ON GAPS GENERATED BY A RANDOM SPACE FILLING PROCEDURE

by

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§. 1. Formulation of the model

In paper [1] A. RÉNYI dealt with a one-dimensional random space filling procedure.[1]

This procedure consists in placing successive disjoint unit intervals on the interval $(0, x)$, according to the following rules:

a) for $x \leqq 1$ no interval can be placed,

b) for $x > 1$ the left endpoint of the first unit interval is uniformly distributed on the interval $(0, x - 1)$,

c) if k unit intervals $(k = 1, 2, \ldots)$ are already placed, the left endpoint of the $(k + 1)$-st unit interval will be uniformly distributed on the subdomain of the interval $(0, x - 1)$ by which no intersection with the former k intervals can be obtained,

d) the procedure will be finished when there remains no possibility of placing a further unit interval without intersection.

The number of the placed unit intervals is a random variable ν_x. It is proved that

$$\text{(1)} \qquad \lim_{x \to +\infty} \frac{\mathbf{E}(\nu_x)}{x} = C = \int_0^\infty \exp\left\{-2 \int_0^t \frac{1 - e^{-u}}{u} du\right\} dt = 0{,}748 \ldots$$

and

$$\mathbf{D}(\nu_x) = O(\sqrt{x}) \qquad (x \to +\infty)$$

(An equivalent formulation of this model, describing the problem from the aspect of random experimentation is given in this section below).

Generalizations of the problem are treated in papers [2], [3] and [4].

This paper contains further investigations concerning the original model, namely the distribution of the gaps (i.e. those parts of the interval $(0, x)$ which are not covered by unit intervals) is considered.

First of all, an exact formulation of the model is given since the problems to be treated have rather complicated structures.

Let $\{t_1, t_2, \ldots, t_n, \ldots\}_x = T_x$ $(x > 1)$ be a sequence of independent observations of the random variable uniformly distributed on the interval $(0, x - 1)$. The set consisting of the elements T_x for a fixed x will be denoted by

[1] This is a special case of a three-dimensional problem raised by W. SCHMETTERER.

τ_x. Obviously τ_x can be interpreted as a random number generator. For every fixed T_x a set of indices $i_1, i_2, \ldots, i_\nu$ will be defined in the following manner. Let

$$i_1 = 1, \quad i_k = \min_i B_k(i) \qquad (k = 2, 3, \ldots)$$

where

$$B_k(i) = \left\{ i : \bigcup_{j=1}^{k-1} [(t_{i_j}, t_{i_j} + 1) \cap (t_i, t_i + 1)] = 0 \right\}.$$

The stopping rule is given by

$$\nu = \max_k \{k : i_k < +\infty\}.$$

(For the sake of brevity the index x of ν_x is sometimes omitted.)

The numbers $t_{i_1}, t_{i_2}, \ldots, t_{i_\nu}$ denote the left endpoints of disjoint unit intervals and, according to the stopping rule, no more unit interval could be placed without intersection; thus the interval $(0, x)$ is *"filled"*.

Let us denote the ordered set of $\{t_{i_1}, t_{i_2}, \ldots, t_{i_\nu}\}$ by $\{t_1^*, t_2^*, \ldots, t_\nu^*\}$. The numbers

$$I_x^{(1)} = t_1^*, \quad I_x^{(k)} = t_k^* - (t_{k-1}^* + 1) \quad (k = 2, 3, \ldots, \nu), \quad I_x^{(\nu+1)} = x - (t_\nu^* + 1)$$

will be called *"gaps"*, generated by the described random space filling procedure[2].

