ON RANDOM SETS

by

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P. ERDŐS raised the following combinatorial problem: Let $N(n)$ subsets of a set $\mathscr{H}$ of n elements be chosen at random, independently and so that at every choice any subset is chosen with the same probability $\frac{1}{2^n}$; how large must be $N(n)$ to ensure with probability near to 1 (as n tends to infinity) that among these $N(n)$ sets there is a pair such that one is a subset of the other.

A. RÉNYI [1] solved this problem and obtained (among others) the following results:

I. Let $P_n(N(n))$ denote the probability of the event that there exist among the $N(n)$ subsets chosen at random at least two, one being a subset of the other. Then for any fixed $c > 0$, if

$$\lim_{n\to\infty} \frac{N(n)}{\left(\frac{2}{\sqrt{3}}\right)^n} = c\,,$$

then

$$\lim_{n\to\infty} P_n(N(n)) = 1 - e^{-c^2}. \tag{1}$$

II. Let γ_n denote the number of pairs of subsets A_i, A_j $(i < j)$ among the chosen sets $A_1, A_2, \ldots, A_{N(n)}$ such that $A_i \subseteq A_j$ or $A_j \subseteq A_i$.[1]

Suppose

$$N(n) \sim \left(\frac{2}{\sqrt{3}}\right)^n \omega(n) \;{}^{2}$$

where $\lim_{n\to\infty} \omega(n) = \infty$, but $\lim_{n\to\infty} \sqrt[n]{\omega(n)} = 1$. Then for any fixed ε $(0 < \varepsilon < 1)$

$$\lim_{n\to\infty} \mathbf{P}(|\gamma_n - \omega^2(n)| < \omega(n)^{1+\varepsilon}) = 1\,.$$

($\mathbf{P}(\ldots)$ denotes the probability of the event in the brackets.)

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[1] $A_1 \subseteq A_2$ denotes that A_1 is a subset of A_2, $A_1 + A_2$ denotes the union of the sets A_1, A_2, while A_1A_2 denotes the intersection of the sets A_1, A_2.

[2] The sign $\sim$ denotes asymptotic equality.

Rényi also obtained results (analogous to (1)) concerning other relations than $A_i \subseteq A_j$, namely

a) $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_r$ (see theorem 6 of [1])

b) $A_3 = A_1 + A_2$ or $A_3 = A_1 A_2$ (see theorem 8 of [1]).

The generalization of this problem — raised by A. Rényi — is as follows: Let us choose at random $N(n, R) \overset{\text{def.}}{=} N$ subsets of a set $\mathscr{H}$ of n elements independently and so that at each choice any subset of $\mathscr{H}$ is chosen with the same probability, i.e. with probability $\frac{1}{2^n}$. Let be given an arbitrary relation

$$R(X_1, X_2, \ldots, X_r) \tag{2}$$

of r set variables, defined on subsets of $\mathscr{H}$, which can be expressed by Boolean operations and which can be fulfilled by r different subsets.[3] How large must be N and of what type must be the relation, that for sufficiently large n with probability near to 1 there should exist among these N subsets at least one r-tuple of sets, fulfilling the given relation?

This paper is devoted to this problem. After giving some notations, we shall prove a lemma concerning the probability that the relation R holds for a random r-tuple of sets (analogous to the lemma 2 of [1]). There will be given a class $\mathcal{R}$ of relations („regular" relations, see def. 1, (8)) such that if $R \in \mathcal{R}$ and if N is suitably chosen ($N \sim (C(R))^n \omega(n)$ where $C(R) > 1$ is a constant, depending only upon the relation R, and $\omega(n) \to \infty$ as $n \to \infty$), then there can be found among the N randomly chosen subsets of $\mathscr{H}$ with probability tending to 1 as $n \to \infty$ at least one r-tuple of sets for which the relation R holds. Our result is valid e.g. for the following relations:

$$R(A_1, A_2, A_3) \equiv A_1, A_2, A_3 \text{ are pairwise disjoint.}$$

$$R(A_1, A_2, A_3) \equiv (A_3 = \bar{A}_1 + \bar{A}_2) \qquad \text{etc.}$$

There will be given a class $\mathcal{R}_1 (\subset \mathcal{R})$ of relations („strictly regular" relations, see def. 1, (9)) such that if $R \in \mathcal{R}_1$ and if $N \sim c(C(R))^n$, where $c > 0$ fixed, then the probability that among the N randomly chosen subsets of $\mathscr{H}$ there can be found at least one r-tuple of sets, for which the relation R holds, tends to $1 - e^{-c^r}$ as $n \to \infty$.

It will be mentioned furthermore, that the „balancedness" of the relation (see def. 2) is a necessary condition of the relation R being fulfilled with positive probability by at least one r-tuple of sets of the N randomly chosen subsets if $N \sim c(C(R))^n$.

Notations. Let $X_1, X_2, \ldots, X_r$ be an ordered r-tuple of subsets of the set $\mathscr{H}$ of n elements. Let us consider all possible set-theoretical products of r factors such that the j-th factor is either equal to X_j or to $\bar{X}_j$ where $\bar{X}_j$ denotes the complementary set of X_j with respect to $\mathscr{H}$, and let us call these products *the atoms of the r set variables* $X_1, X_2, \ldots, X_r$, or simply: *atoms*. The number of possible atoms is obviously 2^r (some of them may be equal to 0, when for $X_1, X_2, \ldots, X_r$ some relation holds). Let these atoms be numbered in some way from 1 to 2^r, the m-th will be denoted by $E_m(X_1, X_2, \ldots, X_r)$.

[3] It will be said that the relation $R(X_1, X_2, \ldots, X_r)$ holds for $X_1^0, X_2^0, \ldots, X_r^0$ if $X_1^0, X_2^0, \ldots, X_r^0$ fulfill R in the given order.

