A CENTRAL LIMIT THEOREM FOR THE SUM OF A RANDOM NUMBER OF INDEPENDENT RANDOM VARIABLES

by

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§. 1. Introduction

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ denote throughout the present paper a sequence of independent and identically distributed random variables with mean value 0 and variance 1. Let us put

$$\zeta_n = \xi_1 + \xi_2 + \cdots + \xi_n \tag{1}$$

and

$$\eta_n = \frac{\zeta_n}{\sqrt{n}}. \tag{2}$$

Then by the simplest case of the central limit theorem

$$\lim_{n\to+\infty} \mathbf{P}(\eta_n < x) = \Phi(x) \qquad (-\infty < x < +\infty). \tag{3}$$

Here and in what follows $\mathbf{P}(\,.\,)$ denotes the probability of the event in the brackets and $\Phi(x)$ the standard form of the normal distribution function, i.e.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy. \tag{4}$$

In the present paper we shall investigate the limiting distribution of the random variables η_{ν_n} for $n \to +\infty$, where ν_n $(n = 1, 2, \ldots)$ is a sequence of positive integer-valued random variables. We mention that in this paper nothing is supposed about the dependence of ν_n on the random variables ξ_k.

The first results of this kind have been obtained by F. J. ANSCOMBE [1]. We mention here a special case of a more general result of the mentioned author.

Theorem 1. *If ν_n is a sequence of positive integer-valued random variables such that ν_n/n converges for $n \to +\infty$ in probability to a constant $c > 0$, then*

$$\lim_{n\to+\infty} \mathbf{P}(\eta_{\nu_n} < x) = \Phi(x) \tag{5}$$

holds.

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Recently A. RÉNYI [2] generalized Theorem 1. He investigated the case when $\frac{\nu_n}{n}$ tends in probability to a positive random variable λ having a discrete distribution. He proved the following

Theorem 2. *If ν_n $(n = 1, 2, \ldots)$ is a sequence of positive integer-valued random variables such that ν_n/n converges in probability to a positive random variable λ having a discrete distribution, then* (5) *holds.*

In paper [6] we set ourself as an aim to investigate in a following paper the case when ν_n/n converges in probability to a positive variable λ having arbitrary distribution.

According to this program, the aim of the present paper is to generalize RÉNYI's result by omitting the restriction that the positive random variable λ has a distribution of the discrete type.

We prove the following

Theorem 3.[2] *Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of independent and identically distributed random variables with mean value* 0 *and variance* 1.

If ν_n $(n = 1, 2, \ldots)$ is a sequence of positive integer-valued random variables such that ν_n/n converges in probability to a positive random variable λ, then (5) *holds.*

§ 2. Some theorems and an inequality of Kolmogorov type

We denote by $A \circ B$ the symmetric difference $(A - B) + (B - A)$ of the random events A and B in the following.

Lemma 1. *Let $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ be a sequence of random variables and suppose that τ_n converges in probability to a random variable τ. Let further a_1 and a_2 $(a_1 < a_2)$ be continuity points of the distribution function $F(x)$ of the random variable τ. Let A denote the event $\{a_1 \leqq \tau < a_2\}$ and A_n the event $\{a_1 \leqq \tau_n < a_2\}$. Then $\mathbf{P}(A_n \circ A) \to 0$ if $n \to +\infty$.*

Proof. Given any $\varepsilon > 0$ we can choose (because of the continuity of $F(x)$ at the points a_1 and a_2) a positive number δ such that $|F(x) - F(a_i)| < \varepsilon$ if $|x - a_i| \leqq \delta$ $(i = 1, 2)$.

We have the following sequence of equalities and inequalities

$$\begin{aligned}\mathbf{P}(A_n \bar{A}) &= \mathbf{P}(A_n \bar{A}, \tau < a_1 - \delta) + \mathbf{P}(A_n \bar{A}, \tau \geqq a_2 + \delta) + \\ &+ \mathbf{P}(A_n \bar{A}, a_1 - \delta \leqq \tau < a_1) + \mathbf{P}(A_n \bar{A}, a_2 \leqq \tau \leqq a_2 + \delta) \leqq \\ &\leqq \mathbf{P}(|\tau_n - \tau| > \delta) + F(a_1) - F(a_1 - \delta) + F(a_2 + \delta) - F(a_2)\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}(\bar{A}_n A) &= \mathbf{P}(\bar{A}_n A, a_1 + \delta \leqq \tau < a_2 - \delta) + \mathbf{P}(\bar{A}_n A, a_1 \leqq \tau < a_1 + \delta) + \\ &+ \mathbf{P}(\bar{A}_n A, a_2 - \delta \leqq \tau < a_2) \leqq \mathbf{P}(|\tau_n - \tau| > \delta) + \\ &+ F(a_1 + \delta) - F(a_1) + F(a_2) - F(a_2 - \delta)\,.\end{aligned}$$

Taking into account that τ_n converges in probability to τ we can choose the positive integer n_0 such that if $n \geqq n_0$ we have $\mathbf{P}(|\tau_n - \tau| > \delta) < \varepsilon$.

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[2] Professor RÉNYI kindly informed me that in the paper "On the strong law of large numbers for a class of stochastic processes" written by J. R. BLUM, D. L. HANSON, and L. H. KOOPMANS the same theorem is proved independently of me. This paper is to be appear in "Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete". (Springer-verlag. Berlin — Göttingen — Heidelberg. 1962).

Thus we have $\mathbf{P}(A_n\bar{A}) < 3\varepsilon$ and $\mathbf{P}(\bar{A}_n A) < 3\varepsilon$ and so $\mathbf{P}(A_n \circ A) < 6\varepsilon$. This gives our assertion.