The probability spaces $\{\tau_x, S_x, \mathbf{P}_x\}$ occuring in problems concerning the gaps can be characterized in the following way. Let be

$$\tau_x(k) = \{T_x : \nu_x = k\} \qquad (k = 0, 1, 2, \ldots).$$

Obviously $\tau_x(m) \cap \tau_x(n) = 0$ $(m \neq n)$ and $\bigcup_k \tau_x(k) = \tau_x$. Furthermore, the equality

$$\sum_{j=0}^{k+1} I_x^{(j)} = x - k \qquad (\tau_x(k) \neq 0, \; T_x \in \tau_x(k)) \tag{2}$$

holds. Let be

$$Z_x(k) =$$
$$= \left\{ z : z = (z_1, z_2, \ldots, z_{k+1}), \; 0 \leqq z_j \leqq 1 \; (j = 1, 2, \ldots, k+1), \; \sum_{j=0}^{k+1} z_j \leqq x - k \right\},$$

i.e. $Z_x(k)$ is the common part of a $(k+1)$-dimensional simplex and of a unit hypercube. (The numbers z_j $(j = 1, 2, \ldots, k+1)$ mean the gaps generated by an element T_x, $T_x \in \tau_x(k)$).

The set $\tau_x(k)$ is transformed by the random space filling procedure onto the domain $Z_x(k)$ in such a way that the inverse images of disjoint subsets of $Z_x(k)$ are disjoint in $\tau_x(k)$. Let $S_x(k)$ denote the σ-algebra of those subsets

[2] I.e. as there is no possibility of misunderstanding, "gap" may denote either an interval or its length.

of $\tau_x(k)$ which are the inverse images of the Borel sets of $Z_x(k)$. Then the σ-algebra

$$S_x = \bigcup_k S_x(k)$$

will be suitable for the problem treated in § 2. (We remark that for the problem treated in § 4, S_x can be given in a simpler way; in this case the Borel sets of $Z_x(k)$ only in respect of z_1 are to be considered).

The probability measure $\mathbf{P}_x$ is determined by the particular problems considered (i.e. by S_x); namely, roughly speaking, each T_x is "equally probable".

§. 2. Limiting distribution of a gap chosen at random

The first problem treated is the determination of the limiting distribution of a gap chosen at random on the filled interval $(0, x)$. More precisely, the distribution function

$$G_x(h) = \mathbf{P}(I_x < h) \qquad (0 < h \leq 1)$$

will be considered (for $x \to +\infty$), where I_x is selected with probability $1/\nu+1$ out of the gaps $I_x^{(1)}, I_x^{(2)}, \ldots, I_x^{(\nu+1)}$ generated by the random sequence T_x.

Definitions and notations. Let $\vartheta_x(h)$ denote the random number of gaps not smaller than h $(0 < h \leq 1)$, occurring on the filled interval $(0, x)$. Let

$$\xi_x(h) = \nu_x + \vartheta_x(h) \tag{3}$$

and

$$\varrho_x(h) = \frac{\xi_x(h) + 1}{\nu_x + 1}. \tag{4}$$

In the following we shall use the notations

$$\mathbf{E}(\nu_x) = M(x),\ \mathbf{D}(\nu_x) = D(x),\ \mathbf{E}(\xi_x(h)) = M_h(x),\ \mathbf{D}(\xi_x(h)) = D_h(x).$$[3]

In dealing with the asymptotic behaviour of the model (for $x \to +\infty$) we shall need three lemmas.

Lemma 1. *The function $M_h(x)$ satisfies the functional equation*

$$M_h(x+1) = \frac{2}{x}\int_0^x M_h(t)\,dt + 1 \qquad (x > 0) \tag{5}$$

and the initial condition

$$M_h(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < h \\ 1 & \text{for} \quad h \leq x \leq 1\,. \end{cases} \tag{6}$$

Lemma 2.

$$\lim_{x \to +\infty} \frac{M_h(x)}{x} = C(h) = 2\int_0^\infty \exp\left\{-ht - 2\int_0^t \frac{1-e^{-u}}{u}\,du\right\}dt \qquad (0 < h \leq 1). \tag{7}$$

[3] The random variables $\xi_x(1)$ and ν_x differ only on a set of measure 0 and therefore their corresponding moments are equal.

