ON A PROBLEM OF TURÁN

by

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Abstract

TURÁN has asked the following question: What is the smallest number, $f(n)$, such that for any system φ of more than $f(n)$ triplets formed from n given elements there are always four elements all four triplets of which occur in φ. The following closely related problem is solved: What is the least number, $g(n)$, such that for any system φ of more than $g(n)$ triplets formed from n given elements there are always four elements each pair of which occurs in some triplet of φ. It is shown that $g(n) = \frac{n^3}{27}, \frac{(n+2)(n-1)^2}{27}$, or $\frac{(n-2)(n+1)^2}{27}$, according as n is congruent to 0, 1, or 2, respectively, modulo 3.

In a recent paper P. ERDŐS [1] has stated an unsolved problem due to TURÁN, namely, what is the smallest number, $f(n)$, such that for every system φ of more than $f(n)$ triplets formed from n given elements there are always four elements all four triplets of which occur in φ. We will treat the following closely related question: Given an ordinary graph on n points (with no loops or multiple edges), with the property that each of its edges belongs to at least one triangle, i.e. a complete subgraph of order 3. What is the least number, $g(n)$, such that every such graph containing more than $g(n)$ triangles contains at least one complete quadrilateral, i.e. a complete subraph of order 4. The main object of this note is to prove the following result:

Theorem.

$$g(n) = \begin{cases} \frac{n^3}{27}, & \text{if } n \equiv 0(3)\,; \\ \frac{(n+2)(n-1)^2}{27}, & \text{if } n \equiv 1(3)\,; \\ \frac{(n-2)(n+1)^2}{27}, & \text{if } n \equiv 2(3)\,. \end{cases} \tag{1}$$

We first derive a lower bound for Q, the number of complete quadrilaterals in a graph of the given type on n points, in terms of T and E, the number of triangles and edges, respectively, contained in the graph.

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Let the points of the graph be labelled $P_1, P_2, \ldots, P_n$. For every pair of distinct points, (P_i, P_j), which are joined by an edge, let $t(i, j)$ denote the number of triangles in the graph which contain the edge joining P_i to P_j.

$$\Sigma\, t(i, j) = 3T , \tag{2}$$

where the summation (here and elsewhere in general), is over all sets of points in the graph for which the expression being summed is defined.

If the distinct points P_i, P_j and P_k form a triangle in the graph define $w(i, j, k)$ and $r(i, j, k)$ as follows:

$$w(i, j, k) = t(i, j) + t(j, k) + t(i, k) , \tag{3}$$

and

$$r(i, j, k) = w(i, j, k) - 2(\text{number of points } P_l,\ l \neq i, j, k, \text{ such that } P_l \text{ is joined by an edge to } P_i, P_j, \text{ and } P_k). \tag{4}$$

It is easily seen that

$$w(i, j, k) = r(i, j, k) + 2Q(i, j, k) , \tag{5}$$

where $Q(i, j, k)$ denotes the number of complete quadrilaterals containing the points P_i, P_j, and P_k.

Summing (5) over all triangles in the graph gives

$$8Q = \Sigma\, t^2(i, j) - \Sigma\, r(i, j, k) , \tag{6}$$

since each complete quadrilateral is counted four times and the edge joining P_i to P_j contributes, $t(i, j)$ times, an amount equal to $t(i, j)$ to the sum of the $w(i, j, k)$'s.

There are T non-zero terms in the sum of the $r(i, j, k)$'s each of which is less than or equal to n, the number of points in the graph, from (4). Also, a lower bound for the sum of the squares of the $t(i, j)$'s is obtained by setting each of the E non-zero terms equal to $\frac{3T}{E}$, from (2).

Hence,

$$8Q \geqq E\left(\frac{3T}{E}\right)^2 - nT = T\left(\frac{9T}{E} - n\right). \tag{7}$$

We now return to the original problem.

Consider first the case where $n \equiv 1(3)$.

If $E > \frac{n^2 - 1}{3}$ in such a graph then, by a theorem of TURÁN [3], it contains at least one complete quadrilateral regardless of the value of T.

If $E = \frac{n^2 - 1}{3}$ then, appealing to TURÁN's theorem again, there is, in the sense of isomorphism, only one such graph having this many edges and no complete quadrilaterals, and this graph contains $\frac{(n + 2)(n - 1)^2}{27}$ triangles. If $E \leqq \frac{n^2 - 1}{3} - 1 = \frac{n^2 - 4}{3}$ then it follows from (7) that if $T > \frac{n(n^2 - 4)}{27}$ then $Q > 0$.

Thus, no matter how many edges the graph has, if $n \equiv 1(3)$, it can have at most $\frac{(n+2)(n-1)^2}{27}$ triangles without containing a complete quadrilateral, and there is essentially only one graph with this many triangles and no complete quadrilaterals.

This completes the proof of part of the theorem stated in (1). The remaining parts may be demonstrated in a similar fashion except that when $n \equiv 0(3)$ it is necessary to consider only two cases. We remark that the same sort of argument may be used to show that

$$3T \geqq \frac{E}{n}(4E - n^2)\,. \tag{8}$$

This has also been proved by NORDHAUS and STEWART, (see [2]).

Substituting this inequality into (7) yields the following lower bound for the number of complete quadrilaterals a graph on n points contains, in terms of the number of its edges:

$$6Q \geqq \frac{E}{n^2}(4E - n^2)(3E - n^2)\,. \tag{9}$$

More generally if S_i denotes the number of complete subgraphs of order i in a graph on n points the type of argument used above yields the following inequality:

$$k(k-2)S_k \geqq S_{k-1}\left[\frac{(k-1)^2 S_{k-1}}{S_{k-2}} - n\right]. \tag{10}$$

(Received November 20, 1961)

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ОБ ОДНОЙ ЗАДАЧЕ TURÁN-А

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Резюме

TURÁN ставил следующий вопрос: Какое число $f(n)$ является наименьшим с таким свойством, что если из n заданных элементов образовать какую либо систему φ из более чем $f(n)$ триплетов, тогда всегда существует

четыре элемента такие, что все четыре триплета образованные из них находятся в φ. Здесь решается следующий, близкий к приведеннему вопрос: Какое число $g(n)$ является найменьшим с таким свойством, что если из n заданных элементов образовать какую либо систему φ из более чем $g(n)$ примеров, тогда всегда существует четыре элемента такие, что каждая пара образованная из них находится в некотором триплете системы φ. Здесь показано, что

$$g(n) = \frac{n^3}{27}, \quad \frac{(n+2)(n-1)^2}{27} \quad \text{или} \quad \frac{(n-2)(n+1)^2}{27}$$

смотря тому, является ли n сравнивым с 0, 1 или 2 соответственно, по модулю 3.