We shall use this Lemma 1 in the following form:

Consequence. If $[a_1, b_1), [a_2, b_2), \ldots, [a_k, b_k)$ are disjoint intervals $(k = 1, 2, \ldots)$, a_i, b_i $(i = 1, 2, \ldots k)$ are continuity points of the distribution function of τ and $A^{(i)}$ denotes the event $\{a_i \leqq \tau < b_i\}$ and $A_n^{(i)}$ the event $\{a_i \leqq \tau_n < b_i\}$ $(i = 1, 2, \ldots, k)$ then for any fixed positive integer k

$$\sum_{i=1}^{k} \mathbf{P}(A_n^{(i)} \circ A^{(i)}) \to 0, \text{ if } n \to +\infty.$$

The following Lemma is almost trivial.

Lemma 2. *If $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ is a sequence of random variables which converges in probability to a random variable τ and a is a continuity point of the distribution function τ such that $\mathbf{P}(\tau < a) < \varepsilon$ (resp. $1 - \mathbf{P}(\tau < a) < \varepsilon$) then there exists a positive integer n_0 such that for $n \geqq n_0$*

$$\mathbf{P}(\tau_n < a) < 2\varepsilon. \quad (\textit{resp.}\ 1 - \mathbf{P}(\tau_n < a) < 2\varepsilon).$$

Proof. The assertion follows at once from the fact that a is a continuity point of the distribution function of τ and from the following equalities and inequalities:

$$\mathbf{P}(\tau_n < a) = \mathbf{P}(\tau_n < a, |\tau_n - \tau| \leqq \delta) + \mathbf{P}(\tau_n < a, |\tau_n - \tau| > \delta) \leqq$$
$$\leqq \mathbf{P}(\tau - \delta < a) + \varepsilon, \text{ if } n \geqq n_0.$$

The case when $1 - \mathbf{P}(\tau < a) < \varepsilon$, can be treated similarly.

We formulate a theorem of A. Rényi [4].

Theorem 4. *If τ_n is a sequence of independent random variables such that putting*

$$\sigma_n = \frac{1}{B_n} \sum_{k=1}^{n} \tau_k, \quad \textit{where} \quad B_n \to +\infty$$

the distribution of the random variable σ_n tends to a limiting distribution, then the conditional distribution of σ_n under any condition having positive probability, tends to the same limiting distribution.

Theorem 5. *Let $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ be a sequence of random variables such that for $n = 1, 2, \ldots$ $\mathbf{M}(\tau_n)$ exists*[3] *and*

$$\mathbf{M}(\tau_n) \to M \qquad (|M| < +\infty)$$

as $n \to +\infty$.

Let us suppose that

a)

$$\lim_{N \to +\infty} \int_{|\tau_n| > N} |\tau_n| \, d\mathbf{P} = 0$$

[3] Here and in what follows $\mathbf{M}(\cdot)$ and $\mathbf{M}(\cdot|A)$ denote the mathematical expectation and the conditional mathematical expectation resp. of the random variable in the brackets.

uniformly in n.

b)
$$\lim_{n \to +\infty} |\mathbf{P}(\tau_n < x | A) - \mathbf{P}(\tau_n < x)| = 0$$

where A *is an event such that* $1 > \mathbf{P}(A) > 0$.

Then
$$\lim_{n \to +\infty} \mathbf{M}(\tau_n | A) = M.$$

Proof. We have
$$\mathbf{M}(\tau_n | A) = \frac{1}{\mathbf{P}(A)} \left\{ \int\limits_{(A \cap \{|\tau_n| \leqq N\})} \tau_n \, d\mathbf{P} + \int\limits_{(A \cap \{|\tau_n| > N\})} \tau_n \, d\mathbf{P} \right\}$$

where N is an arbitrary positive number.

Condition a) ensures that for any $\varepsilon > 0$ we can choose a number $N = N(\varepsilon, A)$ such that

$$\left| \frac{1}{\mathbf{P}(A)} \int\limits_{(A \cap \{|\tau_n| > N(\varepsilon)\})} \tau_n \, d\mathbf{P} \right| \leqq \frac{1}{\mathbf{P}(A)} \int\limits_{(|\tau_n| > N(\varepsilon))} |\tau_n| \, d\mathbf{P} < \varepsilon \tag{6}$$

holds for $n = 1, 2, \ldots$.

Thus we have

$$\left| \mathbf{M}(\tau_n | A) - \frac{1}{\mathbf{P}(A)} \int\limits_{(A \cap (|\tau_n| \leqq N(\varepsilon)))} \tau_n \, d\mathbf{P} \right| < \varepsilon. \quad (n = 1, 2, \ldots). \tag{7}$$

It is easy to see that for $n = 1, 2, \ldots$

$$\frac{1}{\mathbf{P}(A)} \int\limits_{\{A \cap (|\tau_n| \leqq N(\varepsilon))\}} \tau_n \, d\mathbf{P} \geqq \sum_{|kh_0| \leqq N(\varepsilon)} kh_0 \, \mathbf{P}(kh_0 \leqq \tau_n < (k+1) h_0 | A)$$

where h_0 is a fixed positive number and $h_0 < \varepsilon$.

Let
$$\varepsilon' = \frac{\varepsilon}{N(\varepsilon) \left[\dfrac{N(\varepsilon)}{h_0} \right]},$$

Then it follows from condition b) that we can choose an index $n_0 = n_0(\varepsilon') = n_0(\varepsilon)$ such that for $n \geqq n_0$ the following inequality holds:

$$|\mathbf{P}(kh_0 \leqq \tau_n < (k+1) h_0 | A) - \mathbf{P}(kh_0 \leqq \tau_n < (k+1) h_0)| < \varepsilon'$$

for any integer k such that $|k h_0| \leqq N(\varepsilon)$.