Lemma 3.

$$D_h(x) = O(\sqrt{x}) \qquad (0 < h \leqq 1\,;\; x \to +\infty)\,.$$

The first assertion can be easily seen; let us namely consider the filling procedure of the interval $(0, x+1)$ generated by the random sequence T_{x+1} $(x > 0)$. It follows from the model that for $t_1 = t$ the equation

$$(8) \qquad M_h(x+1 \mid t) = M_h(t) + M_h(x-t) + 1$$

holds, where $M_h(x+1 \mid t)$ denotes the conditional expectation of $\xi_{x+1}(h)$; t is uniformly distributed on $(0, x)$ and therefore

$$(9) \qquad M_h(x+1) = \frac{1}{x}\int_0^x M_h(x+1 \mid t)\,dt\,.$$

The equation (5) follows by integrating (8) and considering (9). The initial condition (6) obviously follows from the model.

Lemma 2 is a special case of Theorem 4 in [4] but a shorter proof, similar to that of Lemma 6 can be also given; both of these proofs make use of Lemma 1.

Lemma 3 is a simple consequence of the results obtained by A. RÉNYI (see [1], pp. 121—123).

Theorem 1.

$$(10) \qquad \lim_{x\to+\infty} G_x(h) = G(h) = 2 - C^{-1}\,C(h)$$

for every h $(0 < h \leqq 1)$ *where* $C(h)$ *is defined by* (7) *and* $C = C(1)$ (see (1)).

Proof. The quotient

$$\frac{\vartheta_x(h)}{\nu_x + 1}$$

is the ratio of the number of gaps not smaller than h, relative to the number of all gaps. The equality

$$(11) \qquad \mathbf{P}(I_x \geqq h) = \mathbf{E}\left(\frac{\vartheta_x(h)}{\nu_x + 1}\right)$$

obviously follows from the given definitions. Considering (3) and (4)

$$\frac{\vartheta_x(h)}{\nu_x + 1} = \varrho_x(h) - 1\,,$$

and thus (11) can be written in the form

$$G_x(h) = 2 - \mathbf{E}(\varrho_x(h))\,.$$

The random variables $\varrho_x(h)$ $(0 \leqq x < +\infty)$ are uniformly bounded (namely as $0 \leqq \vartheta_x(h) \leqq \nu_x + 1$, $1 \leqq \varrho_x(h) \leqq 2$); in completing the proof it suffices to show[4] that

$$\lim_{x\to+\infty} \text{st.}\, \varrho_x(h) = C^{-1}\,C(h)$$

[4] See e.g. exercise 17. of Chap. XI. in [5].

as from this

$$\lim_{x \to +\infty} \mathbf{E}(\varrho_x(h)) = C^{-1} C(h)$$

follows[5].

Let A and $A(h)$ denote the events

$$|\nu_x - M(x)| \leqq \lambda_1 D(x)$$

and

$$|\xi_x(h) - M_h(x)| \leqq \lambda_2 D_h(x),$$

respectively, where λ_1 and λ_2 are fixed but arbitrarily chosen positive numbers; $\bar{A}$ and $\bar{A}(h)$ denote the complements of A and $A(h)$, respectively. According to Chebyshev's inequality

$$\mathbf{P}(\bar{A}) < \frac{1}{\lambda_1^2}, \quad \mathbf{P}(\bar{A}(h)) < \frac{1}{\lambda_2^2}$$

hold. Let x_0 be a number such that for $x < x_0$

$$M(x) + 1 > \lambda_1 D(x);$$

(x_0 can be chosen in such a way, as $M(x) = O(x)$, $D(x) = O(\sqrt{x})$). $\varrho_x(h)$ can be written in the form

$$\varrho_x(h) = \frac{\xi_x(h) - M_h(x) + M_h(x) + 1}{\nu_x - M(x) + M(x) + 1},$$

thus, for $x > x_0$ we obtain the estimate

$$1 - \frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} < \mathbf{P}(A \cap A(h)) \leqq$$

