ON THE METHOD OF CHARACTERISTICS

by

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1. Introduction

The classical method of characteristics, discovered by J. MASSAU, is one of the standard numerical methods for solving initial value problems for quasi-linear hyperbolic systems involving n unknown functions and two independent variables. Since an initial value problem for a general nonlinear system can always be transformed into a quasilinear one containing a larger number of equations and unknowns (see, for example, [1] p. 35), the method is applicable to any nonlinear hyperbolic system in two independent variables.

Although the method of characteristics has been much used in practice (see, for example [5], p. 133), as far as the author knows, no attempt has been made to estimate the error of the process in the general case; even the convergence of the process for $n > 2$ has never been proved.

In the present work we shall prove that under some, rather trivial assumptions the error is of order $O(h)$, where h is the maximum distance between two adjacent grid points on the initial curve. Thus the order of the error is the same as in the finite difference method of COURANT, ISAACSON and REES (see [2]). The method of characteristics, however, may be more advantageous in certain cases, since practically no restrictive conditions are required to ensure stability and convergence.

In the special case $n = 2$ a simpler proof of this result is given in [3].

2. The initial value problem for quasilinear hyperbolic systems in two independent variables

Let us consider the following quasilinear system, written in matrix form:

$$\boldsymbol{u}_x + \mathbf{A}\,\boldsymbol{u}_y + \boldsymbol{h} = \mathbf{O}\,, \tag{2.1}$$

where $\boldsymbol{u}_x$ and $\boldsymbol{u}_y$ are partial derivatives of the column vector $\boldsymbol{u} = [u_i]$ whose components u_i are unknown functions of the variables x and y; $\mathbf{A} = [a_{ij}]$ is a $n \times n$ matrix, and $\boldsymbol{h} = [h_i]$ is a column vector; the elements a_{ij} and h_i may depend on x, y and $\boldsymbol{u}$ (but not on $\boldsymbol{u}_x$ and $\boldsymbol{u}_y$).

We assume that the system (2.1) is of hyperbolic type in an open region Γ of the $n+2$-dimensional $x, y, \boldsymbol{u}$-space, that is, the matrix $\mathbf{A}$ has n real eigenvalues $\lambda_1(x, y, \boldsymbol{u}) \leq \lambda_2(x, y, \boldsymbol{u}) \leq \lambda_n(x, y, \boldsymbol{u})$ at every point of Γ, and to these eigen-

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values correspond n linearly independent eigenrows $\boldsymbol{b}_1^*(x, y, \boldsymbol{u})$, $\boldsymbol{b}_2^*(x, y, \boldsymbol{u})$, $\ldots$, $\boldsymbol{b}_n^*(x, y, \boldsymbol{u})$ such that

$$\boldsymbol{b}_i^*(\mathbf{A} - \lambda_i \mathbf{E}) = \mathbf{0}^* , \tag{2.2}$$

where $\mathbf{E}$ is the unit matrix, and $\mathbf{0}^*$ is the zero row. The eigenvalues λ_i, the eigenrows $\boldsymbol{b}_i^*$, and the vector $\boldsymbol{h}$ should have bounded second partial derivatives in Γ. Moreover, let $\lambda_n - \lambda_1 > c_1$ and $|\det \mathbf{B}| > c_2$, where[2] $\mathbf{B} = \begin{bmatrix} \boldsymbol{b}_1^* \\ \vdots \\ \boldsymbol{b}_n^* \end{bmatrix}$.

We suppose for convenience that the initial curve is a segment AB of the y-axis (this assumption can always be satisfied by introducing new coordinates instead of x and y). The components $u_{0i}(y)$ of the initial (vector) function $\boldsymbol{u}_0(y)$ should have derivatives that satisfy a Lipschitz condition in AB. Finally, we assume that every point $(0, y, u_{01}(y), \ldots, u_{0n}(y))$ for which $(0, y) \in AB$ lies in the region Γ.