Using this inequality we obtain for $n \geqq n_0$

$$\frac{1}{\mathbf{P}(A)} \int\limits_{\{A \cap (|\tau_n| \leqq N(\varepsilon))\}} \tau_n \, d\mathbf{P} \geqq \sum_{|kh_0| \leqq N(\varepsilon)} kh_0 \, \mathbf{P}(kh_0 \leqq \tau_n < (k+1) h_0) - \varepsilon.$$

Thus from (7) we see that for $n \geqq n_0$

(8) $$\mathbf{M}(\tau_n|A) \geqq \sum_{|kh_0| \leqq N(\varepsilon)} kh_0 \mathbf{P}(kh_0 \leqq \tau_n < (k+1) h_0) - 2\varepsilon .$$

On the other hand it follows from (6) that

$$|\mathbf{M}(\tau_n) - \int_{(|\tau_n| \leqq N(\varepsilon))} \tau_n \, d\mathbf{P}| < \varepsilon \qquad (n = 1, 2, \ldots)$$

and we have for $n = 1, 2, \ldots$

$$\Big| \int_{(|\tau_n| \leqq N(\varepsilon))} \tau_n \, d\mathbf{P} - \sum_{|kh_0| \leqq N(\varepsilon)} kh_0 \mathbf{P}(kh_0 \leqq \tau_n < (k+1) h_0) \Big| \leqq h_0 < \varepsilon .$$

Confering these last two inequalities with (8) we obtain for $n \geqq n_0$

(9) $$\mathbf{M}(\tau_n|A) \geqq \mathbf{M}(\tau_n) - 4\varepsilon .$$

It follows from (9) that

(10) $$\liminf_{n \to +\infty} \mathbf{M}(\tau_n \,|\, A) \geqq M .$$

Now condition b) ensures that

$$\lim_{n \to +\infty} | \mathbf{P}(\tau_n < x \,|\, \bar{A}) - \mathbf{P}(\tau_n < x) | = 0 .$$

Similarly, as in the preceding argumentation, one shows that

(11) $$\liminf_{n \to +\infty} \mathbf{M}(\tau_n \,|\, \bar{A}) \geqq M .$$

Now from (11) and from the equality

$$\mathbf{M}(\tau_n) = \mathbf{M}(\tau_n \,|\, A) \mathbf{P}(A) + \mathbf{M}(\tau_n \,|\, \bar{A}) \mathbf{P}(\bar{A})$$

we deduce that

(12) $$\limsup_{n \to +\infty} \mathbf{M}(\tau_n \,|\, A) \leqq M .$$

(10) and (12) together give our assertion.

Theorem 6. *Let $\vartheta_1, \vartheta_2, \ldots$ be a sequence of independent random variables with mean value* 0 *and variances $D_1, D_2, \ldots$. Put*

$$\sigma_n = \frac{\vartheta_1 + \vartheta_2 + \ldots + \vartheta_n}{B_n}$$

(where $B_n^2 = D_1^2 + D_2^2 + \ldots + D_n^2$ and $B_n \to +\infty$). Let us suppose that the distribution function $F_n(x)$ of the random variable σ_n tends to a non-degenerate limiting distribution function $F(x)$. Let us suppose further that

$$\mathbf{M}(\sigma_n^2) = \int_{-\infty}^{+\infty} x^2 \, dF(x) .$$

Then under any condition A, having positive probability, we have

$$\mathbf{M}(\sigma_n^2 \,|\, A) \to 1 \qquad (as\ n \to +\infty) .$$

Proof. We prove that under conditions of our theorem

$$\lim_{N\to+\infty} \int_{|x|\geqq N} x^2\, dF_n(x) = 0$$

holds uniformly in n.

For this purpose let us choose the positive number $N(\varepsilon)$ according to the given positive number ε such that for $N \geqq N(\varepsilon)$

$$\int_{|x|\geqq N} x^2\, dF(x) < \frac{\varepsilon}{4}$$

be satisfied. Then we have for these N

$$\left| 1 - \int_{|x|<N} x^2\, dF(x) \right| < \frac{\varepsilon}{4}.$$

On the other hand, since $F_n(x)$ converges to $F(x)$ at all continuity points of $F(x)$, we have for $n \geqq n_0(\varepsilon)$

$$\left| \int_{|x|\leqq N} x^2\, d\, F_n(x) - \int_{|x|\leqq N} x^2\, dF(x) \right| < \frac{\varepsilon}{4}.$$

Thus for $n \geqq n_0(\varepsilon)$ and for $N \geqq N(\varepsilon)$ the following inequality holds

$$\left| 1 - \int_{|x|\leqq N} x^2\, dF_n(x) \right| < \frac{\varepsilon}{2}. \tag{13}$$

Further from the condition

$$\int_{-\infty}^{+\infty} x^2\, dF_n(x) = \int_{-\infty}^{+\infty} x^2\, dF(x)$$

we deduce that

$$\left| \int_{-\infty}^{+\infty} x^2\, dF_n(x) - \int_{-\infty}^{+\infty} x^2\, dF(x) \right| < \frac{\varepsilon}{2}, \tag{14}$$

if $n \geqq n_1(\varepsilon)$. Let $n(\varepsilon) = \max\left(n_0(\varepsilon), n_1(\varepsilon)\right)$. Confering (13) with (14) we conclude

$$0 \leqq \int_{|x|\geqq N} x^2\, dF_n(x) < \varepsilon,$$

if $n \geqq n(\varepsilon)$ and $N \geqq N(\varepsilon)$. This essentially means the asserted uniform integrability.