(12)

$$\leqq \mathbf{P}\left(\frac{M_h(x)\left(1 + \dfrac{1 - \lambda_2 D_h(x)}{M_h(x)}\right)}{M(x)\left(1 + \dfrac{1 + \lambda_1 D(x)}{M(x)}\right)} < \varrho_x(h) < \frac{M_h(x)\left(1 + \dfrac{1 + \lambda_2 D_h(x)}{M_h(x)}\right)}{M(x)\left(1 + \dfrac{1 - \lambda_1 D(x)}{M(x)}\right)}\right).$$

Considering the asymptotic behaviour of the functions $M(x)$, $M_h(x)$, $D(x)$ and $D_h(x)$ (see Lemma 2 and Lemma 3) (12) can be written in the form

$$\mathbf{P}\left(\frac{M_h(x)}{M(x)}(1 + \varepsilon_1) < \varrho_x(h) < \frac{M_h(x)}{M(x)}(1 + \varepsilon_2)\right) > 1 - \frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}, \tag{13}$$

where $\varepsilon_1 = \varepsilon_1(x; h, \lambda_1, \lambda_2)$, $\varepsilon_2 = \varepsilon_2(x; h, \lambda_1, \lambda_2)$, $|\varepsilon_1| \to 0$ and $|\varepsilon_2| \to 0$ as $x \to +\infty$. It follows from (13) that

$$\mathbf{P}\left(\left|\varrho_x(h) - \frac{M_h(x)}{M(x)}\right| > \frac{M_h(x)}{M(x)} \max(|\varepsilon_1|, |\varepsilon_2|)\right) < \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2};$$

this means (considering (1) and (7)) that $\varrho_x(h)$ converges in probability to $C^{-1}C(h)$ as it was to be proved.

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[5] The symbol lim st. denotes convergence in probability.

Evaluated values of $G(h)$ and $C(h)$ are tabulated in § 5.

Theorem 2.

$$m = \lim_{x\to+\infty} \mathbf{E}(I_x) = C^{-1} - 1 .$$

Proof. It is obvious (see (2)) that for every k ($k = 0, 1, 2, \ldots$), $\tau_x(k) \neq 0$ and for every $T_x \in \tau_x(k)$

$$(k+1)\,\mathbf{E}(I_x \mid T_x) = x - k ,$$

and from this

$$\mathbf{E}(I_x) = \mathbf{E}\left(\frac{x-\nu_x}{\nu_x+1}\right) \tag{14}$$

follows. In the following the proof is similar to that of Theorem 1. It can be easily shown that

$$\lim_{x\to+\infty} \text{st.}\, \frac{x-\nu_x}{\nu_x+1} = C^{-1} - 1 \tag{15}$$

since (see [1])

$$\lim_{x\to+\infty} \text{st.}\, \frac{\nu_x}{x} = C .$$

The random variables $(x-\nu_x)(\nu_x+1)^{-1}$ are bounded for $0 \leq x < +\infty$, namely

$$0 \leq \frac{x-\nu_x}{\nu_x+1} \leq {}^1/_2 ;$$

from this fact and (15), under consideration of (14), the assertion follows.

Another proof can be given by starting from

$$m = \int_0^1 y G'(y)\, dy .$$

(This method is used in the proof of the following theorem.)

Theorem 3.

$$\sigma^2 = \lim_{x\to+\infty} \mathbf{D}^2(I_x) =$$

$$= -2 + C^{-1}\left(6 - 8\int_0^\infty \left(\frac{1-e^{-t}}{t}\right)^2 \exp\left\{-2\int_0^t \frac{1-e^{-u}}{u}\,du\right\} dt\right) - C^{-2} .$$

Proof.