(The numbering may be done e.g. as follows: let be

$$E_{m+1}(X_1, X_2, \ldots, X_r) = X_1^{\delta_1} X_2^{\delta_2} \ldots X_r^{\delta_r}, \qquad (m = 0, 1, \ldots, 2^r - 1) \tag{3}$$

where

$$X_j^{\delta_j} = \begin{cases} X_j, & \text{if } \delta_j = 0 \\ \overline{X}_j, & \text{if } \delta_j = 1 \end{cases}$$

in case

$$m = \sum_{j=1}^{r} \delta_j 2^{j-1}$$

where δ_j equals 0 or 1. E.g. in case $r = 2$

$$E_1(X_1, X_2) = X_1 X_2 \, ; \; E_2(X_1, X_2) = X_1 \overline{X}_2 \, ; \; E_3(X_1, X_2) = \overline{X}_1, X_2 \, ;$$
$$E_4(X_1, X_2) = \overline{X}_1, \overline{X}_2 \, .$$

It is well known ([2]), that any relation $R(X_1, X_2, \ldots, X_r)$ which may be expressed by Boolean operations, can be written in disjunctive normal form (in short: normal form), i.e. in the following form:

$$\sum_{m \in \Gamma} E_m(X_1, X_2, \ldots, X_r) = \mathscr{H}, \tag{4}$$

where Γ is a set of indices m. ($\Gamma \subset \{1, 2, \ldots, 2^r\}$). In other words, the normal form of R means that the following two conditions are equivalent:

(a) $\qquad R(X_1, X_2, \ldots, X_r)$ holds,

(b) $\qquad \sum_{m \in \Gamma} E_m(X_1, X_2, \ldots, X_r) = \mathscr{H}.$

Let the number $\nu(\Gamma)$ of elements in Γ be denoted by s, i.e. s is the number of atoms occurring in the normal form of R. Obviously $1 \leq s \leq 2^r$, and when the relation does not hold identically, then $s < 2^r$. As an example, let the relation $R(X_1, X_2, X_3)$ be the following:

$$R(X_1, X_2, X_3) \equiv X_1 + X_2 \subseteq X_3 .$$

The normal form of this relation is:

$$X_1 X_2 X_3 + X_1 \overline{X}_2 X_3 + \overline{X}_1 X_2 X_3 + \overline{X}_1 \overline{X}_2 X_3 + \overline{X}_1 \overline{X}_2 \overline{X}_3 = \mathscr{H}.$$

Let $(i_1, i_2, \ldots, i_l)$ be an l-tuple $(l \leq r)$ of the numbers $1, 2, \ldots, r$. Let us form from $X_{i_1}, X_{i_2}, \ldots, X_{i_l}$ all possible 2^l atoms. Let

$$s_q(i_1, i_2, \ldots, i_l) \qquad (q = 1, 2, \ldots, 2^l) \tag{5}$$

denote the number of atoms occurring in the normal form of R, containing as component the atom $E_q(X_{i_1}, X_{i_2}, \ldots, X_{i_l})$.[4]

[4] For a given relation $R(X_1, X_2, \ldots, X_r)$, the number $s_q(i_1, i_2, \ldots, i_l)$ depends obviously only upon the serial number (in R) of the sets occurring in the atom $E_q(X_{i_1}, X_{i_2}, \ldots, X_{i_l})$ and not upon the sets themselves.

Obviously

$$0 \leqq s_q(i_1, i_2, \ldots, i_l) \leqq 2^{r-l},$$

and

$$\sum_{q=1}^{2^l} s_q(i_1, i_2, \ldots, i_l) = s. \tag{6}$$

Let us put further

$$\sum_{q=1}^{2^l} s_q^2(i_1, i_2, \ldots, i_l) \overset{\text{def.}}{=} S^2(i_1, i_2, \ldots, i_l). \tag{7}$$

Definition 1. If for the relation $R(X_1, X_2, \ldots, X_r)$

$$\operatorname*{Max}_{(i_1, i_2, \ldots, i_l)} S^2(i_1, i_2, \ldots, i_l) \leqq s^{2-\frac{l}{r}} \qquad (l = 1, 2, \ldots, r-1) \tag{8}$$

holds, $R(X_1, X_2, \ldots, X_r)$ will be called a *regular* relation. If for the relation $R(X_1, X_2, \ldots, X_r)$

$$\operatorname*{Max}_{(i_1, i_2, \ldots, i_l)} S^2(i_1, i_2, \ldots, i_l) < s^{2-\frac{l}{r}} \qquad (l = 1, 2, \ldots, r-1) \tag{9}$$

holds, $R(X_1, X_2, \ldots, X_r)$ will be called a *strictly regular* relation.

Examples. I. The relation

$$R(A_{i_1}, A_{i_2}, \ldots, A_{i_t};\ A_{j_1}, A_{j_2}, \ldots, A_{j_t}) \equiv$$

$$\equiv A_{i_1} \subseteq A_{j_1} \text{ and } A_{i_2} \subseteq A_{j_2} \text{ and } \ldots \text{ and } A_{i_t} \subseteq A_{j_t}$$

is regular, but not strictly regular. Namely, it is easy to see, that $s = 3^t$ and

$$\operatorname*{Max}_{(h_1, h_2, \ldots h_{2l-1})} S^2(h_1, h_2, \ldots, h_{2l-1}) = 3^{l-1} \cdot 3^{(t-l)2} + 3^{l-1}(2.3^{t-l})^2 =$$

$$= 5.3^{2t-l-1} < 3^{t\left(2-\frac{2l-1}{2t}\right)};$$

further

$$\operatorname*{Max}_{(h_1, h_2, \ldots, h_{2l})} S^2(h_1, h_2, \ldots, h_{2l}) = 3^l(3^{t-l})^2 = 3^{t\left(2-\frac{2l}{2t}\right)}.$$

II. The following relations are strictly regular:

1. $R(A_1, A_2, A_3) \equiv A_1 + A_2 = A_3$, $\quad (s = 4)$;
2. $R(A_1, A_2, A_3) \equiv A_3 \subseteq A_1 + A_2$, $\quad (s = 7)$;
3. $R(A_1, A_2, A_3) \equiv A_1, A_2, A_3$ are pairwise disjoint, $(s = 4)$.

III. The relation

$$R(A_1, A_2, A_3, A_4) \equiv A_1 \subseteq A_2 \subseteq (A_3 + A_4) \qquad (s = 10)$$

is not regular.

Definition 2. Let $R_1(X_{i_1}, X_{i_2}, \ldots, X_{i_t})$ (where $t \leqq r$ and $(i_1, i_2, \ldots, i_t)$ is a t-tuple of the numbers $1, 2, \ldots, r$) and $R(X_1, X_2, \ldots, X_r)$ be two relations. We shall write

$$R_1(X_{i_1}, X_{i_2}, \ldots, X_{i_t}) \leqq R(X_1, X_2, \ldots, X_r),$$

if the relation $R_1(X_{i_1}, X_{i_2}, \ldots, X_{i_t})$ holds whenever $R(X_1, X_2, \ldots, X_r)$ holds; in this case we shall say, that $R_1(X_{i_1}, X_{i_2}, \ldots, X_{i_t})$ is a *subrelation* of the relation $R(X_1, X_2, \ldots, X_r)$. Let $K(R)$ denote for a relation R the number $1/\sqrt[r]{s}$ where r is the number of sets occurring in the relation R and s is the number of atoms in the normal form of the relation R. The number $K(R)$ will be called the *balance number* of the relation R.