We are now in the position to prove the assertion of our theorem. According to the preceding note we can choose for any n and for any positive ε a positive number $N(\varepsilon)$ such that

$$0 \leqq \int_{|x| \geqq N(\varepsilon)} x^2 \, dF_n(x) \leqq \varepsilon \qquad (n = 1, 2, \ldots)$$

and

$$0 \leqq \int_{|x| \geqq N(\varepsilon)} x^2 \, dF(x) \leqq \varepsilon .$$

Using this inequality we obtain for any fixed condition A, having positive probability,

(15)

$$0 \leqq \int_{|x| \geqq N(\varepsilon)} x^2 \, dF_n(x \mid A) \leqq \frac{1}{\mathbf{P}(A)} \int_{|x| \geqq N(\varepsilon)} x^2 \, dF_n(x) \leqq \frac{\varepsilon}{\mathbf{P}(A)}, \qquad (n = 1, 2, \ldots).$$

Thus it is sufficient to deal with the convergence of the integrals

$$\int_{|x| \leqq N(\varepsilon)} x^2 \, d\, F_n(x \mid A) .$$

Now according to the assertion of Theorem 4 we have

$$F_n(x \mid A) \to F(x)$$

at all continuity points of $F(x)$.

This means that

$$\int_{|x| \leqq N(\varepsilon)} x^2 \, d\, F_n(x \mid A) \to \int_{|x| \leqq N(\varepsilon)} x^2 \, dF(x) . \tag{16}$$

Taking into account the choosing of $N(\varepsilon)$ and confering (15) with (16) we obtain the assertion of our Theorem.

Theorem 7. *Let $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ be a sequence of independent random variables with mean values $\mathbf{M}(\tau_i) = M_i$ and variances $\mathbf{D}(\tau_i) = D_i$. Let further*

$$\sigma_n = \frac{\tau_1 + \ldots + \tau_n - (M_1 + \ldots + M_n)}{S_n}$$

where

$$S_n = \sqrt{D_1^2 + D_2^2 + \ldots + D_n^2} .$$

Let us suppose that the distribution function $F_n(x)$ of random variable σ_n converges to the non-degenerate distribution function $F(x)$ with variance 1. Let further C be an arbitrary random event having positive probability. Then there exists an integer $n_0 = n_0(C)$ such that for $n \geqq n_0$

$$\mathbf{P}\left(\max_{1 \leqq k \leqq n} \left| \sum_{i=1}^{k} (\tau_i - M_i) \right| \geqq \lambda S_n, C \right) \leqq \frac{3 \sqrt{\mathbf{P}(C)}}{\lambda^2}$$

where λ is an arbitrary positive number.

Proof. Let $\vartheta_k = \sum_{i=1}^{k} (\tau_i - M_i)$ and let γ denote the indicator of the event C, i.e.

$$\gamma = \begin{cases} 1, & \text{if } C \text{ occurs} \\ 0, & \text{otherwise.} \end{cases}$$

Let further A_k $(k = 2, \ldots n)$ denote the following event:

$$|\vartheta_i| \leqq \lambda S_n, \qquad (i = 1, 2, \ldots, k-1) \text{ and } |\vartheta_k| \geqq \lambda S_n,$$

and A_1 the following event: $|\vartheta_1| \geqq \lambda S_n$,
and let α_k be the indicator of the event A_k. Then

$$\alpha_i \alpha_j = 0, \quad \text{if} \quad i \neq j, \quad 0 \leqq \sum_{k=1}^{n} \alpha_k \leqq 1 \quad \text{and}$$

$$\mathbf{M}\left[\left(\sum_{k=1}^{n} \alpha_k\right)\gamma\right] = \mathbf{P}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (\tau_i - M_i) \right| \geqq \lambda S_n, C\right).$$

Let us investigate the mean value of $\gamma \vartheta_n^2$. It is easy to see that

$$\mathbf{M}(\gamma \vartheta_n^2) \geqq \sum_{k=1}^{n} \mathbf{M}(\gamma \alpha_k \vartheta_k^2) + 2 \sum_{k=1}^{n-1} \mathbf{M}(\gamma \alpha_k \vartheta_k (\vartheta_n - \vartheta_k)).$$

The second sum of the right-hand side can be written as follows

$$\mathbf{M}\left(\gamma \sum_{k=1}^{n-1} \alpha_k \vartheta_k (\vartheta_n - \vartheta_k)\right) = \mathbf{M}\left(\gamma \sum_{j=2}^{n} \beta_j\right),$$

where

$$\beta_j = (\tau_j - M_j) \sum_{k=1}^{j-1} \alpha_k \vartheta_k. \qquad (j = 2, 3, \ldots, n).$$

The system $\{\beta_j\}$ is suborthonormal $(j = 2, 3, \ldots, n)$.

For, if $j \neq i$ and $j < i$, then $(\tau_i - M_i)$ is independent of all the random variables playing role in the product $\beta_j \beta_i$, and thus we have $\mathbf{M}(\beta_j \beta_i) = 0$.

On the other hand

$$\mathbf{M}(\beta_j^2) = \mathbf{M}((\tau_j - M_j)^2)\, \mathbf{M}\left(\left(\sum_{k=1}^{j-1} \alpha_k \vartheta_k\right)^2\right) \leqq D_j^2 S_n^2 < +\infty.$$

Thus applying the inequality of Cauchy and Bessel resp. we obtain

$$\left| \mathbf{M}\left(\gamma \sum_{j=2}^{n} \beta_j\right) \right| = \left| \sum_{j=2}^{n} \mathbf{M}\left(\gamma \frac{\beta_j}{\sqrt{\mathbf{M}(\beta_j^2)}} \sqrt{\mathbf{M}(\beta_j^2)}\right) \right| \leqq$$

$$\leqq \sqrt{\sum_{j=2}^{n} \mathbf{M}^2\left(\gamma \frac{\beta_j}{\sqrt{\mathbf{M}(\beta_j^2)}}\right) \sum_{j=2}^{n} \mathbf{M}(\beta_j^2)} \leqq \sqrt{\mathbf{M}(\gamma^2)} \sqrt{S_n^2 \sum_{j=1}^{n} D_j^2} \leqq \sqrt{\mathbf{P}(C)}\, S_n^2.$$