Let us put $\lambda_{\max} = \max[0, \max \lambda_n(x,y,\boldsymbol{u})]$ and $\lambda_{\min} = \min[0, \min\limits_{(x,y,\boldsymbol{u}) \in \Gamma} \lambda_1(x,y,\boldsymbol{u})]$. By the existence theorem of A. SCHMIDT (see [4]) from the preceding assumptions it follows that our initial value problem has a unique solution inside a trapezoid[3] Δ, bounded by the segment AB, two sides with slopes $\lambda_{\max}$ and $\lambda_{\min}$ through A and B respectively, and a line parallel to the y-axis; moreover, the partial derivatives of the solution satisfy a Lipschitz condition inside Δ.

3. The method of characteristics

If $\boldsymbol{u} = \boldsymbol{u}(x, y)$ is a solution of the system (2.1), then along the line element $dy = \lambda_i\, dx$ we have

$$\frac{d\boldsymbol{u}}{dx} = \boldsymbol{u}_x + \lambda_i \boldsymbol{u}_y = -(\mathbf{A} - \lambda_i \mathbf{E})\, \boldsymbol{u}_y - \boldsymbol{h} ,$$

whence by (2.2)

$$\boldsymbol{b}_i^* \frac{d\boldsymbol{u}}{dx} = -\boldsymbol{b}_i^* \boldsymbol{h} = d_i . \tag{3.1}$$

The simplest variant of MASSAU's method of characteristic can now be described as follows. We choose a sequence of grid points (that are not necessarily equally spaced) on the initial segment. These points will be called grid points at the 0-th level. The values of $\boldsymbol{u}$ at these points can be determined from the initial condition. If P_0 is a grid point at the 0-th level then we put $y_0(P_0) = y(P_0)$. Let now P, P' be two adjacent grid points at the ν-th level such that $y_0(P) > y_0(P')$. We assume that the approximate values of $\boldsymbol{u}$ at these points,

$$\overline{\boldsymbol{u}}(P) = \overline{\boldsymbol{u}}(x(P), y(P)) \quad \text{and} \quad \overline{\boldsymbol{u}}(P') = \overline{\boldsymbol{u}}(x(P'), y(P'))$$

are already computed. If Q is the intersection of the line with slope $\bar{\lambda}_1(P) = \lambda_1(x(P), y(P), \overline{\boldsymbol{u}}(P))$ through P and the line with slope $\bar{\lambda}_n(P) = \lambda_n(x(P), y(P), \overline{\boldsymbol{u}}(P))$ through P', then Q will be called a grid point at the $\nu+1$-th level, and we shall put $y_0(Q) = y_0(P)$. It will be proved in the sequel that

[2] In this paper $c_1, \ldots, c_{19}$ are positive constants that depend on the bounds and the Lipschitz constants of a_{ij}, h_i, $\boldsymbol{u}$, $\boldsymbol{u}_x$ and $\boldsymbol{u}_y$.

[3] In what follows we shall regard the half-plane $x > 0$ only.

if the maximum distance between two adjacent grid points at the 0-th level is small, and the points P, P' lie near to the y-axis, then Q always (exists and) lies right to the points P and P'. The coordinates of Q can be determined from the equations

$$y(Q) - y(P) = \lambda_1(P)\,(x(Q) - x(P)) \tag{3.2.a}$$

$$y(Q) - y(P') = \bar{\lambda}_n(P)\,(x(Q) - x(P')) \,. \tag{3.2.b}$$

The components of the vector $\overline{\boldsymbol{u}}(Q)$ are computed from the equations

$$\overline{\boldsymbol{b}}_i^*(P)\,(\overline{\boldsymbol{u}}(Q) - \overline{\boldsymbol{u}}(P_i)) = \bar{d}_i(P)\,(x(Q) - x(P_i))\,, \tag{3.3}$$

where $\overline{\boldsymbol{b}}_i^*(P) = \boldsymbol{b}_i^*(x(P),\ y(P),\ \overline{\boldsymbol{u}}(P))$, $\bar{d}_i(P) = d_i(x(P),\ y(P),\ \overline{\boldsymbol{u}}(P))$ and P_i is the intersection of the line with slope $\bar{\lambda}_i(P)$ through Q and the line PP'. (Thus $P_1 = P$ and $P_n = P'$). Since Q lies right to the points P and P' all points P_i lie between P and P' (see Fig. 1). The vectors $\overline{\boldsymbol{u}}(P_i)$ are defined by linear interpolation between P and P':