A relation R is called *balanced*, if for any relation R_1 for which

$$R_1 \leqq R$$

holds, the corresponding balance numbers fulfil the inequality

$$K(R_1) \leqq K(R)\,.$$ [5]

Remark. If

$$\sum_{m \in \Gamma} E_m(X_1, X_2, \ldots, X_r) = \mathscr{H}$$

is the normal form of the relation $R(X_1, X_2, \ldots, X_r)$ of r variables, then the relation $R_1(X_1, X_2, \ldots, X_r)$, — the normal form of which is

$$\sum_{m \in \Gamma_1} E_m(X_1, X_2, \ldots, X_r) = \mathscr{H}$$

where $\Gamma \subseteq \Gamma_1$, — is a subrelation of R and all subrelations $R_1(X_1, X_2, \ldots, X_r)$ of $R(X_1, X_2, \ldots, X_r)$ can be obtained in this way.

Let $R_2(X_{i_1}, X_{i_2}, \ldots, X_{i_t})$ $(\{X_{i_1}, X_{i_2}, \ldots, X_{i_t}\} \subset \{X_1, X_2, \ldots, X_r\})$ be any subrelation of R. The normal form of R_2 can be obtained as follows: we add atoms (of the r set variables $X_1, X_2, \ldots, X_r$) to the normal form of R until it is possible to bracket out all the atoms $E_q(X_{i_{t+1}}, X_{i_{t+2}}, \ldots, X_{i_r})$ $(q = 1, 2, \ldots, 2^{r-t})$ (here $X_{i_{t+1}}, X_{i_{t+2}}, \ldots, X_{i_r}$ are those of the sets $X_1, X_2, \ldots, X_r$ which are not contained among $X_{i_1}, X_{i_2}, \ldots, X_{i_t}$), i.e. until the normal form of R can be written in the form:

$$\sum_{q=1}^{2^{r-t}} E_q(X_{i_{t+1}}, X_{i_{t+2}}, \ldots, X_{i_r}) \sum_{m \in \Gamma_2} E_m(X_{i_1}, X_{i_2}, \ldots, X_{i_t}) = \mathscr{H}$$

where obviously the first factor is equal to $\mathscr{H}$. Thus we get a relation R^*:

$$\sum_{m \in \Gamma_2} E_m(X_{i_1}, X_{i_2}, \ldots, X_{i_t}) = \mathscr{H},$$

which clearly is a subrelation of R. Now evidently R_2 is a subrelation of R^* and thus R_2 can be obtained from R^* by eventually adding some atoms of the variables $X_{i_1}, X_{i_2}, \ldots, X_{i_t}$ to the normal form of R^*.

It can be easily shown that every regular relation is balanced. As a matter of fact if $R_1(X_{i_1}, X_{i_2}, \ldots, X_{i_t})$ is any subrelation of the regular relation

[5] The concept of a balanced relation is very similar to that of a balanced graph. (See [3], p. 22.).

$R(X_1, X_2, \ldots, X_r)$ where $1 \leqq i_1 < i_2 < \ldots < i_l \leqq r$, further if t denotes the number of atoms in the normal form of R_1 and s in that of R, we have evidently

$$t \geqq \sum_{\substack{s_q(i_1, i_2, \ldots, i_l) > 0 \\ (q = 1, 2, \ldots, 2^l)}} 1 .$$

Thus by the inequality of Cauchy we obtain

$$s^2 = \left(\sum_{q=2}^{2^l} s_q(i_1, i_2, \ldots, i_l) \right)^2 = \left(\sum_{s_q(i_1, i_2, \ldots, i_l) > 0} s_q(i_1, i_2, \ldots, i_l) \right)^2 \leqq t \sum_{q=1}^{2^l} s_q^2(i_1, i_2, \ldots, i_l)$$

and therefore as R is by supposition regular we obtain

$$s^2 \leqq ts^{2 - \frac{l}{r}}$$

and thus

$$K(R_1) = \frac{1}{\sqrt[l]{t}} \leqq \frac{1}{\sqrt[r]{s}} = K(R) .$$

This proves our assertion.

On the other hand a balanced relation is not necessarily regular. For example the following relations are balanced, but not regular:

(1) $$R(A_1, A_2, A_3) \equiv (A_1 + A_2) \subseteq A_3 , \qquad (s = 5) ;$$

(2) $$R(A_1, A_2, A_3, A_4) \equiv (A_1 + A_2) \subseteq A_3 A_4 , \qquad (s = 7) .$$

To give an example of a relation which is not balanced, let us consider the relation

$$R_i(A_0, A_1, \ldots, A_{i-1}) \equiv A_0 \supseteq A_1 ; \; A_0 \supseteq A_2 ; \ldots ; \; A_0 \supseteq A_{i-1} .$$

$R_i(A_0, A_1, \ldots, A_{i-1})$ is not balanced if $i \geqq 4$.

As a matter of fact it is easy to see that for the balance number of R_i we have

$$K(R_i) = \frac{1}{\sqrt[i]{2^{i-1} + 1}} .$$

Further

$$R_1 \leqq R_2 \leqq \ldots \leqq R_{i-1} < R_i$$

holds, but among the numbers $K(R_i)$ $(i = 1, 2, \ldots)$ $K(R_3)$ is the largest.

In the following we prove a lemma concerning the probability, that the given relation (2) holds for an r-tuple of subsets $A_1, A_2, \ldots, A_r$ of the set $\mathscr{H}$ chosen at random and independently in the above described sense.

Lemma.