Taking into account this inequality we can write

$$\mathbf{M}(\gamma \vartheta_n^2) \geqq \sum_{k=1}^{n} \mathbf{M}(\gamma \alpha_k) \lambda^2 S_n^2 - 2 \sqrt{\mathbf{P}(C)}\, S_n^2 . \tag{17}$$

On the other hand according to Theorem 6 we have

$$\lim_{n\to+\infty} \mathbf{M}(\gamma \sigma_n^2) = \lim_{n\to+\infty} \mathbf{M}(\sigma_n^2 \mid \gamma = 1)\, \mathbf{P}(C) = \mathbf{P}(C).$$

Thus there is an index $n_0 = n_0(C)$ such that if $n \geqq n_0$

$$\mathbf{M}(\gamma \sigma_n^2) \leqq \sqrt{\mathbf{P}(C)}$$

or

$$\mathbf{M}(\gamma \vartheta_n^2) \leqq S_n^2 \sqrt{\mathbf{P}(C)} . \tag{18}$$

Confering (17) with (18) we get the assertion.

Remark. Theorem 7 is in some sense an extension of the celebrated Kolmogorov inequality. Its deficiency however is that we postulate the convergence of the distribution functions $\mathbf{P}(\sigma_n < x)$ to a non-degenerate distribution function $F(x)$. This requirement is eliminated in a theorem of A. Rényi [5] but on the other hand the existence of the fourth moment of random variables τ_n is postulated in his theorem. We give here a generalized form of the theorem of A. Rényi, where only the existence of the $2 + p$-adic (p is an arbitrary positive number) moment of the random variables τ_n is required.

Theorem 8. *Let $\tau_1, \tau_2, \ldots, \tau_n$ be independent random variables and suppose that their moment of order $2 + p$ (p is an arbitrary positive number) exists. Let further C be any event of positive probability. If M_i, S_n, λ denote the same quantities as in the preceding theorem then the following inequality holds for any $n = 1, 2, \ldots$*

$$\mathbf{P}\left(\max_{1\leqq k\leqq n} \left| \sum_{i=1}^{k} (\tau_i - M_i) \right| \geqq \lambda S_n , C \right) \leqq$$

$$\leqq \frac{\mathbf{P}(C)^{\frac{p}{2+p}}}{\lambda^2} \left[\frac{B_p \sum_{j=1}^{n} \left(|\tau_j - M_j|^{2+p} \right)^{\frac{2}{2+p}}}{S_n^2} + 2 \right]$$

where B_p is a positive constant depending only on the parameter p.

The proof of this theorem can be performed in somewhat similar way as that of the theorem of A. Rényi.

§ 3. Proof of theorem 3

Let us consider the distribution function of the positive random variable λ. Given any $\varepsilon > 0$ we can choose a sufficiently small $a > 0$ and a sufficiently large $b > 0$ such that a and b are continuity points of the distribution function of λ and

$$\mathbf{P}(a \leqq \lambda < b) > 1 - \varepsilon .$$

Then taking into account that ν_n/n converges in probability to λ, there exists by virtue of Lemma 2 a positive integer n_0 such that for $n \geqq n_0$

$$\mathbf{P}\left(a \leqq \frac{\nu_n}{n} < b\right) > 1 - 2\,\varepsilon\,.$$

Denoting by A_n the event $a \leqq \frac{\nu_n}{n} < b$ we can write

$$\mathbf{P}(\eta_{\nu_n} < x) = \mathbf{P}(\eta_{\nu_n} < x,\, A_n) + \mathbf{P}(\eta_{\nu_n} < x,\, \bar{A}_n)\,.$$

The second member of the right-hand side is smaller than 2ε, and thus we have only to deal with the first member. Let us divide the interval $[a, b)$ into k subintervals by the points $a_1 < a_2 < \ldots < a_{k-1}$ $(a < a_1,\, a_{k-1} < b)$ and suppose that they are continuity points of the distribution function of λ. We introduce the following notations:

$$a_0 = a,\; a_k = b,\; \text{and}\; A_n^{(i)} = \left\{a_{i-1} \leqq \frac{\nu_n}{n} < a_i\right\} \quad (i = 1, 2, \ldots, k)\,.$$

Then $\sum_{i=1}^{k} A_n^{(i)} = A_n$ and $A_n^{(i)} \cdot A_n^{(j)} = \bigcirc$, if $i \neq j$.

We can write

$$\mathbf{P}(\eta_{\nu_n} < x, A_n) = \sum_{i=1}^{k} \mathbf{P}(\eta_{\nu_n} < x, A_n^{(i)}). \tag{19}$$

We have clearly

$$\eta_{\nu_n} = \eta_{[na_{i-1}]} + \sqrt{\frac{[na_{i-1}]}{\nu_n}}\left(\frac{\zeta_{\nu_n} - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}}\right) + \eta_{[na_{i-1}]}\left(\sqrt{\frac{[na_{i-1}]}{\nu_n}} - 1\right). \tag{20}$$

$$(i = 1, 2, \ldots, k)\,.$$

Let us denote by $C(n, k, i, \varrho)$ the event that

$$\left|\sqrt{\frac{[na_{i-1}]}{\nu_n}}\left(\frac{\zeta_{\nu_n} - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}}\right) + \eta_{[na_{i-1}]}\left(\sqrt{\frac{[na_{i-1}]}{\nu_n}} - 1\right)\right| < 2\,\varrho \quad (i = 1, 2, \ldots, k).$$

Let us choose the positive number ϱ such that the following inequality

$$|\Phi(x) - \Phi(x \pm 2\,\varrho)| < \varepsilon$$

be satisfied.

Let further the positive integer k be choosen such that

$$\frac{12\,(b-a)}{\varrho^2 \sqrt{k}} < \frac{\varepsilon}{2}$$

be satisfied.