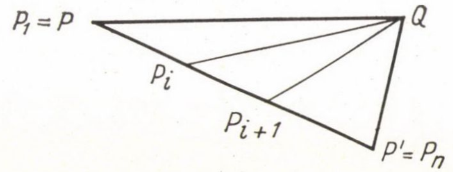

Figure 1.

$$\overline{\boldsymbol{u}}(P_i) = \overline{\boldsymbol{u}}(P) + t(P_i)\,(\overline{\boldsymbol{u}}(P') - \overline{\boldsymbol{u}}(P))\,, \tag{3.4}$$

where

$$t(P_i) = \frac{y(P_i) - y(P)}{y(P') - y(P)}\,; \qquad 0 \leqq t(P_i) \leqq 1\,.$$

It is clear that by successive application of this method we can find the values of $\overline{\boldsymbol{u}}$ in a system of irregularly spaced grid points, bounded by the segment AB and two broken lines that approximate the "highest" characteristic curve through A and the "lowest" characteristic curve through B respectively (see Fig. 2). Thus the method of characteristics automatically yields the domain of determinacy of the segment AB.

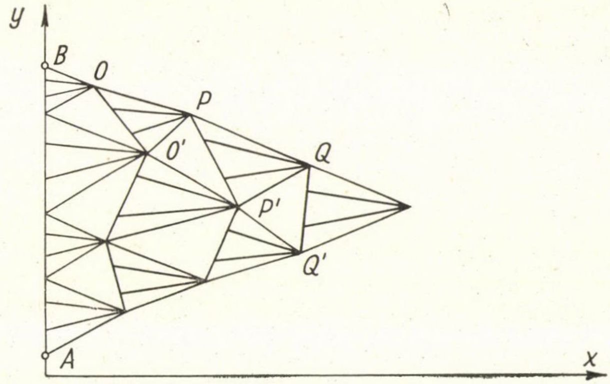

Figure 2.

Let us denote by h and k the maximum resp. minimum distance between two adjacent grid points on the initial segment. We shall prove the following theorem:

If the assumptions of section 2 are satisfied, and $h = O(k)$ then as long as the grid point R lies in a trapezoid Δ' (that may be narrower than the trapezoid Δ of the existence theorem) we have

$$\boldsymbol{u}(R) - \overline{\boldsymbol{u}}(R) = \boldsymbol{O}(h),$$

where $\boldsymbol{O}(h)$ denotes a vector whose components are of order $O(h)$.[4]

4. Two lemmas

The proof of the above theorem will be based on two lemmas. The validity of these lemmas will be proved in section 6.

First lemma: As long as the grid point R lies inside the trapezoid Δ', the point $(x(R), y(R), \overline{u}_1(R), \ldots, \overline{u}_n(R))$ lies in the region Γ. ($\overline{u}_1(R), \ldots, \overline{u}_n(R)$ are the components of the vector $\overline{\boldsymbol{u}}(R)$).

Second lemma: If R and R' are two adjacent grid points at the same level inside Δ' such that $y_0(R) > y_0(R')$ then

$$c_3 h < y(R) - y(R') - \bar{\lambda}_i(R)\,(x(R) - x(R')) < c_4 h \tag{4.1}$$

for $i = 1, 2, \ldots, n$.

From these lemmas we can immediately derive upper and lower bounds for the difference of the abscissas, and upper bounds for the difference of the ordinates of corresponding grid points at two consecutive levels. Namely, (3.2.a) may be rewritten as

$$y(Q) - y(P') + y(P') - y(P) = \bar{\lambda}_1(P)\,[x(Q) - x(P') + x(P') - x(P)],$$

whence by (3.2.b) we have

$$(\bar{\lambda}_n(P) - \bar{\lambda}_1(P))\,(x(Q) - x(P')) = \bar{\lambda}_1(P)\,(x(P') - x(P)) - \\ - (y(P') - y(P)). \tag{4.2}$$

Similarly

$$(\bar{\lambda}_1(P) - \bar{\lambda}_n(P))\,(x(Q) - x(P)) = \bar{\lambda}_n(P)\,(x(P) - x(P')) - \\ - (y(P) - y(P')). \tag{4.3}$$

Since the eigenvalues λ_i are bounded in Γ, from the first lemma it follows that $\bar{\lambda}_n(P) - \bar{\lambda}_1(P) < c_5$, provided the points P and P' lie in Δ'. Thus by the second lemma, (4.2) and (4.3) we obtain:

$$x(Q) - x(P) > c_6 h\,;\; x(Q) - x(P') > c_6 h \tag{4.4}$$

so that the point Q lies right to the points P and P'.