(10) $$\mathbf{P}(R(A_1, A_2, \ldots, A_r) \text{ holds}) = \left(\frac{s}{2^r} \right)^n .$$

Proof. Let $a_1, a_2, \ldots, a_n$ denote the elements of $\mathscr{H}$. Let be

$$\varepsilon_k(j) = \begin{cases} 1, & \text{if } a_k \in A_j \\ & \text{if } a_k \notin A \end{cases} \qquad \begin{pmatrix} j = 1, 2, \ldots, r ; \\ k = 1, 2, \ldots, n \end{pmatrix} .$$

It is shown in [1] (lemma 1), that the random variables $\varepsilon_k(j)$ $(k = 1, 2, \ldots, n)$ are independent and they take on the values 0 and 1 each with probability $\frac{1}{2}$. It is obvious that the random choice of a subset A_j corresponds to a sequence of n experiments for the random variables $\varepsilon_1(j), \varepsilon_2(j), \ldots, \varepsilon_n(j)$. The probability of any fixed result of such a sequence is equal to $\frac{1}{2^n}$. It is obvious furthermore, that the relation R holds if and only if a corresponding relation R_ε concerning the random variables $\varepsilon_k(1), \varepsilon_k(2), \ldots, \varepsilon_k(r)$ holds for $k = 1, 2, \ldots, n$. However the probability of the latter can be obtained — for a given k — by determining the number s' of sequences $\varepsilon_k(1), \varepsilon_k(2), \ldots, \varepsilon_k(r)$ for which the relation R_ε holds and dividing this number by 2^r (the number of all possible sequences of r elements each 0 or 1). On account of the above mentioned independence the probability that R_ε holds for every k, i.e. that R holds, is equal to $\left(\frac{s'}{2^r}\right)^n$. But $s' = s$, since the sequence $\varepsilon_k(1), \varepsilon_k(2), \ldots, \varepsilon_k(r)$ corresponds to the atom $A_1^{1-\varepsilon_k(1)} \cdot A_2^{1-\varepsilon_k(2)} \ldots A_r^{1-\varepsilon_k(r)}$ $(k = 1, 2, \ldots, n)$ of the normal form of R (see (3)), which proves the lemma.

Theorem 1. *Let* $\eta(R, N, n) \overset{\text{def.}}{=} \eta$ *denote the number of ordered r-tuples,* $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ *among the randomly, independently chosen* N *subsets* $A_1, A_2 \ldots, A_N$ *of the set* $\mathscr{H}$ *of n elements, for which the relation* $R(A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ *holds. If the relation is regular* (see (8)), *and if*

$$N \sim \left(\frac{2}{\sqrt[r]{s}}\right)^n \sqrt[r]{\omega(n)} \tag{11}$$

where

$$\lim_{n \to \infty} \omega(n) = \infty,$$

then

$$\lim_{n \to \infty} \mathbf{P}(\eta = 0) = 0. \tag{12}$$

By other words, there exists, with probability tending to 1 *when n tends to infinity, at least one r-tuple of sets for which the relation R holds.*

Proof. Let us denote by $\mathbf{M}(\xi)$ resp. $\mathbf{D}^2(\xi)$ the mean value resp. variance of the random variable ξ. It is sufficient to prove that

$$\lim_{n \to \infty} \frac{\mathbf{D}^2(\eta)}{\mathbf{M}^2(\eta)} = 0, \tag{13}$$

since

$$\begin{aligned}\mathbf{P}(\eta = 0) &< \mathbf{P}\left(\eta < \frac{\mathbf{M}(\eta)}{2}\right) < \mathbf{P}\left(|\eta - \mathbf{M}(\eta)| \geq \frac{\mathbf{M}(\eta)}{2}\right) = \\ &= \mathbf{P}\left(|\eta - \mathbf{M}(\eta)| \geq \frac{\mathbf{M}(\eta)}{2\,\mathbf{D}(\eta)} \cdot \mathbf{D}(\eta)\right) \leq \frac{4\,\mathbf{D}^2(\eta)}{\mathbf{M}^2(\eta)}.\end{aligned} \tag{14}$$

The latter follows from Chebyshev's inequality. Hence from (13) and (14), we obtain (12). We prove (13).

Let us put

(15) $$\varepsilon_{i_1 i_2 \ldots i_r} = \begin{cases} 1, & \text{if } R(A_{i_1}, A_{i_2}, \ldots, A_{i_r}) \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$$

Obviously

(16) $$\eta = \sum_{\substack{1 \leq i_j \leq N \\ i_j \neq i_k \\ j = 1, 2, \ldots, r}} \varepsilon_{i_1 i_2 \ldots i_r},$$

(i.e. the summation runs over all ordered r-tuples from the integers $1, 2, \ldots, N$). We shall make use of the following asymptotical formula

(17) $$\binom{M-k}{p} \sim \frac{M^p}{p!}$$

which is valid if $k \geqq 0, p \geqq 1$ are fixed integers, and $M \to +\infty$.

For the mean value of η we have from (16), (10), (17) and (11)

(18) $$\mathbf{M}(\eta) = \binom{N}{r} r! \left(\frac{s}{2^r}\right)^n \sim \omega(n).$$

Now let us estimate the variance of η, making use of the identity:

(19) $$\mathbf{D}^2(\eta) = \mathbf{M}(\eta^2) - \mathbf{M}^2(\eta).$$

Since by (15) $\mathbf{M}(\varepsilon^2_{i_1 i_2 \ldots i_r}) = \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r})$ and since

$$\mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r} \cdot \varepsilon_{j_1 j_2 \ldots j_r}) = \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r})\, \mathbf{M}(\varepsilon_{j_1 j_2 \ldots j_r})$$

in case $(i_1, i_2, \ldots, i_r)$ and $(j_1, j_2, \ldots, j_r)$ are disjoint (because of independence), thus we have

(20) $$\mathbf{M}(\eta^2) = \mathbf{M}(\eta) + \sum_{\substack{i_h \neq j_k \\ (h,k = 1, 2, \ldots, r)}} \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r})\, \mathbf{M}(\varepsilon_{j_1 j_2 \ldots j_r}) + \sum{}' \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r} \cdot \varepsilon_{j_1 j_2 \ldots j_r})$$

where the sum $\sum'$ contains those terms $\mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r}\, \varepsilon_{j_1 j_2 \ldots j_r})$ for which some of the i_h-s are equal to some of the j_k-s. Since furthermore

(21) $$\sum_{\substack{i_h \neq j_k \\ (h,k = 1, 2, \ldots, r)}} \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r})\, \mathbf{M}(\varepsilon_{j_1 j_2 \ldots j_r}) \leq \left(\sum_{\substack{1 \leq i_j \leq N \\ i_j \neq i_k \\ j = 1, 2, \ldots, r}} \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r}) \right)^2 = \mathbf{M}^2(\eta),$$

hence from (19), (20), (21) we have

(22) $$\mathbf{D}^2(\eta) \leq \mathbf{M}(\eta) + \sum{}' \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r}\, \varepsilon_{j_1 j_2 \ldots j_r}).$$