$$\lim_{x\to+\infty} E(I_x^2) = \int_0^1 y^2 G'(y)\, dy = 2C^{-1}\int_0^1\left(\int_0^\infty y^2 t \exp\left\{-yt - 2\int_0^t \frac{1-e^{-u}}{u}\right\} dt\right) dy =$$

$$= 2C^{-1}\int_0^\infty \left(\int_0^1 y^2 e^{-yt}\, dy\right) t \exp\left\{-2\int_0^t \frac{1-e^{-u}}{u}\, du\right\} dt =$$

$$= -1 + 4\,C^{-1} \int\limits_0^\infty \frac{1 - e^{-t} - te^{-t}}{t^2} \exp\left\{-2 \int\limits_0^t \frac{1 - e^{-u}}{u}\, du\right\} dt =$$

$$= -1 + 4\,C^{-1} \int\limits_0^\infty \left(\frac{1 - e^{-t}}{t}\right)' \exp\left\{-2 \int\limits_0^t \frac{1 - e^{-u}}{u}\, du\right\} dt =$$

$$= -1 + 4\,C^{-1} - 8\,C^{-1} \int\limits_0^\infty \left(\frac{1 - e^{-t}}{t}\right)^2 \exp\left\{-2 \int\limits_0^t \frac{1 - e^{-u}}{u}\, du\right\} dt\,.$$

Since

$$\sigma^2 = \lim_{x \to +\infty} \mathbf{E}(I_x^2) - m^2,$$

considering Theorem 2, the assertion follows.

§. 3. Two further problems

In this paragraph two problems are simply solved by applying the results of § 2.

1. What is the probability of the event that a point placed at random on the filled interval $(0, x)$ is placed on a gap smaller than h $(0 < h \leq 1)$?

The answer will be given for $x \to +\infty$.

On the filled interval $(0, x)$ the number of those gaps for which the inequality

$$h \leq I_x^{(k)} < h + dh \qquad (k = 1, 2, \ldots, \nu_x + 1) \tag{16}$$

holds, clearly equals

$$\vartheta_x(h) - \vartheta_x(h + dh) = \xi_x(h) - \xi_x(h + dh)\,.$$

The sum of these gaps is approximately equal to $h(\xi_x(h) - \xi_x(h + \mathrm{d}h))$. Thus the probability of placing a point on such a gap is given by

$$p_x(h)\, dh \approx \mathbf{E}\left(\frac{h}{x}(\xi_x(h) - \xi_x(h + dh))\right) = h\left(\frac{M_h(x)}{x} - \frac{M_{h+dh}(x)}{x}\right). \tag{17}$$

Considering (7) we obtain from (17)

$$p(h)\, dh = \lim_{x \to +\infty} p_x(h)\, dh \approx h\,(C(h) - C(h + dh)) \approx -hC'(h)\, dh \tag{18}$$

and since from (10)

$$C'(h) = -C\,G'(h)\,, \tag{19}$$

thus the required probability equals

$$P(h) = \int\limits_0^h p(y)\, dy = C \int\limits_0^h y\, G'(y)\, dy = C(h\, G(h) - \int\limits_0^h G(y)\, dy)\,.$$

2. What is the probability of the event that a segment of length L $(0 < L < 1)$ placed at random on the filled interval $(0, x)$ intersects none of the unit intervals?

Let us consider the interval $(0, x + L)$; the left endpoint of the segment is uniformly distributed on $(0, x)$. Therefore the probability of the event that such a segment will lie on one of the gaps for which (16) holds, equals approximately

$$\frac{h-L}{x}(M_h(x) - M_{h+dh}(x)) \approx \frac{h-L}{h} p_x(h)\, dh .$$

Thus the required probability for the interval $(0, x + L)$ is equal to

$$\int_L^1 \frac{y-L}{y} p_x(y)\, dy$$

and hence for $x \to +\infty$ (considering (18) and (19)) we obtain

$$P_L = \int_L^1 \frac{y-L}{y} p(y)\, dy = -\int_L^1 (y-L)\, C'(y)\, dy = C \int_L^1 (1 - G(y))\, dy .$$

We remark that

$$P(1) = \lim_{L \to +0} P_L = Cm = 1 - C .$$

§. 4. On the distribution of the first gap

By considering I_x, we have investigated so far only the "average properties" of the gaps. A further interesting problem is that of the properties of $I_x^{(k)}$ for a fixed k; for $k > 1$ the tackling of this question seems to be rather complicated, but for $k = 1$, the limiting distribution of $I_x^{(1)}$ can be determined and compared with that of I_x.