[4] It is possible that the distance of a grid point from the y-axis is less than the altitude of Δ', but it lies outside Δ'. The assertion of the theorem may be extended to these grid points if we continue the initial function $\boldsymbol{u}_0(y)$ beyond the segment AB so that the Lipschitz-condition for $\boldsymbol{u}_0'(y)$ should remain valid.

From the condition $\lambda_n - \lambda_1 > c_1$ by the first lemma it follows that $\lambda_n(P) - \bar{\lambda}_1(P) > c_1$; consequently, by the second lemma, (4.2), (4.3), (3.2.a) and (3.2.b) we have

$$x(Q) - x(P) < c_7 h\,;\; x(Q) - x(P') < c_7 h\,;\; |y(Q) - y(P)| < c_8 h \tag{4.5}$$
$$|y(Q) - y(P')| < c_9 h\,.$$

The differences $x(P) - x(P')$ and $y(P) - y(P')$ may be expressed from the system of linear inequalities

$$|y(P) - y(P') - \bar{\lambda}_1(P)\,(x(P) - x(P')| < c_4 h$$
$$|y(P) - y(P') - \bar{\lambda}_n(P)\,(x(P) - x(P')| < c_4 h\,,$$

since the determinant of this system is equal to $\bar{\lambda}_1(P) - \bar{\lambda}_n(P)$ consequently

$$|x(P) - x(P')| < c_{10} h\,;\; |y(P) - y(P')| < c_{11} h\,. \tag{4.6}$$

5. The error of the method of characteristics

Let $\boldsymbol{u}(x, y)$ be any vector function of two variables, whose partial derivatives satisfy a Lipschitz-condition in Δ'. From the construction of the points Q and P_i by (4.5) and (4.6) it follows that

$$\boldsymbol{u}(Q) - \boldsymbol{u}(P_i) = (\boldsymbol{u}_x(P_i) + \bar{\lambda}_i(P)\,\boldsymbol{u}_y(P_i))\,(x(Q) - x(P_i)) + \boldsymbol{O}(x(Q) - x(P_i))^2 =$$
$$= (\boldsymbol{u}_x(P) + \bar{\lambda}_i(P)\boldsymbol{u}_y P))\,(x(Q) - x(P_i)) + \boldsymbol{O}(h^2)\,,$$

provided the points P, P' and Q lie inside Δ'.

If $\boldsymbol{u}(x, y)$ is the solution of our initial value problem, then by (3.1)

$$\boldsymbol{b}_i^*(P)\,(\boldsymbol{u}(Q) - \boldsymbol{u}(P_i)) = d_i(P)\,(x(Q) - x(P_i)) + \tag{5.1}$$
$$+ O(|\lambda_i(P) - \bar{\lambda}_i(P)|\,h + h^2)\,.$$

Let us put

$$\boldsymbol{z} = \boldsymbol{u} - \overline{\boldsymbol{u}}$$

then because of the continuous differentiability of the eigenvalues λ_i and the eigenrows $\boldsymbol{b}_i^*$ by the first lemma we have[5]

$$\bar{\lambda}_i(P) - \lambda_i(P) = O(|\boldsymbol{z}(P)|)\,,$$
$$\overline{\boldsymbol{b}}_i^*(P) - \boldsymbol{b}_i^*(P) = \boldsymbol{O}(|\boldsymbol{z}(P)|)\,.$$