In order to verify (13), it will suffice according to (22) and (18), to show, that

(23) $$\lim_{n \to \infty} \frac{\sum' \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r} \varepsilon_{j_1 j_2 \ldots j_r})}{\mathbf{M}^2(\eta)} = 0.$$

For this there will be needed to compute the probability that the relation R holds simultaneously for both of the r-tuples of subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ and $A_{j_1}, A_{j_2}, \ldots, A_{j_r}$ (i.e. $R(A_{i_1}, A_{i_2}, \ldots, A_{i_r}) R(A_{j_1}, A_{j_2}, \ldots, A_{j_r})$ holds) in case some of the indices $i_1, i_2, \ldots, i_r$ coincide with some of the indices $j_1, j_2, \ldots, j_r$. Let l $(l \leq r)$ denote the number of common elements of $i_1, i_2, \ldots, i_r$ and $j_1, j_2, \ldots, j_r$. Let us suppose e.g., that

$$(*) \qquad i_{h_t} = j_{k_t}, \quad t = 1, 2, \ldots, l; \; 1 \leq h_t, k_t \leq r .$$

The probability

$$\mathbf{P}(R(A_{i_1}, A_{i_2}, \ldots, A_{i_r}) R(A_{j_1}, A_{j_2}, \ldots, A_{j_r}) \text{ holds on condition } (*))$$

can be obtained according to lemma 1, by determining the number σ_l of atoms in the normal form of $R(A_{i_1}, A_{i_2}, \ldots, A_{i_r}) R(A_{j_1}, A_{j_2}, \ldots, A_{j_r})$. This normal form is obviously the product of the normal forms of $R(A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ and $R(A_{j_1}, A_{j_2}, \ldots, A_{j_r})$. The latters can be written by condition (*) (bracketing out the common components, i.e. the components containing the sets $A_{i_{h_1}}, A_{i_{h_2}}, \ldots, A_{i_{h_l}}$) in the following form:

$$(24) \qquad \sum_{m=1}^{2^l} E_m(A_{i_{h_1}}, A_{i_{h_2}}, \ldots, A_{i_{h_l}}) \sum_{t \in \Gamma_{l,m}(h_1, h_2, \ldots, h_l)} E_t(A_{i_{h_{l+1}}}, A_{i_{h_{l+2}}}, \ldots, A_{i_{h_r}}),$$

resp.

$$(25) \qquad \sum_{m=1}^{2^l} E_m(A_{i_{h_1}}, A_{i_{h_2}}, \ldots, A_{i_{h_l}}) \sum_{t \in \Gamma_{l,m}(k_1, k_2, \ldots, k_l)} E_t(A_{j_{k_{l+1}}}, A_{j_{k_{l+2}}}, \ldots, A_{j_{k_r}}),$$

where[6]

$$\nu(\Gamma_{l,m}(h_1, h_2, \ldots, h_l)) = s_m(h_1, h_2, \ldots, h_l),$$

resp.

$$\nu(\Gamma_{l,m}(k_1, k_2, \ldots, k_l)) = s_m(k_1, k_2, \ldots, k_l),$$

(see (5)). Multiplying (24) by (25) we get non-zero members only when multiplying members which contain the same component $E_m(A_{i_{h_1}}, A_{i_{h_2}}, \ldots, A_{i_{h_l}})$. Hence for the number σ_l of atoms of the normal form of

$$R(A_{i_1}, A_{i_2}, \ldots, A_{i_r}) R(A_{j_1}, A_{j_2}, \ldots, A_{j_r})$$

(by (*)) we have:

$$(26) \qquad \sigma_l = \sum_{m=1}^{2^l} s_m(h_1, h_2, \ldots, h_l) \, s_m(k_1, k_2, \ldots, k_l), \qquad (l = 1, 2, \ldots, r)$$

and by lemma 1, we have

$$(27) \quad \mathbf{P}(R(A_{i_1}, A_{i_2}, \ldots, A_{i_r}) R(A_{j_1}, A_{j_2}, \ldots, A_{j_r}) \text{ holds on condition}(*)) = \left(\frac{\sigma_l}{2^{2r-l}}\right)^n$$

$$(l = 1, 2, \ldots, r).$$

[6] $\nu(E)$ denotes — as before — the number of elements of the (finite) set E.

Hence

$$\sum' \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r} \varepsilon_{j_1 j_2 \ldots j_r}) = \sum_{l=1}^{r} \sum_{(*)} \left(\frac{\sigma_l}{2^{2r-l}}\right)^n = \tag{28}$$

$$= \sum_{l=1}^{r-1} \frac{\binom{N}{l}\binom{N-l}{r-l}\binom{N-r}{r-l} r!^2}{2} \left(\frac{\sigma_l}{2^{2r-l}}\right)^n + \frac{\binom{N}{r} r!\,(r!-1)}{2} \left(\frac{\sigma_r}{2^r}\right)^n .$$

According to (17) and (11), we have from (28)

$$\sum' \mathbf{M}(\varepsilon_{i_1 i_2 \ldots i_r} \varepsilon_{j_1 j_2 \ldots j_r}) \sim \sum_{l=1}^{r-1} \left(\frac{\sigma_l}{s^{2-\frac{l}{r}}}\right)^n \omega(n)^{2-\frac{l}{r}} \cdot \frac{r!^2}{2l!\,(r-l)!^2} + \tag{29}$$

$$+ \left(\frac{\sigma_r}{s}\right)^n \omega(n) \frac{r!-1}{2} .$$

According to Cauchy's inequality, by (7) and because of the regularity of the relation R, we have from (26)

$$\sigma_l \leqq S(h_1, h_2, \ldots, h_l)\, S(k_1, k_2, \ldots, k_l) \leqq \tag{30}$$

$$\leqq \operatorname*{Max}_{(\varrho_1, \varrho_2, \ldots, \varrho_l)} S^2(\varrho_1, \varrho_2, \ldots, \varrho_l) \leqq s^{2-\frac{l}{r}}, \quad (l = 1, 2, \ldots, r-1)$$

and

$$\sigma_r \leqq s \tag{31}$$

is trivial. (23) follows immediately from (29), (30) and (31), which proves Theorem 1.