Let us introduce the following notations

$$\mathbf{P}(I_x^{(1)} < h) = G^*(x; h), \quad \mathbf{E}((I_x^{(1)})^j) = M_j^*(x) \qquad (j = 1, 2, \ldots) .$$

Lemma 4. *For every fixed h $(0 < h \leq 1)$, $G^*(x; h)$ satisfies the functional equation*

$$G^*(x+1\,;\, h) = \frac{1}{x} \int_0^x G^*(t\,;\, h)\, dt \qquad (x > 0) \tag{20}$$

and the initial condition

$$G^*(x; h) = \begin{cases} 1 & \text{for } 0 \leq x < h \\ 0 & \text{for } h \leq x \leq 1 . \end{cases} \tag{21}$$

Lemma 5. *The functions $M_j^*(x)$ $(j = 1, 2, \ldots)$ satisfy the functional equations*

$$M_j^*(x+1) = \frac{1}{x} \int_0^x M_j^*(t)\, dt \qquad (x > 0) \tag{22}$$

and the initial conditions

$$M_j^*(x) = x^j \qquad (0 \leqq x \leqq 1). \tag{23}$$

The initial conditions (21) and (23) obviously follow from the model. The equations (20) and (22) can be deduced similarly to (5).

Lemma 6. *If the function $Q(x)$ satisfies the functional equation*

$$Q(x+1) = \frac{1}{x}\int_0^x Q(t)\,dt \qquad (x > 0) \tag{24}$$

and the initial condition

$$Q(x) = q(x) \qquad (0 \leqq x \leqq 1) \tag{25}$$

where $q(x)$ is integrable and

$$0 \leqq q(x) \leqq K \qquad (0 \leqq x \leqq 1), \tag{26}$$

then

$$\lim_{x\to+\infty} Q(x) = \int_0^\infty A(t)\ \exp\left\{-\int_0^t \frac{1-e^{-u}}{u}\,du\right\}dt,$$

where

$$A(t) = \int_0^1 q(x)\,e^{-tx}\,dx \qquad (0 \leqq t < +\infty).$$

Proof. Let us introduce the Laplace-transform of $Q(x)$,

$$\varphi(s) = \int_0^\infty Q(x)\,e^{-sx}\,dx \qquad (\operatorname{Re} s > 0).$$

$\varphi(s)$ exists, since it follows from (24), (25) and (26) that

$$Q(x) \leqq K \qquad (0 \leqq x < +\infty)$$

and thus

$$\varphi(s) \leqq K\int_0^\infty e^{-sx}\,dx = \frac{K}{s}.$$

As

$$\int_0^\infty \left(\int_0^x Q(t)\,dt\right) e^{-sx}\,dx = \frac{\varphi(s)}{s} \tag{27}$$

and

$$\int_0^\infty x\,Q(x+1)\,e^{-sx}\,dx = -\frac{d}{ds}\big(e^s(\varphi(s) - A(s))\big), \tag{28}$$

irom (24), (27) and (28) the ordinary differential equation

$$(29) \qquad \frac{d}{ds}\left(e^s(\varphi(s)-A(s))\right)+\frac{\varphi(s)}{s}=0$$

is obtained. By substituting

$$(30) \qquad \psi(s)=e^s(\varphi(s)-A(s)),$$

(29) can be written in the form

$$(31) \qquad \psi'(s)+\frac{e^{-s}}{s}\psi(s)+\frac{A(s)}{s}=0.$$

It is easy to show that

$$(32) \qquad \lim_{s\to+\infty}\psi(s)=0\,;$$

namely

$$\psi(s)=e^s\int_1^\infty Q(x)\,e^{-sx}\,dx=\int_0^\infty Q(x+1)\,e^{-sx}\,dx\leqq\frac{K}{s}.$$

