équation différentielle $u'' + a(t) = 0$". *Annales Polon. Math.* **5** (1958) 77—93.
[3] FOIAS, C.—GUSSI, G.—POENARU, V.: "Une méthode directe dans l'étude des équations aux dérivées partielles". *Math. Nachr.* **15** (1956) 89—116.
[4] BIHARI, I.: "Oscillation and monotonity theorems". *Acta Math. Hung. Acad.* **9** (1958) 83—104.

РАСПРОСТРАНЕНИЕ ОДНОЙ ТЕОРЕМЫ ARMELLINI—TONELLI—SANSONE НА НЕЛИНЕЙНЫЕ УРАВНЕНИЯ $u'' + a(t)f(u) = 0$

I. BIHARI

Резюме

В статье дается доказательство следующего факта. Всякое решение нелинейного уравнения

$$u'' + a(t)\,f(u) = 0$$

стремится к нулю при $t \to \infty$, если только $a(t)$ положительная, непрерывная неубывающая функция, $\log a(t)$ стремится «регулярно» к $+\infty$ при $t \to +\infty$, $f(u)$ является неубывающей нечёткой функцией, далее, $\frac{f(u)}{u}$ не убывающая функция при $u > 0$, $\frac{f(u)}{u} = O(t)$ $(u \to 0)$ и, наконец, $|f(u_1) - f(u_2)| \leqq \omega(|u_1 - u_2|)$, где $\omega(z)$ неубывающая положительная функция и

$$\int_0^{u_0} \frac{dz}{\omega(z)} = \infty \qquad (u_0 \to 0).$$