Theorem 2. *Let $P_n(N, R)$ denote the probability that among the randomly chosen N subsets $A_1, A_2, \ldots, A_N$ of the set $\mathcal{H}$ of n elements, there exists at least one ordered r-tuple of sets: $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ for which the relation $R(A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ holds. If the relation is strictly regular* (see (9)) *and if*

$$\lim_{n\to\infty} \frac{N}{\left(\frac{2}{\sqrt[r]{s}}\right)^n} = c \qquad (c > 0 \textit{ fixed}), \tag{32}$$

then

$$\lim_{n\to\infty} P_n(N, R) = 1 - e^{-c^r}. \tag{33}$$

We wish to remark, that the proof is very similar to that of Theorem 4 of [1], nevertheless for the sake of completeness it will be given in detail. The proof is based on the sieving theorem, given in [1], which makes use of a graph-theoretical lemma. We shall give here the lemma and the sieving theorem without proof; the proof can be found in [1].

A graph-theoretical lemma. (See [1], p. 89.). Let H be a finite set. N_H denotes the number of elements of a set H. H^2 denotes the set of all possible unordered pairs of different elements of H. If $N_H = m$, then obviously $N_{H^2} = \binom{m}{2}$. Let $E \subset H^2$ be any subset of H^2. The pair of sets $(H, E) = G$ is called a (finite) graph; the elements of H are called the *vertices* and the elements of E the *edges* of G. If V_i, V_j are vertices of G, and (V_i, V_j) is an edge of G, then it is said that V_i and V_j are connected by an edge in G. If $G = (H, E)$, we put $N_G = N_H$ and $M_G = N_E$.

Let A be a subset of H. Let the graph $AG \overset{\text{def.}}{=} (A, A^2E)$. If $M_{AG} = j$, then it is said that the set A contains j edges of G.

Put

$$N_G(j, \alpha) + \sum_{\substack{A \subset H \\ N_A = \alpha \\ M_{AG} \leqq j}} 1 \,; \qquad N_G(j, 0) = 1 \,.$$

By other words, $N_G(j, \alpha)$ is equal to the number of subsets of H which have α elements and contain not more than j edges of the graph G. Let further be

$$N_G^{(1)}(j, 2\alpha + 1) = \sum_{\beta=0}^{\alpha} N_G(j, 2\beta + 1)$$

and

$$N_G^{(0)}(j, 2\alpha) = \sum_{\beta=0}^{\alpha} N_G(j, 2\beta) \,.$$

Thus $N_G^{(1)}(j, m)$ (resp. $N_G^{(0)}(j, m)$) denotes the number of subsets of H which contain an odd (resp. even) number $\leqq m$ of elements of H and contain not more than j edges of G.

For any nonnegative integers α, β, put

$$\delta_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta \,, \\ 0, & \text{if } \alpha \neq \beta \,. \end{cases}$$

If $G = (H, E)$ is an arbitrary finite graph, $m = N_G$, then for any nonnegative integral value of α, the inequalities

$$N_G^{(0)}(1, 2\alpha + 2) - \delta_{m0} \geqq N_G^{(1)}(0, 2\alpha + 1)$$

and

$$N_G^{(1)}(1, 2\alpha + 1) \geqq N_G^{(0)}(0, 2\alpha) - \delta_{m0}$$

hold.

A sieving theorem. (See [1], pp. 91—92.) Let $\mathfrak{B}$ be a probability field, $B_1, B_2, \ldots, B_m$ arbitrary events. Let G be an arbitrary graph with m vertices. Let them be labelled and identified with the integers $1, 2, \ldots, m$. Let H be the set of the numbers $1, 2, \ldots, m$. Let $H^{(1)}$ denote the set of those subsets of H which contain no edges of G in case the number of their elements is even, and which contain at most one edge of G in case the number of their elements is odd. Let $H^{(0)}$ denote the set of those subsets of H which do not contain edges of G in case the number of their elements is odd, and which contain at most one edge of G in case the number of their elements is even.

Let be $S_0^{(1)} = 1$ and

$$S_\alpha^{(1)} = \sum_{\substack{1 \leq i_1 < i_2 < \ldots < i_\alpha \leq m \\ (i_1, i_2, \ldots, i_\alpha) \in H^{(1)}}} \mathbf{P}(B_{i_1} B_{i_2} \ldots B_{i_\alpha}), \text{ if } \alpha = 1, 2, \ldots, m,$$

further $S_0^{(0)} = 1$ and

$$S_\alpha^{(0)} = \sum_{\substack{1 \leq i_1 < i_2 < \ldots < i_\alpha \leq m \\ (i_1, i_2, \ldots, i_\alpha) \in H^{(0)}}} \mathbf{P}(B_{i_1} B_{i_2} \ldots B_{i_\alpha}), \text{ if } \alpha = 1, 2, \ldots, m.$$

Then the inequalities

$$(34) \qquad \sum_{\alpha=0}^{2\beta+1} (-1)^\alpha S_\alpha^{(1)} \leq \mathbf{P}(\bar{B}_1 \bar{B}_2 \ldots \bar{B}_m) \leq \sum_{\alpha=0}^{2\beta} (-1)^\alpha S_\alpha^{(0)}, \quad (\beta = 0, 1, 2, \ldots)$$

hold.

Proof of theorem 2. Let $B(i_1, i_2, \ldots, i_r)$ denote the event, that the relation R holds for the randomly chosen subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ (in this order). Then obviously

$$(35)\ P_n(N, R) = \mathbf{P}\Bigg(\sum_{\substack{1 \leq i_j \leq N \\ j=1,2,\ldots,r}} B(i_1, i_2, \ldots, i_r)\Bigg) = 1 - \mathbf{P}\Bigg(\prod_{\substack{1 \leq i_j \leq N \\ j=1,2,\ldots,r}} \overline{B(i_1, i_2, \ldots, i_r)}\Bigg).$$

Let us denote by Q the set of ordered different r-tuples[7] $(i_1, i_2, \ldots i_r)$, formed from the numbers $1, 2, \ldots, N$. Let the vertices of the graph G be the elements of Q and let the vertices $(i_1^{(\alpha_1)}, i_2^{(\alpha_1)}, \ldots, i_r^{(\alpha_1)})$ and $(i_1^{(\alpha_2)}, i_2^{(\alpha_2)}, \ldots, i_r^{(\alpha_2)})$ be connected if they are not disjoint, i.e. if $i_h^{(\alpha_1)} = i_k^{(\alpha_2)}$ for certain values of h resp. k. (Suppose

$$i_{h_1}^{(\alpha_1)} = i_{k_1}^{(\alpha_2)}, \ldots, i_{h_l}^{(\alpha_1)} = i_{k_l}^{(\alpha_2)} \qquad (l = 1, 2, \ldots, r).)$$