Thus from (5.1) we get if we replace $\boldsymbol{b}_i^*$ by $\overline{\boldsymbol{b}}_i^*$

$$\overline{\boldsymbol{b}}_i^*(P)\,(\boldsymbol{u}(Q) - \boldsymbol{u}(P_i)) = d_i(P)\,(x(Q) - x(P_i)) + O(|\boldsymbol{z}(P)|\,h + h^2)\,.$$

Subtracting (3.3) from this we obtain

$$\overline{\boldsymbol{b}}_i^*(P)\,(\boldsymbol{z}(Q) - \boldsymbol{z}(P_i)) = (d_i(P) - \bar{d}_i(P))\,(x(Q) - x(P_i)) +$$
$$+ O(|\boldsymbol{z}(P)|\,h + h^2) = O(|\boldsymbol{z}(P)|\,h + h^2)\,,$$

[5] If $\boldsymbol{a}$ is a vector, then by $|\boldsymbol{a}|$ we denote the maximum absolute value of the components of $\boldsymbol{a}$.

or after some rearrangements

$$\bar{\boldsymbol{b}}_i^*(P)\,\boldsymbol{z}(Q) = \bar{\boldsymbol{b}}_i^*(O)\,\boldsymbol{z}(P_i) + \big(\bar{\boldsymbol{b}}_i^*(P) - \bar{\boldsymbol{b}}_i^*(O)\big)\boldsymbol{z}(P_i) + O\big(|\boldsymbol{z}(P)|\,h + h^2\big) \tag{5.2}$$

where O is the grid point corresponding to P at the $\nu - 1$-th level (see Fig. 2; it is convenient to introduce grid points at the $\nu - 1$-th level, since the lemmas will be proved by induction on ν).

By the well-known estimation of the remainder-term in the linear interpolation formula from (4.6) we have

$$\boldsymbol{u}(P_i) = \boldsymbol{u}(P) + t(P_i)\big(\boldsymbol{u}(P') - \boldsymbol{u}(P)\big) + \boldsymbol{O}(h^2). \tag{5.3}$$

Subtracting (3.4) from (5.3) we obtain

$$\boldsymbol{z}(P_i) = \boldsymbol{z}(P) + t(P_i)\big(\boldsymbol{z}(P') - \boldsymbol{z}(P)\big) + \boldsymbol{O}(h^2).$$

Insertion of this into (5.2) yields

$$\begin{aligned}\bar{\boldsymbol{b}}_i^*(P)\,\boldsymbol{z}(Q) &= \bar{\boldsymbol{b}}_i^*(O)\,\boldsymbol{z}(P) + t(P_i)\big(\bar{\boldsymbol{b}}_i^*(O)\,\boldsymbol{z}(P') - \bar{\boldsymbol{b}}_i^*(O)\,\boldsymbol{z}(P)\big) + \\ &+ \big(\bar{\boldsymbol{b}}_i^*(P) - \bar{\boldsymbol{b}}_i^*(O)\big)\boldsymbol{z}(P_i) + O\big(|\boldsymbol{z}(P)|\,h + h^2\big).\end{aligned} \tag{5.4}$$

Because of the continuous differentiability of $\boldsymbol{u}$ using (4.5) it follows that

$$\begin{aligned}\bar{\boldsymbol{u}}(P) - \bar{\boldsymbol{u}}(O) &= \bar{\boldsymbol{u}}(P) - \boldsymbol{u}(P) + \boldsymbol{u}(P) - \boldsymbol{u}(O) + \boldsymbol{u}(O) - \bar{\boldsymbol{u}}(O) = \\ &= \boldsymbol{O}\big(|\boldsymbol{z}(O)| + |\boldsymbol{z}(P)| + h\big).\end{aligned} \tag{5.5}$$

Hence by (4.5) and the first lemma

$$\bar{\boldsymbol{b}}_i^*(P) - \bar{\boldsymbol{b}}_i^*(O) = \boldsymbol{O}\big(|\boldsymbol{z}(O)| + |\boldsymbol{z}(P)| + h\big).$$

Similarly we have

$$\bar{\boldsymbol{b}}_i^*(O) - \bar{\boldsymbol{b}}_i^*(O') = \boldsymbol{O}\big(|\boldsymbol{z}(O)| + |\boldsymbol{z}(O')| + h\big).$$