On the basis of (19) we can write

$$\begin{aligned}\mathbf{P}(\eta_{\nu_n} < x, A_n) &= \sum_{i=1}^{k} \mathbf{P}(\eta_{\nu_n} < x, A_n^{(i)}, C(n, k, i, \varrho)) + \\ &+ \sum_{i=1}^{k} \mathbf{P}(\eta_{\nu_n} < x, A_n^{(i)}, \overline{C(n, k, i, \varrho)})\,.\end{aligned} \tag{21}$$

Applying now the inequalities

$$\mathbf{P}(ABC) \geqq \mathbf{P}(AB) + \mathbf{P}(AC) - \mathbf{P}(A) = \mathbf{P}(AB) - \mathbf{P}(A\overline{C}), \quad \text{and} \quad \mathbf{P}(AB) \leqq \mathbf{P}(A)$$

and taking into account (20) we have

$$\begin{aligned}\sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x - 2\varrho, A_n^{(i)}) - \sum_{i=1}^{k} \mathbf{P}\big(A_n^{(i)}, \overline{C(n,k,i,\varrho)}\big) \leqq \\ \leqq \mathbf{P}(\eta_{\nu_n} < x, A_n) \leqq \sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x + 2\varrho, A_n^{(i)}) + \sum_{i=1}^{k} \mathbf{P}\big(A_n^{(i)}, \overline{C(n,k,i,\varrho)}\big).\end{aligned} \tag{22}$$

It follows from the consequence of Lemma 1 that for any $\varepsilon > 0$ there exists a positive integer n_1 ($n_1 \geqq n_0$) such that if $A^{(i)}$ denotes the event $(a_{i-1} \leqq \lambda < a_i)$ $(i = 1, 2, \ldots, k)$ then

$$\sum_{i=1}^{k} \mathbf{P}(A_n^{(i)} \circ A^{(i)}) < \frac{\varepsilon}{2} \quad \text{if } n \geqq n_1. \tag{23}$$

Obviously we have for any three event

$$|\,\mathbf{P}(AB) - \mathbf{P}(AC)\,| \leqq \mathbf{P}(B \circ C)$$

and thus for $n \geqq n_1$

$$\begin{aligned}\sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x \pm 2\varrho, A^{(i)}) - \varepsilon \leqq \sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x \pm 2\varrho, A_n^{(i)}) \leqq \\ \leqq \sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x \pm 2\varrho, A^{(i)}) + \varepsilon.\end{aligned} \tag{24}$$

We obtain thus the following estimate for $\mathbf{P}(\eta_{\nu_n} < x, A_n)$

$$\begin{aligned}\sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x - 2\varrho, A^{(i)}) - \varepsilon - \sum_{i=1}^{k} \mathbf{P}\big(A_n^{(i)}, \overline{C(n,k,i,\varrho)}\big) \leqq \mathbf{P}(\eta_{\nu_n} < x, A_n) \leqq \\ \leqq \sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x + 2\varrho, A^{(i)}) + \varepsilon + \sum_{i=1}^{k} \mathbf{P}\big(A_n^{(i)}, \overline{C(n,k,i,\varrho)}\big)\end{aligned} \tag{25}$$

if $n \geqq n_1$.
Since the distribution of random variables $\eta_{[na_{i-1}]}$ converges to (4), we have by Theorem 4

$$\begin{aligned}\lim_{n \to +\infty} \sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x \pm 2\varrho, A^{(i)}) = \Phi(x \pm 2\varrho) \sum_{i=1}^{k} \mathbf{P}(A^{(i)}) = \\ = \Phi(x \pm 2\varrho)\, \mathbf{P}(a \leqq \lambda < b)\end{aligned} \tag{26}$$

if k is the fixed integer.

This means that there exists a positive integer n_2 ($n_2 \geqq n_1$) such that if $n \geqq n_2$,

$$\left|\sum_{i=1}^{k} \mathbf{P}(\eta_{[na_{i-1}]} < x \pm 2\varrho, A^{(i)}) - \Phi(x \pm 2\varrho)\, \mathbf{P}(a \leqq \lambda < b)\right| < \varepsilon.$$

From this inequality and from (25) we have

$$\Phi(x-2\varrho)\,\mathsf{P}(a\leqq\lambda<b)-2\varepsilon-\sum_{i=1}^{k}\mathsf{P}(A_n^{(i)},\overline{C(n,k,i,\varrho)})\leqq\mathsf{P}(\eta_{\nu_n}<x,A_n)\leqq$$

(27)

$$\leqq\Phi(x+2\varrho)\,\mathsf{P}(a\leqq\lambda<b)+2\varepsilon+\sum_{i=1}^{k}\mathsf{P}(A_n^{(i)},C(n,k,i,\varrho))$$

if $n\geqq n_2$. We have

$$\mathsf{P}(a\leqq\lambda<b)>1-\varepsilon$$

and thus for $n\geqq n_2$ and the chosen ϱ

$$\Phi(x)-4\varepsilon-\sum_{i=1}^{k}\mathsf{P}(A_n^{(i)},\overline{C(n,k,i,\varrho)})\leqq\mathsf{P}(\eta_{\nu_n}<x,A_n)\leqq$$

(28)

$$\leqq\Phi(x)+3\varepsilon+\sum_{i=1}^{k}\mathsf{P}(A_n^{(i)},\overline{C(n,k,i,\varrho)})\,.$$

The proof will be completed if we prove that for any $\varepsilon>0$ there exists n_3 ($n_3\geqq n_2$) such that for $n\geqq n_3$

$$\sum_{i=1}^{k}\mathsf{P}(A_n^{(i)},\overline{C(n,k,i,\varrho)})<\varepsilon\,.$$