According to (34), the following inequalities hold:

$$(36) \qquad \mathbf{P}\Bigg(\prod_{\substack{1 \leq i_j \leq N \\ j=1,2,\ldots,r}} \overline{B(i_1, i_2, \ldots, i_r)}\Bigg) \leq \sum_{\alpha=0}^{2\beta} (-1)^\alpha S_\alpha^{(0)} \quad (\beta = 0, 1, 2, \ldots)$$

and

$$(37) \qquad \mathbf{P}\Bigg(\prod_{\substack{1 \leq i_j \leq N \\ j=1,2,\ldots,r}} \overline{B(i_1, i_2, \ldots, i_r)}\Bigg) \geq \sum_{\alpha=0}^{2\beta+1} (-1)^\alpha S_\alpha^{(1)} \quad (\beta = 0, 1, 2, \ldots).$$

The numbers $S_\alpha^{(0)}$ in (36) are defined as follows: $S_0^{(0)} = 1$; and

$$S_\alpha^{(0)} = \sum\nolimits^{(0)} \mathbf{P}\big(B(i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)})\, B(i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)}) \ldots B(i_1^{(\alpha)}, i_2^{(\alpha)}, \ldots, i_r^{(\alpha)})\big),$$

$$\alpha = 1, 2, \ldots$$

[7] Two r-tuples, $(i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)}$ and $(i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)})$ are not different, if for the corresponding sets $A_{i_1^{(1)}}, A_{i_2^{(1)}}, \ldots, A_{i_r^{(1)}}$ and $A_{i_1^{(2)}}, A_{i_2^{(2)}}, \ldots, A_{i_r^{(2)}}$ the normal form of $R(A_{i_1^{(1)}}, A_{i_2^{(1)}}, \ldots, A_{i_r^{(1)}})$ and $R(A_{i_1^{(2)}}, A_{i_2^{(2)}}, \ldots, A_{i_r^{(2)}})$ contain the same atoms.

where the summation in $\sum^{(0)}$ is taken over all combinations of order α chosen of different r-tuples $(i_1, i_2, \ldots, i_r)$ $(1 \leqq i_j \leqq N; j = 1, 2, \ldots, r)$ such that the r-tuples $(i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)}), (i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)}), \ldots, (i_1^{(\alpha)}, i_2^{(\alpha)}, \ldots, i_r^{(\alpha)})$ are all disjoint in case α is odd, while at most two r-tuples have common elements (say l $(l = 1, 2, \ldots, r)$) in case α is even.

The numbers $S_\alpha^{(1)}$ in (37) are defined as follows: $S_0^{(1)}, = 1$; and

$$S_\alpha^{(1)} = \sum^{(1)} \mathbf{P}(B(i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)})\, B(i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)}) \ldots B(i_1^{(\alpha)}, i_2^{(\alpha)}, \ldots, i_r^{(\alpha)})), \qquad \alpha = 1, 2, \ldots$$

where the summation in $\sum^{(1)}$ is taken over all combinations of order α chosen of different r-tuples $(i_1, i_2, \ldots, i_r)$ such that the r-tuples $(i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)})$, $(i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)}), \ldots, (i_1^{(\alpha)}, i_2^{(\alpha)}, \ldots, i_r^{(\alpha)})$ are all disjoint in case α is even, while at most two r-tuples have common elements in case α is odd.

According to lemma 1,

$$\mathbf{P}(B(i_1, i_2, \ldots, i_r)) = \left(\frac{s}{2^r}\right)^n, \tag{38}$$

hence

$$S_{2\varrho+1}^{(0)} = \frac{\binom{N}{r} r! \binom{N-r}{r} r! \ldots \binom{N - 2\varrho r}{r} r!}{(2\varrho + 1)!} \left(\frac{s}{2^r}\right)^{n(2\varrho+1)}; \tag{39}$$

$$S_{2\varrho}^{(1)} = \frac{\binom{N}{r} r! \binom{N-r}{r} r! \ldots \binom{N - 2\varrho r + r}{r} r!}{(2\varrho)!} \left(\frac{s}{2^r}\right)^{n2\varrho}. \tag{40}$$

Further

$$S_{2\varrho}^{(0)} = \frac{\binom{N}{r} r! \binom{N-r}{r} r! \ldots \binom{N - 2\varrho r + r}{r} r!}{(2\varrho)!} \left(\frac{s}{2^r}\right)^{n2\varrho} + R_{2\varrho}^{(0)} \tag{41}$$

where

$$R_{2\varrho}^{(0)} = \sum{}^{*} \mathbf{P}(B(i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)})\, B(i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)}) \ldots B(i_1^{(2\varrho)}, i_2^{(2\varrho)}, \ldots, i_r^{(2\varrho)})).$$

In $\sum^{*}$ the summation is taken over all combinations of order 2ϱ chosen of different r-tuples $(i_1, i_2, \ldots, i_r)$ such that among the r-tuples $(i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)})$, $(i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)}), \ldots, (i_1^{(2\varrho)}, i_2^{(2\varrho)}, \ldots, i_r^{(2\varrho)})$ there are exactly two, which are not disjoint. Let these two r-tuples (containing l elements in common $(l = 1, 2, \ldots, r)$) be $(i_1^{(\varrho_1)}, i_2^{(\varrho_1)}, \ldots, i_r^{(\varrho_1)})$ and $(i_1^{(\varrho_2)}, i_2^{(\varrho_2)}, \ldots, i_r^{(\varrho_2)})$, and suppose, that

$$i_{h_j}^{(\varrho_1)} = i_{k_j}^{(\varrho_2)}, \qquad j = 1, 2, \ldots, l;\ 1 \leqq h_j, k_j \leqq r;\ 1 \leqq \varrho_1 < \varrho_2 \leqq 2\varrho. \tag{**}$$

