ON THE PROBLEM OF MIKUSIŃSKI'S LOGARITHM

by

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A very important question in the application of MIKUSIŃSKI's operational calculus is, to settle the question, whether a certain operator is logarithm or not. This problem was solved by MIKUSIŃSKI [1] for S^α and by WLOKA [4] for $e^{\gamma s}$. In this paper there is given a neccessary and sufficient condition for an operator, to be a logarithm of a certain type. We shall give also conditions, sufficient only, which can be easily applied.

Concerning to the used definitions and theorems, we refer to MIKUSIŃSKI's book [1]. However, we give some of the most important notations and notions as follows.

Notations. N1. A Greek letter means a real number and a Roman letter means an operator. N2. Product is always the structure product, generated from the convolution. N3. C is the ring of continuous functions in $[0, \infty)$ with the convolution product and with the topology generated by the quasi-uniform convergence. (It is called C-convergence). N4. If $f, g \in C$, then we shall write

$$\|f\| \leqq \|g\|$$

if and only if

$$\sup_{t<t_0} |f(t)| \leqq \sup_{t<t_0} |g(t)|$$

for every t_0.

Definition of the exponential function. The exponential function $\exp(-\lambda\omega)$ is an operational function which satisfies the differential equation[1]

$$x'(\lambda) + \omega\, x(\lambda) = 0 \qquad x(0) = 1 \tag{1}$$

for $\lambda > 0$.

Definition of the logarithm. The operator ω is called a logarithm, if

$$\exp(-\lambda\omega)$$

exists.

Definition of bounded logarithm. We say, that a logarithm ω is bounded, if the exponential function $\exp(-\lambda\omega)$ is bounded in the following sense:

[1] We remember [1] that this equation has at most one solution and so the definition is correct.

There exists $f \in C$ that $f \exp(-\lambda\omega) \in C$ for every $\lambda \geqq 0$ and

$$\|f \exp(-\lambda\omega\| \leqq \|f\|.$$

We remark, that even from

$$\|f \exp(-\lambda\omega)\| \leq e^{\beta\lambda}\|f\| \qquad \beta > 0 \tag{2}$$

follows, that the logarithm $\omega + \beta$ is bounded.

In this paper, we are going to give a characterization of the bounded logarithms.

Examples. I. If $a \in C$ then

$$\exp(-\lambda a) = \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^k a^k}{k!}$$

and if $f \in C$ then (1) will be in the form of the integro-differential equation

$$\frac{\partial}{\partial\lambda} x(\lambda, t) + \int_0^t a(t-\tau)\, x(\lambda, \tau)\, d\tau = 0 \qquad x(0, t) = f.$$

II. The operator S is a logarithm. If $f \in C$ than

$$f \exp(-\lambda S) = \begin{cases} 0 & \text{if} \quad t - \lambda < 0 \\ f(t-\lambda) & \text{if} \quad t - \lambda \geqq 0. \end{cases} \tag{3}$$

If $f, f' \in C$ and $f(0) = 0$ then (1) is in the form of the partial differential equation

$$\frac{\partial}{\partial\lambda} x(\lambda, t) + \frac{\partial}{\partial t} x(\lambda, t) = 0 \qquad \begin{aligned} x(0, t) &= f \\ x(\lambda, 0) &= 0. \end{aligned}$$

III. The operator e^{-S} is also a logarithm. If $f \in C$, then (1) will be in the form of the difference-differential equation

$$\frac{\partial}{\partial\lambda} x(\lambda, t) + x(\lambda, t-1) = 0 \qquad x(0) = f$$

(where

$$x(\lambda, t-1) \equiv 0 \quad \text{if} \quad t - 1 < 0)$$

S is bounded logarithm and e^{-S} satisfies (2) as will be seen in corollary 1 and 2.