The solution of the equation (31) under the initial condition (32) is

$$(33) \qquad \psi(s)=\frac{1}{s}\int_s^\infty A(t)\exp\left\{-\int_s^t\frac{1-e^{-u}}{u}\,du\right\}dt.$$

From (33) and (30) the relation

$$(34) \qquad \lim_{s\to+0}(s\,\psi(s))=\lim_{s\to+0}(s\,\varphi(s))=\int_0^\infty A(t)\exp\left\{\int_0^t\frac{1-e^{-u}}{u}\,du\right\}dt$$

ollows, since

$$0\leqq\lim_{s\to+0}(se^s A(s))\leqq\lim_{s\to+0}\left(Kse^s\int_0^1 e^{-sx}\,dx\right)=K\lim_{s\to+0}(e^s-1)=0\,.$$

In completing the proof, we shall make use of the following Tauberian theorem (see [6] Theorem 108 and [1]): If $\alpha(x)$ is a monotonically increasing function $(0<x<+\infty)$, $\beta>0$ and

$$\lim_{s\to+\infty}s^\beta\int_0^\infty e^{-sx}\,d\alpha(x)=c$$

then

$$\lim_{x\to+\infty}\frac{\alpha(x)}{x^\beta}=\frac{c}{\Gamma(\beta+1)}\,.$$

Let us put $\alpha(x)=\int_0^x Q(t)\,dt$ and $\beta=1$; considering that in this case

$$\int_0^\infty e^{-sx}\,d\alpha(x)=\varphi(s)\,,$$

we obtain

$$\lim_{s\to+0}(s\,\varphi(s)) = \lim_{x\to+\infty}\frac{\int_0^x Q(t)\,dt}{x} = \lim_{x\to+\infty} Q(x)\,. \tag{35}$$

Comparing (35) with (34), our assertion follows.

Applying Lemmas 4, 5 and 6, the following theorems are obtained:

Theorem 4.

$$\lim_{x\to+\infty} G^*(x;h) = G^*(h) = \int_0^\infty \frac{1-e^{-ht}}{t}\exp\left\{-\int_0^t \frac{1-e^{-u}}{u}du\right\}dt \qquad (0<h\leq 1)\,.$$

Theorem 5.

$$m^* = \lim_{x\to+\infty} M_1^*(x) = \int_0^\infty \frac{1-e^{-t}-te^{-t}}{t^2}\exp\left\{-\int_0^t \frac{1-e^{-u}}{u}du\right\}dt\,.$$

Theorem 6.

$$m_2^* = \lim_{x\to+\infty} M_2^*(x) = 2\int_0^\infty \frac{1-e^{-t}-te^{-t}-\frac{1}{2}t^2e^{-t}}{t^3}\exp\left\{-\int_0^t \frac{1-e^{-u}}{u}du\right\}dt\,.$$

Thus the asymptotic variance of $I_x^{(1)}$ equals:

$$(\sigma^*)^2 = \lim_{x\to+\infty}\mathbf{D}^2(I_x^{(1)}) = m_2^* - (m^*)^2.$$

Obviously, also other proofs can be given for Theorem 5 and Theorem 6, by making use of Theorem 4 and the relations

$$m^* = \int_0^1 y\,G^{*\prime}(y)\,dy,\; m_2^* = \int_0^1 y^2\,G^{*\prime}(y)\,dy\,.$$

§. 5. Experimental results

The computing of $G(h)$ for a fixed h requires the evaluation of the double integral $C(h)$. This laborious work can be avoided by applying the Monte Carlo method[6], the great advantage of which consists in obtaining estimates of $G(h)$ for all $h\,(0<h\leq 1)$ simultaneously. Of course, this method yields only approximate values with a random fluctuation.

The stochastic model was given (see § 1). We have performed ten experiments on the interval (0, 100).[7] The results are tabulated in Table 1; the accuracy is about 10^{-2}. The average value of $10^{-2}\nu_{100}$ was 0.746.

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[6] See e.g. [7].

[7] We made use of the table of random numbers of [8].