Inserting these inequalities into (5.4) we get

$$\begin{aligned}\bar{\boldsymbol{b}}_i^*(P)\,\boldsymbol{z}(Q) &= \bar{\boldsymbol{b}}_i^*(O)\,\boldsymbol{z}(P) + t(P_i)\big(\bar{\boldsymbol{b}}_i^*(O')\,\boldsymbol{z}(P') - \bar{\boldsymbol{b}}_i^*(O)\,\boldsymbol{z}(P)\big) + \\ &+ O\big(|\boldsymbol{z}(P)|\,|\boldsymbol{z}(P_i)| + |\boldsymbol{z}(P_i)|\,h + |\boldsymbol{z}(O)|\,|\boldsymbol{z}(P_i)| + |\boldsymbol{z}(O)|\,|\boldsymbol{z}(P')| + \\ &+ |\boldsymbol{z}(P')|\,h + |\boldsymbol{z}(O')|\,|\boldsymbol{z}(P')| + |\boldsymbol{z}(P)|\,h + h^2\big).\end{aligned} \tag{5.6}$$

Denote by M_ν the maximum of $|\,\boldsymbol{z}(R)\,|$ for all grid points R at the ν-th level, and by N_ν the maximum of $|\,\boldsymbol{b}_i^*(S)\,\boldsymbol{z}(R)\,|$ for $i = 1, 2, \ldots, n$ and for all pairs of corresponding grid points S, R at the $\nu - 1$-th and the ν-th level respectively. Since $c_2 < |\det \boldsymbol{B}\,| < c_{12}$ in Γ, from the first lemma it follows that

$$M_\nu = O(N_\nu); \quad N_\nu = O(M_\nu). \tag{5.7}$$

(5.6) and (5.7)yield

$$N_{\nu+1} \leqq N_\nu + c_{13}(N_\nu^2 + N_\nu N_{\nu-1} + N_\nu h + h^2). \tag{5.8}$$

We shall show that if E_ν is the solution of the linear difference problem

$$E_{\nu+1} = E_\nu + c_{14}E_\nu h; \quad E_0 = c_{15}h$$

with suitably chosen constants c_{14} and c_{15}, then $E_\nu \geqq N_\nu$.

Let us denote by d the altitude of the trapezoid Δ'. From (4.4) it follows that $\nu < c_{16}\, dh^{-1}$, therefore

$$E_\nu = E_0(1 + c_{14}h^\nu) < c_{15}he^{c_{14}h\nu} < c_{15}he^{c_{14}c_{16}d}. \tag{5.9}$$

Thus, if $c_{15} > 1$; $c_{14} > 4c_{15}(c_{13} + 1)$ and

$$d < \frac{\log(c_{13} + 1) - \log c_{13}}{c_{14}c_{16}}$$

then $c_{13} + 1 > c_{13}e^{c_{14}c_{16}d}$; $c_{14}h > 4\,c_{13}c_{15}he^{c_{14}c_{16}d} > 4\,c_{13}E_\nu$; $E_\nu > h$ and, consequently

$$E_{\nu+1} \geqq E_\nu + c_{13}(E_\nu^2 + E_\nu E_{\nu-1} + E_\nu h + h^2). \tag{5.10}$$

It is easy to see that $N_0 = 0$ and $N_1 < c_{17}\, h$. Thus if $c_{15} > c_{17}$, then from (5.8) and (5.10) it follows that $E_\nu \geqq N_\nu$. Finally, by (5.7) and (5.9) we have

$$E_\nu = O(h); \quad N_\nu = O(h); \quad M_\nu = O(h). \tag{5.11}$$

This proves our theorem in section 3 provided the two lemmas of section 4 are true. We must still prove these lemmas.

6. Proof of the lemmas.

The two lemmas will be proved simultaneously by induction.

Let P_0, P_0' be two adjacent grid points on the y-axis. From the condition $h = O(k)$ it follows that $c_{18}\, h < y(P_0) - y(P_0') \leqq h$. Thus the second lemma is true on the initial segment provided $c_4 > 1$ and $c_{18} > c_3$. Since the first lemma is trivial, both lemmas are valid at the 0-th level.