Lemma. *If $\{\omega_n\}$ is a sequence of logarithms, $\lim\limits_n \omega_n = \omega$ and there exists a continuous function $M(\lambda, t)$ that*

$$\sup_{\lambda < \lambda_0} \|f \exp(-\lambda\omega_n)\| < \|M(\lambda_0)\| \tag{4}$$

then ω is also a logarithm and

$$\lim_n \exp(-\lambda\omega_n) = \exp(-\lambda\omega). \tag{5}$$

Proof. Let $g \in C$ and $\{g\,\omega_n\}$ be C-convergent. Then

$$f^2 g \exp(-\lambda\omega_n) - f^2 g \exp(-\lambda\omega_m) =$$

$$(6)\qquad = \int_0^\lambda \frac{d}{d\mu}\{gf \exp(-[\lambda-\mu]\,\omega_m)\cdot f\exp(-\mu\omega_n)\}\,d\mu =$$

$$= \int_0^\lambda (\omega_m - \omega_n)\,g\cdot f\exp(-[\lambda-\mu]\,\omega_m)\cdot f\exp(-\mu\omega_n)\,d\mu$$

and from (4) and (6)

$$\sup_{\lambda<\lambda_0} \|f^2 g\exp(-\lambda\omega_n) - f^2 g\exp(-\lambda\omega_m)\| \leqq \|\omega_m g - \omega_n g\|\,\lambda_0\,t^2\,\|M(\lambda_0)\|^2$$

hence $\{f^2 g\exp(-\lambda\omega_n)\}$ C-converges uniformly in $[0, \lambda_0]$. So (5) holds.

Theorem 1. *If* $\exp(-\lambda\omega)$ *is bounded then*

$$(*)\qquad \left(\frac{\alpha}{\omega+\alpha}\right)^k f\in C \quad\text{and}\quad \left\|\left(\frac{\alpha}{\omega+\alpha}\right)^k f\right\| \leqq \|f\| \qquad \alpha>0,\ k=1,2,\ldots$$

holds.

Proof.

$$\Big\|\int_n^m e^{-\alpha\lambda} f\exp(-\lambda\omega)\,d\lambda\Big\| \leqq \|f\|\int_n^m e^{-\alpha\lambda}\,d\lambda.$$

So

$$\int_0^\infty e^{-\alpha\lambda} f\exp(-\omega\lambda)\,d\lambda$$

exists and

$$(7)\qquad \Big\|\alpha\int_0^\infty e^{-\alpha\lambda} f\exp(-\lambda\omega)\,d\lambda\Big\| \leqq \|f\|.$$

The equation

$$(8)\qquad (\omega+\alpha)\int_0^m e^{-\alpha\lambda} f\exp(-\omega\lambda)\,d\lambda = \int_0^m (\omega+\alpha)\,e^{-\alpha\lambda} f\exp(-\omega\lambda)\,d\lambda$$

hold, because of the continuity of the product. The right hand side of (8) can be written in the form

$$-\int_0^m \frac{d}{d\lambda}[e^{-\alpha\lambda} f\exp(-\omega\lambda)]\,d\lambda$$

from which follows

$$(9)\qquad \alpha\int_0^\infty e^{-\alpha\lambda}\exp(-\omega\lambda)\,d\lambda = \frac{\alpha}{\omega+\alpha}.$$

From the identity

$$\frac{1}{\alpha-\beta}\left(\frac{1}{\omega+\beta} - \frac{1}{\omega+\alpha}\right) = \frac{1}{\omega+\alpha}\cdot\frac{1}{\omega+\beta}$$

and from (9) follows that

$$-\frac{d}{d\alpha}\frac{1}{\omega+\alpha}=\frac{1}{(\omega+\alpha)^2}$$

and by the use of induction we get

$$(10)\qquad (-1)^k\frac{\alpha^k}{k!}\frac{d^k}{d\alpha^k}\frac{1}{\omega+\alpha}=\frac{\alpha^k}{(\omega+\alpha)^{k+1}}\,.$$

From the identities

$$(-1)^k\frac{d^k}{d\alpha^k}\left(\frac{1}{\omega+\alpha}f\right)=\int_0^\infty e^{-\alpha\lambda}\lambda^k f\exp(-\omega\lambda)\,d\lambda\,,$$

$$(11)\qquad \frac{\alpha^k}{k!}\int_0^\infty e^{-\alpha\lambda}\lambda^k\,d\lambda=1$$

and from (10) follows

$$(12)\qquad (-1)^{k-1}\frac{\alpha^k}{(k-1)!}\int_0^\infty e^{-\alpha\lambda}\lambda^{k-1}f\exp(-\omega\lambda)\,d\lambda=\left(\frac{\alpha}{\omega+\alpha}\right)^k f\,.$$

From (12) and (11) we get (*).