Table 1.

h	$G(h)$	$C(h)$	h	$G(h)$	$C(h)$
0.05	0.166	1.370	0.55	0.765	0.923
0.10	0.269	1.293	0.60	0.791	0.903
0.15	0.346	1.236	0.65	0.813	0.887
0.20	0.425	1.177	0.70	0.837	0.869
0.25	0.493	1.126	0.75	0.865	0.848
0.30	0.555	1.079	0.80	0.898	0.823
0.35	0.604	1.043	0.85	0.919	0.808
0.40	0.642	1.014	0.90	0.946	0.787
0.45	0.688	0.980	0.95	0.976	0.765
0.50	0.724	0.953	1.00	1.000	0.747

By numerical integration we have obtained from the above values the following estimates:

$$\int_0^1 G(y)\,dy \approx 0{,}662, \qquad \int_0^1 yG(y)\,dy \approx 0{,}402.$$

Thus we have the estimates:

$$m = 1 - \int_0^1 G(y)\,dy \approx 0{,}338,$$

$$\sigma^2 = 2\int_0^1 (1-y)\,G(y)\,dy - \Big(\int_0^1 G(y)\,dy\Big)^2 \approx 0{,}082,$$

and, considering Theorem 2,

$$C = \frac{1}{1+m} \approx 0{,}747. \tag{36}$$

The estimate (36) is in good agreement with (1). This fact shows the relatively good accuracy of the experimental method. It is interesting that σ^2 is near to the variance (1/12) of the random variable uniformly distributed on the interval (0,1).

To obtain good estimates for the values of $G^*(h)$ by the Monte Carlo method, requires by far more experiments. This work was not done.

Finally, in Table 2 we have tabulated the estimates obtained for the values of P_L.

Table 2.

L	P_L	L	P_L	L	P_L
0.1	0.211	0.4	0.122	0.7	0.060
0.2	0.176	0.5	0.100	0.8	0.039
0.3	0.146	0.6	0.080	0.9	0.019

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REFERENCES

[1] RÉNYI A.: „Egy egydimenziós véletlen térkitöltési problémáról." *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **3** (1958) 109—127.
[2] PALÁSTI, I.: „On some random space filling problems." *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **5** (1960) A. 353—360.
[3] BÁNKÖVI, G.: „Evaluation of integrals by Monte Carlo methods based on the one-dimensional random space filling." *A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei* **5** (1960) A. 339—352.
[4] BÁNKÖVI G.—DOBÓ A.: „Egydimenziós véletlen térkitöltés változó hosszúságú szakaszokkal". *A Magyar Tudományos Akadémia III. (Matematikai és Fizikai) Osztályának Közleményei* **11** (1961) 399—415.
[5] RÉNYI A.: *Valószínűségszámítás.* Tankönyvkiadó, Budapest, 1954.
[6] HARDY, G. H.: *Divergent series.* Oxford University Press, 1949.
[7] MEYER, H. A. (ed.): *Symposium on Monte Carlo methods.* Wiley, New York, 1956.
[8] HALD, A.: *Statistical Tables and Formulas.* Wiley, New York, 1952.

О ПРОМЕЖУТКАХ, СОЗДАННЫХ ОДНОЙ ПРОЦЕДУРОЙ СЛУЧАЙНОГО ЗАПОЛНЕНИЯ ПРОСТРАНСТВА

G. BÁNKÖVI

Резюме

В работе [1] A. RÉNYI исследовал случайное заполнение отрезка $(0, x)$ единичными отрезками. Он определил математическое ожидание асимптотического числа (при $x \to +\infty$) расположенных отрезков.

В этой работе исследуются промежутки между отрезками. В § 1 дается точная формулировка модели. В § 2 определяется предельное распределение промежутка, выбранного из всех промежутков случайным образом. В § 3 решаются две задачи относительно заполненного отрезка $(0, x)$. В § 4 определяется предельное распределение первого промежутка. В § 5 сообщаются результаты опытов проведенных методом Монте-Карло.