Now let us assume both lemmas are proved for all grid points at the 0-th, ..., ν-th level inside Δ' and let Q be a grid at the $\nu + 1$-th level. From (5.11) it follows that

$$\boldsymbol{u}(Q) - \overline{\boldsymbol{u}}(Q) = O(h). \tag{6.1}$$

(It should be observed that in the proof of the inequalities (5.11) for the grid points at the $\nu + 1$-th level the two lemmas have been used only at the 0-th, ..., ν-th levels). If Γ' is a closed subregion of Γ containing the points $(0, y, \boldsymbol{u}_0(y))$, and the altitude d of the trapezoid Δ' is small enough, then as long as the point (x, y) lies in Δ', the point $(x, y, \boldsymbol{u}(x, y))$ lies in Γ'. In particular, the point $(x(Q), y(Q), \boldsymbol{u}(Q))$ lies in Γ'. But then from (6.1) it follows that if h is small enough, the point $(x(Q), y(Q), \overline{\boldsymbol{u}}(Q))$ lies in Γ.

Now let P, P' be at the μ-th level, and let Q, Q' be the corresponding grid points at the $\mu+1$-th level, where $\mu \leqq \nu$. Subtracting (3.2.a) from

$$y(Q') - y(P') = \bar{\lambda}_1(P')\bigl(x(Q') - x(P')\bigr)$$

we get

$$y(Q') - y(Q) = y(P') - y(P) + \bar{\lambda}_1(P')\bigl(x(Q') - x(P')\bigr) - \\ - \bar{\lambda}_1(P)\bigl(x(Q) - x(P)\bigr). \tag{6.2}$$

From (4.5) and (5.11) it follows that

$$\bar{\lambda}_1(Q) - \bar{\lambda}_1(P) = O(h).$$

Similarly we have

$$\bar{\lambda}_1(P) - \bar{\lambda}_1(P') = O(h).$$

Using these inequalities and (4.4) from (6.2) we get

$$y(Q') - y(Q) - \bar{\lambda}_1(Q)\bigl(x(Q') - x(Q)\bigr) = \\ = y(P') - y(P) - \bar{\lambda}_1(P)\bigl(x(P') - x(P)\bigr) + O\bigl(h\bigl(x(Q) - x(P)\bigr)\bigr).$$

Summing this inequality for $\mu = 0, 1, \ldots, \nu$, we find that if Q and Q' are at the $\nu+1$-th level, and P_0, P_0' are the corresponding grid points at the 0-th level, then

$$y(Q') - y(Q) - \bar{\lambda}_1(Q)\bigl(x(Q') - x(Q)\bigr) = y(P_0') - y(P_0) + O(hd),$$

whence

$$(c_{18} - c_{19}d)\,h < y(Q) - y(Q') - \bar{\lambda}_1(Q)\bigl(x(Q) - x(Q')\bigr) < (1 + c_{19}d)\,h.$$

Thus the second lemma is true for the points Q, Q' and for $i = 1$, provided d is small enough (but independent of h and ν). The case $i = n$ can be treated in the same way.

(Received september 13, 1961)

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О МЕТОДЕ ХАРАКТЕРИСТИК

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Резюме

В настоящей статье рассматривается численное решение задачи Коши для системы n квазилинейных гиперболических уравнений с двумя независимыми переменными по методу характеристик (по методу Массо). Доказывается следующая теорема:

Пусть матрица $\mathbf{A}$, вектор $\boldsymbol{h}$ и началные условия для системы (2.1) удовлетворяют требованиям, перечисленным в пункте 2, и пусть $h=O(k)$ где h-максимальное, k-минимальное расстояние между двумя соседними точками сетки на начальном отрезке. Тогда для каждой точки сетки R лежащей внутри некоторой трапеции Δ', ограниченной начальным отрезком AB мы имеем

$$\boldsymbol{u}(R)-\overline{\boldsymbol{u}}(R)=\boldsymbol{O}(h),$$

где $\boldsymbol{u}(R)$ — значение решения системы (2.1) в точке R, а $\overline{\boldsymbol{u}}(R)$ — приближенное значение по методу Массо.