REMARKS ON A PAPER OF PÓSA

by

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This note will use the terminology of PÓSA's paper. $G_l^{(n)}$ will denote a graph of n vertices and l edges and $G_l^{(n)}(k)$ denotes a graph of having n vertices l edges and every vertex of which has valency $\geqq k$. ORE [2] proved that if $l \geqq \binom{n-1}{2} + 2$ then every $G_l^{(n)}$ is Hamiltonian, and he showed that the result is false for $l = \binom{n-1}{2} + 1$. Now I prove the following more general

Theorem. *Let* $1 \leqq k < n/2$. *Put*

$$l_k = 1 + \max_{k \leqq t < \frac{n}{2}} \left[\binom{n-t}{2} + t^2 \right] =$$

(1)
$$= 1 + \max \left[\binom{n-k}{2} + k^2, \binom{n - \left[\frac{n-1}{2}\right]}{2} + \left[\frac{n-1}{2}\right]^2 \right].$$

Then every $G_{l_k}^{(n)}(k)$ *is Hamiltonian. There further exists a* $G_{l_k-1}^{(n)}(k)$ *which is not Hamiltonian.*

First of all observe that by the theorem of DIRAC (see the preceding paper of PÓSA) if every vertex of G has valency $\geqq n/2$ then G is Hamiltonian, thus the condition $1 \leqq k < n/2$ can be assumed without loss of generality.

Next a simple computation shows that $\binom{n-t}{2} + t^2$ decreases for $1 \leqq t \leqq (n-2)/3$ and increases for $(n-2)/3 < t < n/2$, which proves the second equality of (1).

Now we are ready to prove our Theorem. If our $G^{(n)}(k)$ is not Hamiltonian then by the Theorem of PÓSA there exists a t, $k \leqq t < n/2$ so that $G^{(n)}(k)$ has at least t vertices $x_1, \ldots, x_t$ of valency not exceeding t. The number of edges of $G^{(n)}(k)$ which are not incident to any of the vertices $x_1, \ldots, x_t$ is clearly at most $\binom{n-t}{2}$ (i. e. if the vertices of $G^{(n)}(k)$ are $x_1, \ldots, x_n$ we obtain $\binom{n-t}{2}$ edges if every two of the vertices x_{j_1} and x_{j_2}, $t < j_1 < j_2 \leqq n$ are connected by an edge). The number of edges incident to one of the vertices $x_1, \ldots, x_t$ is at most t^2 (since each of them has valency $\leq t$). Thus our $G^{(n)}(k)$ has at most $\binom{n-t}{2} + t^2$ edges for some $k \leq t < n/2$ i. e. it can have at most $l_k - 1$ edges which proves (1).

To complete our proof we show that (1) is best possible. Let the vertices of $G^{(n)}_{l_t-1}(t)$ be $x_1, \ldots, x_n$. Its edges are:

$$(x_{j_1}, x_{j_2}), \quad t < j_1 < j_2 \leq n \text{ and } (x_i, x_j), \quad 1 \leq i \leq t < j \leq 2t < n .$$

A simple argument shows that our $G^{(n)}_{l_t-1}(t)$ is not Hamiltonian (it clearly has $l_t - 1$ edges). It is easy to see that every $G^{(n)}_{l_t-1}(t)$ which is not Hamiltonian has this structure. (If $t = (n-1)/2$ (n odd) by Pósa's theorem we can assume that there are $t + 1 = (n + 1)/2$ vertices of valency $\leq t$ but by $\binom{t+1}{2} + t^2 = \binom{t}{2} + t(t+1)$ we do not obtain a better result by utilising this $(t+1)$-st vertex).

It is easy to see that the argument of Pósa's paper gives the following

Theorem. *Let $G^{(n)}$ be a graph and assume that for every $1 \leq k < (n-1)/2$ $G^{(n)}$ has at most k vertices of valency $\leq k$. Then $G^{(n)}$ has an open Hamilton line. The theorem is best possible.*

The proof can be left to the reader of Pósa's paper. Using this result we obtain by the same argument as used in this paper that every $G^{(n)}_{\mu_k}(k)$ with

$$\mu_k = 1 + \max_{k \leq t < \frac{n-1}{2}} \left[\binom{n-t-1}{2} + t(t+1) \right]$$

has an open Hamilton line. The theorem is best possible. Finally we mention that by the method of this paper we can prove the following sharpening of Lemma (3.2) of [1]. Let $G^{(n)}$ be a graph with the vertices $x_1, \ldots, x_n$ and $2 \leq k < n/2$. Assume that $v(x_1) \geq k$ and that there is a circuit containing the vertices $x_2, x_3, \ldots, x_n$. Then if $G^{(n)}$ has $\geq l_k$ edges it is Hamiltonian. The result is best possible. We leave the simple proof to the reader.

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ЗАМЕЧАНИЯ ОБ ОДНОЙ СТАТЬЕ PÓSA

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Для тех графов, которые не содержат петель и кратных ребер, исходя из одной теоремы PÓSA [3] автор доказывает следующее

Теорема. *Пусть* $1 \leqq k < \frac{n}{2}$ *и*

$$l_k = 1 + \max_{k \leqq t < \frac{n}{2}} \left[\binom{n-t}{2} + t^2 \right] =$$

$$= 1 + \max \left[\binom{n-k}{2} + k^2, \binom{n - \left[\frac{n-1}{2}\right]}{2} + \left[\frac{n-1}{2}\right]^2 \right].$$

Тогда в каждом графе G, который имеет n точек и в котором степень каждой точки $\geqq k$ и число ребер равно l_k, существует Гамильтонова линия, то есть окружность, содержающая все точки G. Теорема точна.