A THEOREM CONCERNING HAMILTON LINES

by

L. PÓSA

In this note we only consider graphs without loops and multiple edges. G. A. DIRAC proved the following theorem [1].

Let $G^{(n)}$ $(n \geqq 3)$ *be a graph of n vertices. Assume that the valency of every vertex is* $\geqq n/2$. *Then* $G^{(n)}$ *is Hamiltonian.*

The valency $v(x)$ of a vertex x is the number of edges incident to it. A graph is said to be Hamiltonian if it contains a Hamilton line (i. e. a circuit which contains every vertex of the graph).

Several sharpenings of this theorem are known ([2], [4], [5]). We now prove the following theorem which contains [2].

Theorem. *Let* $n \geqq 3$. *Assume that for every* k, $1 \leqq k < (n-1)/2$, *the number of vertices of* $G^{(n)}$ *of valency not exceeding k is less than k and for odd n the number of vertices of valency* $(n-1)/2$ *does not exceed* $(n-1)/2$, *then* $G^{(n)}$ *is Hamiltonian.*

Proof. (I). Assume that $G^{(n)}$ satisfies the conditions of the Theorem and is not Hamiltonian. By connecting (by an edge) vertices of $G^{(n)}$ which were not connected in $G^{(n)}$ (by an edge) we obtain a graph $G_*^{(n)}$ which is not Hamiltonian but which becomes Hamiltonian if we connect any two vertices of $G_*^{(n)}$ which are not connected by an edge. Since every complete graph of $n \geqq 3$ vertices is Hamiltonian, our graph $G_*^{(n)}$ exists. (A graph is said to be complete if every two of its vertices are connected by an edge.) Clearly $G_*^{(n)}$ has the same vertices as $G^{(n)}$ and $G_*^{(n)}$ also satisfies the conditions of our Theorem. Further if two vertices of $G_*^{(n)}$ are not connected by an edge they are connected by an open Hamilton line (i. e. by a path which contains every vertex of our graph, once and only once).

(II). First we show that in $G_*^{(n)}$ every vertex of valency $\geqq (n-1)/2$ is connected to every vertex of valency $\geqq n/2$. To show this let a_1 and a_n be two vertices with $v(a_1) \geqq (n-1)/2$, $v(a_n) \geqq n/2$, which are not connected by an edge. Then by (I) there is an open Hamilton line $(a_1, a_2, \ldots, a_n)$ (i. e. the edges of the open Hamilton line are the edges connecting a_i with a_{i+1}, $1 \leqq i \leqq n-1$). Let $a_{i_1}, \ldots, a_{i_k}$ $k = v(a_1)$, $2 = i_1 < \ldots < i_k \leqq n-1$ be the vertices connected with a_1 in $G_*^{(n)}$ by an edge. a_n can not be connected to a_{i_j-1}, $1 \leqq j \leqq k$ by an edge for otherwise $(a_1, \ldots, a_{i_j-1}, a_n, a_{n-1}, a_{n-2}, \ldots, a_{i_j}, a_1)$ would be a Hamilton line of $G_*^{(n)}$. Thus

$$\frac{n}{2} \leqq v(a_n) \leqq n-1-k < \frac{n}{2}.$$

This contradiction establishes our assertion.

(III). From (II) it clearly follows that $G_*^{(n)}$ has vertices of valency $< n/2$ (since the complete graph of $n \geqq 3$ vertices is Hamiltonian). Let m be the maximal valency of these vertices and b_1 a vertex with $v(b_1) = m$. By our assumptions there are at most $m < n/2$ vertices of valency $\leqq m$, thus there are more than m vertices of valency $> m$, (by the maximality of m the valency of these vertices is $\geqq n/2$). Thus there exists a vertex b_n of valency $\geqq n/2$ which is not connected to b_1 by an edge. Hence by (II) $m < (n-1)/2$ and therefore we can assume that the number of vertices of valency $\leqq m$ is less than m.

Let now $(b_1, b_2, \ldots, b_n)$ be an open Hamilton line connecting b_1 and b_n and let $b_{i_1}, \ldots, b_{i_m}$ $(i_1 = 2 < i_2 < \ldots < i_m \leqq n-1)$ be the vertices connected to b_1 by an edge. As in (II) it follows that b_n can not be connected to b_{i_j-1}, $1 \leqq j \leqq m$ by an edge. By our assumption at least one of these vertices must have valency $> m$ and hence valency $\geqq n/2$. But this contradicts (II) and our Theorem is proved.

Remarks (1). Our Theorem is sharp. Let $1 \leqq k < (n-1)/2$ and G_1 be a complete graph of $k+1$ vertices and G_2 a complete graph of $n-k$ vertices. We assume that G_1 and G_2 have exactly one common vertex. Let G be the union of G_1 and G_2. Clearly G is not Hamiltonian (it has a cut point) and G has exactly k vertices of valency k. Now let n odd and $k = (n-1)/2$. We define the graph G as follows: The vertices of G are $x_1, \ldots, x_{2k+1}$ and its edges are (x_i, x_j), $1 \leqq i \leqq k < j \leqq 2k+1$. G is not Hamiltonian and G has exactly $(n+1)/2$ verties of valency $(n-1)/2$.

(2). (I) and (II) gives a very simple proof of Dirac's theorem [1].

(Received August 2, 1962.)

REFERENCES

[1] Dirac, G. A.: „Some theorems on abstract graphs." *Proc. London math. Soc.* (3), **2**(1952)69—81.
[2] Erdős, P.—Gallai, T.: „On maximal paths and circuits of graphs." *Acta Math. Sci. Hung.* **10**(1959)337—356.
[3] Newman, D. J.: „A problem in graph theory." *Amer. Math. Monthly* **65**(1958)611.
[4] Ore, O.: „Note on Hamilton circuits." *Amer. Math. Monthly* **67**(1960)55.
[5] Ore, O.: „Theory of graphs." *Amer. Math. Soc. Colloquium Publ.* **38**(1962)

ТЕОРЕМА, ОТНОСЯЩАЯСЯ К ГАМИЛЬТОНОВЫМ ЛИНИЯМ

L. PÓSA

Резюме

Для графов, не содержащих петель и кратных ребер, имеет силу следующая

Теорема. *Пусть число точек графа G не менее трех. Если для любого k, где $1 \leqq k < (n-1)/2$, число тех точек G, степень которых $\leqq k$ меньше k и, в случае нечетного n, число тех точек, степень которых $\leqq (n-1)/2$, не долее $(n-1)/2$, тогда G обладает Гамильтоновой линией, то есть такой окружностью, которая содержит все точки G. Теорема точна.*