For this purpose we remark that the following inequality holds

$$\mathsf{P}(A_n^{(i)},\overline{C(n,k,i,\varrho)})\leqq\mathsf{P}\left(A_n^{(i)},\left|\sqrt{\frac{[na_{i-1}]}{\nu_n}}\left(\frac{\zeta_{\nu_n}-\zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}}\right)\right|\geqq\varrho\right)+$$

(29)

$$+\mathsf{P}\left(A_n^{(i)},\left|\eta_{[na_{i-1}]}\left(\sqrt{\frac{[na_{i-1}]}{\nu_n}}-1\right)\right|\geqq\varrho\right).$$

First of all we remark that if the event $A_n^{(i)}$ takes place then

$$0\leqq\sqrt{\frac{[na_{i-1}]}{\nu_n}}\leqq1\,.$$

Thus we have only to estimate the sum of the probabilities

$$\mathsf{P}\left(\left|\frac{\zeta_{\nu_n}-\zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}}\right|\geqq\varrho,A_n^{(i)}\right)\tag{30}$$

where ϱ is the fixed positive number, ($i=1,2,\ldots,k$). It can be easily seen that (30) is smaller than

$$\mathsf{P}\left(\max_{na_{i-1}\leqq l\leqq na_i}\left|\frac{\zeta_l-\zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}}\right|\geqq\varrho,A_n^{(i)}\right).\tag{31}$$

It follows from (31) by the aid of the inequality

$$|\mathsf{P}(AB)-\mathsf{P}(AC)|\leqq\mathsf{P}(B\circ C)$$

and by (23) that for $n \geqq n_3(\varepsilon)$

$$\sum_{i=1}^{k} \mathbf{P}\left(\max_{na_{i-1} \leqq l \leqq na_i} \left| \frac{\zeta_l - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}} \right| \geqq \varrho,\ A_n^{(i)} \right) \leqq \tag{32}$$
$$\leqq \sum_{i=1}^{k} \mathbf{P}\left(\max_{na_{i-1} \leqq l \leqq na_i} \left| \frac{\zeta_l - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}} \right| \geqq \varrho,\ A^{(i)} \right) + \frac{\varepsilon}{2}\,.$$

Applying Theorem 7 to the terms of the right-hand side of (32) we obtain for $n \geqq n_4$ $(n_4 \geqq n_3)$

$$\mathbf{P}\left(\max_{na_{i-1} \leqq l \leqq na_i} \left| \frac{\zeta_l - \zeta_{[na_{i-1}]}}{\sqrt{[n(a_i - a_{i-1})]}} \right| \geqq \varrho \sqrt{\frac{[na_{i-1}]}{[n(a_i - a_{i-1})]}},\ A^{(i)} \right) \leqq \tag{33}$$
$$\leqq 3 \sqrt{\mathbf{P}(A^{(i)})}\, \frac{[n(a_i - a_{i-1})]}{\varrho^2\,[na_{i-1}]}\,.$$

Since the set of the continuity points of the distribution function of λ is everywhere dense, we can choose the points of subdivision such that for any $\delta > 0$

$$a_i - a_{i-1} = (b-a)\,\frac{1 + \varepsilon_i \delta}{k} \qquad (i = 1, 2, \ldots, k)$$

holds with some ε_i, $|\varepsilon_i| \leqq 1$.

Putting now this subdivision of $[a, b)$ we obtain from (33) the following inequality

$$\mathbf{P}\left(\max_{na_{i-1} \leqq l \leqq na_i} \left| \frac{\zeta_l - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}} \right| \geqq \varrho,\ A^{(i)} \right) \leqq 3 \sqrt{\mathbf{P}(A^{(i)})}\, \frac{n(b-a)}{\varrho^2\,[na]}\, \frac{1+\delta}{k}\,. \tag{34}$$

Let us choose $n_5 (n_5 \geqq n_4)$ such that for $n \geqq n_5$ the inequality

$$\frac{n}{[na]} < \frac{1}{a - \delta}$$

be satisfied.

Then (34) gives

$$\mathbf{P}\left(\max_{na_{i-1} \leqq l \leqq na_i} \left| \frac{\zeta_l - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}} \right| \geqq \varrho,\ A^{(i)} \right) \leqq \frac{K \sqrt{\mathbf{P}(A^{(i)})}}{\varrho^2 k} \tag{35}$$

where K is a positive constant independent of n. Substituting (35) in (32) we have for

$$\sum_{i=1}^{k} \mathbf{P}\left(\max_{na_{i-1} \leqq l \leqq na_i} \left| \frac{\zeta_l - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}} \right| \geqq \varrho,\ A_n^{(i)} \right) \leqq \frac{K}{\varrho^2 k} \left(\sum_{i=1}^{k} \sqrt{\mathbf{P}(A^{(i)})} \right) + \frac{\varepsilon}{2}\,. \tag{36}$$

The cauchy inequality gives

$$\sum_{i=1}^{k} \sqrt{\mathbf{P}(A^{(i)})} \leqq \sqrt{\sum_{i=1}^{k} \mathbf{P}(A^{(i)})}\, \sqrt{k} \leqq \sqrt{k}\,.$$

Taking into account the choice of k and the preceding estimate we obtain for $n \geqq n_5$

$$\sum_{i=1}^{k} \mathbf{P}\left(\max_{na_{i-1} \leqq l \leqq na_i} \left| \frac{\zeta_l - \zeta_{[na_{i-1}]}}{\sqrt{[na_{i-1}]}} \right| \geqq \varrho, A_n^{(i)} \right) \leqq \varepsilon . \tag{37}$$

Our next aim is to estimate the sum of the probabilities

$$\mathbf{P}\left(\left| \eta_{[na_{i-1}]} \left(\sqrt{\frac{[na_{i-1}]}{\nu_n}} - 1 \right) \right| \geqq \varrho, A_n^{(i)} \right), \qquad (i = 1, 2, \ldots, k). \tag{38}$$

Arguing again similarly as in the preceding estimation (37) we obtain that for $n \geq n_6$ $(n_6 \geq n_5)$ and for the chosen k

$$\sum_{i=1}^{k} \mathbf{P}\left(\left| \eta_{[na_{i-1}]} \left(\sqrt{\frac{[na_{i-1}]}{\nu_n}} - 1 \right) \right| \geqq \varrho, A_n^{(i)} \right) \leq \varepsilon . \tag{38}$$

Confering now (28), (29), (37) and (38) we obtain the assertion of Theorem 3.