Theorem 2. *If* (*) *is true, then* ω *is logarithm and*

$$\|f\exp(-\omega\lambda)\|\leq\|f\|\,.$$

Proof. I.

$\dfrac{\alpha\omega}{\omega+\alpha}$ is a logarithm for every $\alpha>0$.

If $\dfrac{\alpha\omega}{\omega+\alpha}\in C$, then

$$f\exp\left(-\frac{\alpha\omega}{\omega+\alpha}\lambda\right)=e^{-\alpha\lambda}\sum_{k=0}^{\infty}\frac{\alpha^k\lambda^k}{k!}\left(\frac{\alpha}{\omega+\alpha}\right)^k f$$

since in this case the power series of the exponential function is convergent. We show, that this is alwys true if (*) is satisfied. Let

$$(13)\qquad \varphi_n=\varphi_n(\alpha,\lambda)=e^{-\alpha\lambda}\sum_{k=n}^{n}\frac{\alpha^k\lambda^k}{k!}\left(\frac{\alpha}{\omega+\alpha}\right)^k.$$

With the notation (13)

$$\|(\varphi_n-\varphi_m)f\|\leq\|f\|\sum_{k=n}^{m}\frac{\alpha^k\lambda^k}{k!}$$

and

(14) $$\|\varphi_n f\| \leqq \|f\|$$

so

$$\lim_n \varphi_n f = \varphi f$$

uniformly in every finite $[0, \lambda]$. Here

$$\varphi = \varphi(\alpha, \lambda) = e^{-\alpha\lambda} \sum_{k=0}^{\infty} \frac{\alpha^k \lambda^k}{k!} \left(\frac{\alpha}{\omega+\alpha}\right)^k .$$

Since

$$\frac{\partial}{\partial\lambda}\varphi(\alpha, \lambda) = -\alpha\varphi(\alpha, \lambda) + \frac{\alpha^2}{\omega+\alpha}\varphi(\alpha, \lambda) = -\frac{\alpha\omega}{\omega+\alpha}\varphi(\alpha, \lambda)$$

and

$$\varphi(\alpha, 0) = 1$$

thus the statement I is proved.

II. $$\boxed{\lim_{\alpha\to\infty} \frac{\alpha\omega}{\omega+\alpha} = \omega}$$

It is easy to verify, that

$$\left\|\frac{\alpha\omega}{\omega+\alpha} f\cdot g\right\| \leqq t\,\|\omega g\|\,\|f\| .$$

Thus

$$\left\|\left(\frac{\alpha\omega}{\omega+\alpha} - \omega\right) f\cdot g^2\right\| = \left\|\frac{\omega^2}{\alpha+\omega} f g^2\right\| \leqq \frac{t}{\alpha}\,\|\omega^2 g^2\|\,\|f\| .$$

III. $$\boxed{\left\| f\cdot\exp\left(-\frac{\alpha\omega}{\omega+\alpha}\lambda\right)\right\| \leqq \|f\|}$$

This is an immediate consequence of (14).

The theorem follows from I, II and III considering the Lemma. And now, we give two conditions only sufficient conditions for to be a logarithm of bounded type.

Corollary 1. *If* $\frac{1}{\omega} \in C$, *then* $\frac{\alpha}{\omega+\alpha} \in C$ *too. If in addition*

$$\frac{1}{\omega} \geqq 0 ; \quad \frac{\alpha}{\omega+\alpha} \geqq 0 \quad \text{for every } \alpha > 0$$

then (*) *is satisfied.*

Proof. If $\frac{1}{\omega} \in C$, then

$$\frac{\alpha}{\omega + \alpha} = \frac{\frac{\alpha}{\omega}}{1 + \frac{\alpha}{\omega}} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\alpha^k}{\omega^k} \in C .$$

If

(15) $$\frac{1}{\omega} > 0 ; \quad \frac{\alpha}{\omega + \alpha} > 0$$

then

(16) $$\frac{\alpha}{\omega + \alpha} \frac{1}{\omega} \leq \frac{\alpha}{\omega + \alpha} \frac{1}{\omega} + \frac{1}{\omega + \alpha} = \frac{1}{1 + \frac{\alpha}{\omega}} \left(\frac{\alpha}{\omega} + 1 \right) \frac{1}{\omega} = \frac{1}{\omega} .$$

