

REMARK ON A THEOREM OF DÉNES

by

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The aim of this note is to give a slightly more explicit form to the correspondence between labelled trees and cycles which was found by Dénes (1).

Let $X = \{x_i\}$ ($1 \leq i \leq n$) be a set of vertices and

$T = \{t_j = (x_{i_j}, x_{i'_j}) = (x_{i_j}, x_{i_j})\}$, ($1 \leq j \leq n - 1$) be a set of edges so that (X, T) is a tree.

Considering T as an abstract alphabet, we denote by G the set of all words in the letters $t \in T$ that contain each t at most once and we define a mapping from G into the symmetric group of permutations S_n on elements $1, 2, \dots, n$, by associating the transposition $\bar{t}_j = (i_j, i'_j)$ with each $t_j = (x_{i_j}, x_{i'_j})$ and the product $\bar{g} = \bar{t}_{j_1} \bar{t}_{j_2} \dots \bar{t}_{j_k}$ with each $g = t_{j_1} t_{j_2} \dots t_{j_k}$. It is trivial that any two \bar{t}_j 's corresponding to disjoint edges commute. Thus, unless (X, T) is a „bush” not all the associated cycles are distinct.

For any permutation $s \in S_n$ and triple (a, b, c) of distinct elements, we define the indicator $\sigma(s; a, b, c)$ with value zero if a, b , and c do not belong to the same cycle of s and with value ± 1 depending upon whether b is or is not between a and c . Formally,

$\sigma(s; a, b, c) = \sigma(s; b, c, a) = -\sigma(s; a, c, b) = 1$ if $s^n a = b$, $s^{n'} a = c$ with $n \leq n'$ and $s^{n''} a \neq c$ for all $n'' < n'$.

Property 1. For any $g = g't \in G$ and triple (a, b, c) if $\sigma(\bar{g}'; a, b, c) \neq 0$ then $\sigma(\bar{g}; a, b, c) = \sigma(\bar{g}'; a, b, c)$.

Proof. The subgraph $(X, T') \subset (X, T)$ corresponding to the factors of \bar{g}' is a disjoint union of trees $T' = (T'_1, T'_2, \dots, T'_k)$ some of which may be reduced to a single vertex.

By DÉNES' Theorem there is a one to one correspondence between the trees T'_j and the cycles which constitute the factors of the permutation \bar{g}' .

Now $T' \cup \{t\} \subset T$ is also a disjoint union of trees. The new edge $t = (x_i, x_j)$ connects two disconnected components of the graph (T', X) . Furthermore, i and j belong to two different cycles of \bar{g}' say, $(i i_2 i_3 \dots i_m)$ and $(j j_2 j_3 \dots j_p)$. Now $\bar{g} = \bar{g}' i$ is obtained by replacing these two cycles by the single cycle $(i j_2 j_3 \dots j_p j i_2 \dots i_m)$. The result follows immediately. As a consequence, we have:

Property 2. If $t = (x_i, x_j)$ and $t' = (x_j, x_k)$ are two distinct factors of the word $g = g_1 t g_2 t' g_3$, then $\sigma(\bar{g}; i, j, k) = 1$.

Proof. Using the construction which has been given for property 1, it follows that i and j are in one cycle and k in another cycle of $\bar{g}_1 t \bar{g}_2$, the particular cycles being $(i l_2 \dots l_m j l_{m+2} \dots l_{m+m})$ and $(k k_2 \dots k_{m'})$, say. Thus

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$\bar{g}_1 \bar{t} \bar{g}_2 \bar{t}'$ contains the cycle $(jk_2 \dots k_{m'}, kl_{m+2} \dots l_{m+m}, il_2 \dots l_m)$. Hence, $\sigma(\bar{g}_1 \bar{t} \bar{g}_2 \bar{t}'; i, j, k) = 1$ and the results follow from property 1.