§ 4. Some additional remarks

It is easy to see that the above method of proof of Theorem 3 can be applied to the case also when the random variables ξ_k are not identically distributed, but are such that the distribution of the normed sums of random variables ξ_k tends to a non-degenerate limiting distribution. Especially the following generalization of Theorem 3 is valid.

Theorem 9. *Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of independent random variables with mean value* 0 *(this is not an essential restriction) and variance* Var $\xi_k = D_k$. *Let us denote by η_n the expression*

$$\frac{\xi_1 + \ldots + \xi_n}{B_n}$$

where

$$B_n = \sqrt{\sum_{k=1}^{n} D_k^2} .$$

Let us suppose that the distribution of η_n tends to a non-degenerate limiting distribution and that the sequence B_n is in the sense of Karamata *„slowly oscillating" i. e.*

$$B_n = n^\alpha L(n) , \qquad (\alpha > 0)$$

where for any $c > 0$, $L(cn)/L(n) \to 1$, if $n \to +\infty$. Let further ν_n be a sequence of positive integer-valued random variables such that $\frac{\nu_n}{n}$ converges in probability to λ. Then η_{ν_n} has the same limiting distribution as η_n.

The proof of this theorem can be performed in the same way as that of Theorem 3.

We remark also that the supposition that $\frac{\nu_n}{n}$ converges in probability to λ can be replaced by the more general supposition that $\frac{\nu_n}{\omega(n)}$ converges in probability to λ where $\omega(n)$ is an arbitrary positive function tending to infinity for $n \to +\infty$.

(Received October 20, 1961; in revised from August 10, 1962)

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ЦЕНТРАЛЬНАЯ ПРЕДЕЛЬНАЯ ТЕОРЕМА ДЛЯ СУММ СЛУЧАЙНОГО ЧИСЛА НЕЗАВИСИМЫХ СЛУЧАЙНЫХ ВЕЛИЧИН

J. MOGYORÓDI

Резюме

Доказывается следующая

Теорема 3. *Пусть* $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ — *последовательность независимых одинаково распределенных случайных величин, с математическим ожиданием* 0 *и дисперсией* 1.

Если ν_n $(n = 1, 2, \ldots)$ — *последовательность положительных целочисленных случайных величин, таких что* $\frac{\nu_n}{n}$ *стремится по вероятности к некоторой положительной случайной величине* λ, *то мы имеем:*

$$\lim_{n \to +\infty} \mathbf{P}\left(\frac{\xi_1 + \ldots + \xi_{\nu_n}}{\sqrt{\nu_n}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy .$$

Теорема 3 является обобщением результатов Anscombe-а [1], Rényi [2], [3] и автора [6]. Именно в работе Anscombe-а и автора предполагается, что $\frac{\nu_n}{n}$ стремится по вероятности к некоторой положительной константе, а в работах Rényi к некоторой положительной случайной величине λ, имеющей дискретное распределение.

Теорема 9, является легким обобщением Теоремы 3. Метод доказательства Теоремы 3 переносится почти тривиальным образом на доказательство Теоремы 9.

Теорема 3 доказывается с помощью Теоремы 7, которая основывается на следующей теореме:

Теорема 6. *Пусть $\vartheta_1, \vartheta_2, \dots$ последовательность независимых случайных величин с математическим ожиданием 0 и дисперсией $D_1, D_2, \dots$, таких, что функция распределения $F_n(x)$ случайной величины*

$$\sigma_n = \frac{\vartheta_1 + \vartheta_2 + \dots + \vartheta_n}{B_n} \qquad (B_n \to +\infty)$$

стремится к некоторой несобственной функции распределения $F(x)$. Пусть

$$\lim_{n\to+\infty} \mathbf{M}(\sigma_n^2) = M, \quad (|M| < +\infty, \; M = \int_{-\infty}^{+\infty} x^2 d\, F(x))$$

то мы имеем для условных математических ожиданий

$$\lim_{n\to+\infty} \mathbf{M}(\sigma_n^2 | A) = M,$$

где A — произвольное случайное событие, имеющее положительную вероятность.

Теорема 7. *Пусть $\vartheta_1, \vartheta_2, \dots, \vartheta_n, \dots$ — последовательность независимых случайных величин. Предполагается, что математическое ожидание $\mathbf{M}(\vartheta_i) = M_i$ и дисперсия $\mathbf{D}(\vartheta_i) = D_i$ $(i = 1, 2, \dots)$ существуют для всех i.* Возьмем

$$\sigma_n = \frac{\vartheta_1 + \dots + \vartheta_n - (M_1 + \dots + M_n)}{S_n}$$

где

$$S_n = \sqrt{D_1^2 + \dots + D_n^2}.$$

Предположим, что функция распределения $F_n(x)$ случайной величины σ_n стремится к некоторой несобственной функции распределения. Если C — случайное событие, имеющее положительную вероятность, то существует целое число $n_0 = n_0(C)$, такое что для чисел n, больших n_0 мы имеем:

$$\mathbf{P}\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} (\vartheta_i - M_i) \right| \geq \lambda S_n, C \right) \leq \frac{3\sqrt{P(C)}}{\lambda^2}$$

где λ произвольное положительное число.