Because of (15), from (16) follows

$$\left\| \frac{\alpha}{\omega + \alpha} \frac{1}{\omega} \right\| \leq \left\| \frac{1}{\omega} \right\| .$$

From the inequality

$$\left(\frac{\alpha}{\omega + \alpha} \right)^k \frac{1}{\omega} = \left(\frac{\alpha}{\omega + \alpha} \right)^{k-1} \left(\frac{\alpha}{\omega + \alpha} \frac{1}{\omega} \right) \leq \left(\frac{\alpha}{\omega + \alpha} \right)^{k-1} \frac{1}{\omega}$$

by induction follows, that

$$\left\| \left(\frac{\alpha}{\omega + \alpha} \right)^k \frac{1}{\omega} \right\| < \left\| \frac{1}{\omega} \right\| .$$

The second condition (Corollary 2) based on the following lemma.

Lemma 2. Let be C_0 a closed linear subspace of C, ω be a bounded logarithm and

(17) $$e^{-\lambda\omega} C_0 \subseteq C_0 .$$

Then

(18) $$\sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{(\alpha + 1)^{k+1}} e^{-k\omega} f \in C_0$$

for every $f \in C_0$ and $\alpha > 0$.

Proof. From the boundedness of ω follows, that

(19) $$\left\| \sum_{k=n}^{m} (-1)^k \frac{\alpha}{(\alpha + 1)^{k+1}} e^{-k\omega} f \right\| \leq \|f\| \sum_{k=n}^{m} \frac{\alpha}{(\alpha + 1)^{k+1}} \qquad \text{for } \alpha > 0$$

and from (17) follows

$$\sum_{k=0}^{n} (-1)^k \frac{\alpha}{(\alpha+1)^{k+1}} e^{-k\omega} f \in C_0 . \tag{20}$$

From (19) and (20) follows (18).

Corollary 2. If ω is bounded logarithm and (17) is valid, then $e^{-\omega} + 1$ is also a logarithm of bounded type.

Proof. If $f \in C_0$ then

$$\frac{\alpha}{e^{-\omega} + 1 + \alpha} f = \sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{(\alpha+1)^{k+1}} e^{-k\omega} f \tag{21}$$

and

$$\left\| \sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{(\alpha+1)^{k+1}} e^{-k\omega} f \right\| \leqq \|f\| \sum_{k=0}^{\infty} \frac{\alpha}{(\alpha+1)^{k+1}} = \|f\| . \tag{22}$$

From (21) and (22) follows, that the condition (*) is satisfied for $k = 1$. From lemma 2 follows, that

$$\left(\frac{\alpha}{e^{-\omega} + 1 + \alpha}\right)^k f \in C_0 \qquad k = 1, 2, \ldots$$

and from (21) and (22)

$$\left\| \left(\frac{\alpha}{e^{-\omega} + 1 + \alpha}\right)^k f \right\| = \tag{23}$$

$$= \left\| \frac{\alpha}{e^{-\omega} + 1 + \alpha} \left(\frac{\alpha}{e^{-\omega} + 1 + \alpha}\right)^{k-1} f \right\| \leqq \left\| \left(\frac{\alpha}{e^{-\omega} + 1 + \alpha}\right)^{k-1} f \right\| .$$

From the inequality (23) follows, that

$$\left\| \left(\frac{\alpha}{e^{-\omega} + 1 + \alpha}\right)^k f \right\|$$

is a monoton decreasing function of k and so, condition (*) of theorem 1 is valid for every k. Q. E. D.

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О ПРОБЛЕМЕ ЛОГАРИФМОВ МИКУСИНСКОГО

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Резюме

Неравенство (*) является необходимым и достаточным условием того, чтобы уравнение (1) имело такое $\exp(-\lambda\omega)$ решение, которое удовлетворяет следующим условиям:

а) Если $f \in C$, тогда $f \exp(-\lambda\omega) \in C$ для всех положительных λ.

б) $$\sup_{\lambda>0} \sup_{t<t_0} |f \exp(-\lambda\omega)| \leqq \sup_{t<t_0} |f| .$$

В следствиях 1 и 2 автор дает легко применяемые достаточные условия для выполнения (*).