Let $h_i \in G$ denote a product of the set of all edges t_{j_i} incident to x_i . Let us now consider $W = \{w = (h_1, h_2, \dots, h_n)\}$ the set of all n -tuples of elements h_1, \dots, h_n . Clearly the number of elements w in W is

$$\Delta(X, T) \stackrel{\text{def}}{=} \prod_{1 \leq i \leq n} (\deg x_i)!$$

For any given $w \in W$ if $g^* = \prod_{j=1}^{n-1} t_j$ is a product of maximal degree $n - 1$, and if for each i , the symbol t_j corresponding to edges incident to x_i appear in g^* in the same order as in h_i , then we shall say after LYNDRON [2] that the word g^* is a *minimal infiltration* of $w (g^* \in ((w)))$. Clearly if $\deg x_i = 1$, h_i reduces to a single t and may as well be omitted from w .

For instance if $X = (x_1, x_2, x_3, x_4)$ and $T = \{t_1 = (x_1, x_2), t_2 = (x_2, x_3), t_3 = (x_3, x_4)\}$ we have

$$\begin{aligned} W &= \{(t_1, t_1 t_2, t_2 t_3, t_3), (t_1, t_2 t_1, t_2 t_3, t_3), (t_1, t_2 t_1, t_3 t_2, t_3), (t_1, t_1 t_2, t_3 t_2, t_3)\} = \\ &= \{(t_1 t_2, t_2 t_3), (t_2 t_1, t_2 t_3), (t_1 t_2, t_3 t_2), (t_2 t_4 t_3 t_2)\} \end{aligned}$$

and

$$\begin{aligned} ((t_2 t_1, t_2 t_3)) &= \{t_2 t_1 t_3, t_2 t_3 t_1\} \\ ((t_1 t_2, t_3 t_2)) &= \{t_1 t_3 t_2, t_3 t_1 t_2\} \\ ((t_2 t_1, t_3 t_2)) &= \{t_3 t_2 t_1\} \\ ((t_1 t_2, t_2 t_3)) &= \{t_1 t_2 t_3\}. \end{aligned}$$

We now wish to prove:

Property 3. *If $g \in ((w))$ and $g' \in ((w'))$ then $\bar{g} = \bar{g}'$ i. f. f. $w = w'$.*

Proof. By 2) we know that $w \neq w'$ implies $\bar{g} \neq \bar{g}'$ for every $g \in ((w))$ $g' \in ((w'))$ since there must be at least one triple such that $\sigma(g; i, j, k) = -\sigma(g'; i, j, k)$.

To prove the backward implication let us now assume that $f, f' \in ((w))$ and $f \neq f'$.

According to 3) it may be that $f = f'$ i.e. $f = t_{i_1} t_{i_2} \dots t_{i_{n-1}}$, $f' = t_{i_1} t_{i_2} \dots t_{i_{n-1}}$ but that one can be reduced to the other by a certain number of exchanges of adjacent t 's corresponding to disjoint edges in which case $\bar{f} = \bar{f}'$.

Assume there is no such reduction f^* of f to f' . Thus there exists some minimal n such that the left factor of f^* of degree n is different from the left factor of f' , i.e.

$$f^* = g_1(x_j, x_k) g_2; f' = g_1(x_1, x_m) g'_2(x_j, x_k) g'_3.$$

Now g'_1 contains an edge with either x_j or x_k as end point since otherwise n is not minimal. Suppose this edge is (x_j, x_p) . Then $f^* = g_1(x_j, x_k) g'_2(x_j, x_p) g'_3$ and $f' = g_1(x_1, x_m) g'_2(x_j, x_p) g'_3(x_j, x_k) g'_4$ in which case by property 2, $\sigma(f^*; j, k, p) \neq \sigma(f'; j, k, p)$.

This completes the proof and it follows immediately that the number of distinct cycles associated to the tree (X, T) is simply $\Delta(X, T)$.

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ЗАМЕЧАНИЕ ОБ ОДНОЙ ТЕОРЕМЕ DÉNES-A

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Резюме

Авторы устанавливают соответствие между числом деревьев с n пронумерованными вершинами и числом разложений цикла степени $n - 1$ на произведений транспозиций. Они доказывают что эти два числа равны.