According to a previous consideration (see (27))

$$\mathbf{P}(B(i_1^{(\varrho_1)}, i_2^{(\varrho_1)}, \ldots, i_r^{(\varrho_1)})\, B(i_1^{(\varrho_2)}, i_2^{(\varrho_2)}, \ldots, i_r^{(\varrho_2)}) \text{ on condition } (**)) = \left(\frac{\sigma_l}{2^{2r-l}}\right)^n \tag{42}$$

where for σ_l it follows from (26), (30) and from the condition that the relation is strictly regular:

$$(43)\qquad \begin{aligned}\sigma_l &= \sum_{m=1}^{2^l} s_m(h_1, h_2, \ldots, h_l)\, s_m(k_1, k_2, \ldots, k_l) \leqq \\ &\leqq \operatorname*{Max}_{h_1,h_2,\ldots,h_l} S^2(h_1, h_2, \ldots, h_l) < s^{2-\frac{l}{r}} \quad (l = 1, 2, \ldots, r-1)\end{aligned}$$

and it is easy to see, that

$$(44)\qquad \sigma_r < s\,.$$

Hence by (38) and (42) we have:

$$R_{2\varrho}^{(0)} \leqq \sum_{l=1}^{r-1} \frac{\binom{N}{l}\binom{N-l}{r-l}\binom{N-r}{r-l} r!^2}{2} \times$$

$$\times \frac{\binom{N-2r+l}{r} r!\binom{N-3r+l}{r} r!\ldots\binom{N+l+r-2\varrho r}{r} r!}{(2\varrho-2)\,!}\left(\frac{s}{2^r}\right)^{n(2\varrho-2)}\cdot\left(\frac{\sigma_l}{2^{2r-l}}\right)^n +$$

$$+\frac{\binom{N}{r} r!\,(r!-1)}{2}\,\frac{\binom{N-r}{r} r!\binom{N-2r}{r} r!\ldots\binom{N-(2\varrho-2)r}{r} r!}{(2\varrho-2)!}\cdot\left(\frac{s}{2^r}\right)^{n(2\varrho-2)}\left(\frac{\sigma_r}{2^r}\right)^n.$$

(45)

Let now be $N \sim c\left(\frac{2}{\sqrt[r]{s}}\right)^n$, then using again (17), the right hand side of (45) is asymptotically equal to

$$\sum_{l=1}^{r-1} \frac{r!^2 N^{2\varrho r-l}}{2\, l!\,(r-l)!^2\,(2\varrho-2)!}\left(\frac{s}{2^r}\right)^{n(2\varrho-2)}\cdot\left(\frac{\sigma_l}{2^{2r-l}}\right)^n +$$

$$+\frac{(r!-1)\,N^{(2\varrho-1)r}}{2\,(2\varrho-2)!}\left(\frac{s}{2^r}\right)^{n(2\varrho-2)}\cdot\left(\frac{\sigma^r}{2^r}\right)^n \sim \sum_{l=1}^{r-1}\frac{r!^2\, c^{2\varrho r-l}}{2\, l!\,(r-l)!^2\,(2\varrho-2)!}\cdot\left(\frac{\sigma_l}{s^{2-\frac{l}{r}}}\right)^n +$$

$$+\frac{c^{(2\varrho-1)r}\,(r!-1)}{2\,(2\varrho-2)!}\cdot\left(\frac{\sigma_r}{s}\right)^n,$$

from which it follows according to (43) and (44), that

$$(46)\qquad R_{2\varrho}^{(0)} \to 0, \quad \text{if} \quad n \to \infty.$$

Furthermore we have

$$(47)\qquad S_{2\varrho+1}^{(1)} = \frac{\binom{N}{r} r!\binom{N-r}{r} r!\ldots\binom{N-2\varrho r}{r} r!}{(2\varrho+1)!}\cdot\left(\frac{s}{2^r}\right)^{n(2\varrho+1)} + R_{2\varrho+1}^{(1)}$$

where

$$R^{(1)}_{2\varrho+1}=\sum{}^{*}\mathbf{P}\big(B(i^{(1)}_1, i^{(1)}_2, \ldots, i^{(1)}_r)\, B(i^{(2)}_1, i^{(2)}_2, \ldots, i^{(2)}_r) \ldots B(i^{(2\varrho+1)}_1, i^{(2\varrho+1)}_2, \ldots, i^{(2\varrho+1)}_r)\big).$$

In $\sum^*$ the summation is taken over all combinations of order $2\varrho+1$, chosen from different r-tuples $(i_1, i_2, \ldots, i_r)$ in such a way that among the r-tuples $(i^{(1)}_1, i^{(1)}_2, \ldots, i^{(1)}_r)$, $(i^{(2)}_1, i^{(2)}_2, \ldots, i^{(2)}_r)$, $\ldots$, $(i^{(2\varrho+1)}_1, i^{(2\varrho+1)}_2, \ldots, i^{(2\varrho+1)}_r)$ there are exactly two, which are not disjoint. Similarly we have

$$R^{(1)}_{2\varrho+1}\to 0, \quad \text{if} \quad n\to\infty . \tag{48}$$

Hence from formulae (39), (40), (41), (47) it is easy to obtain by means of (46) and (48), that

$$S^{(0)}_j=\frac{c^{rj}}{j!}+o(1) \qquad (j=0,1,\ldots) \tag{49}$$

and

$$S^{(1)}_j=\frac{c^{rj}}{j!}+o(1) \qquad (j=0,1,\ldots) \tag{50}$$

where $o(1)\to 0$, if $n\to\infty$; and from (49) — (50) by (35) — (37), our statement (33) follows.

We have seen that if R is a relation among r sets, the normal form of which contains s atoms, the regularity of the relation is a sufficient condition for

$$\lim_{n\to\infty} P_n\left(c\left(\frac{2}{\sqrt[r]{s}}\right)^n, R\right)>0 \tag{51}$$

for any $c>0$. We shall show now, that the balancedness of R is necessary condition for the validity of (51).

Theorem 3. *Let R be a relation among r sets, the normal form of which contains s atoms. In order that* (51) *should hold, the relation R has to be balanced.*

Proof. Let R_1 be a subrelation of R among $r_1<r$ sets such that the normal form of R_1 contains s_1 elements, and $\frac{1}{\sqrt[r_1]{s_1}}>\frac{1}{\sqrt[r]{s}}$. It follows from (51) that if we choose $N\sim c\left(\frac{2}{\sqrt[r]{s}}\right)^n$ subsets of a set having n elements at random, there exists with probability $\geq p>0$ at least one ordered r-tuple of these sets for which the relation R holds. Then it is obvious that there must exist with probability $\geq p$ at least one r_1-tuple of the N sets $(r_1<r)$ for which the relation R_1 holds.

On the other hand,

$$P_n(N, R_1)\leq \mathbf{M}(\eta_1)$$

where η_1 denotes the number of r_1-tuples among the N sets for which R_1 holds; as further by (18) we have

$$\mathbf{M}(\eta_1)\sim N^{r_1}\left(\frac{s_1}{2^{r_1}}\right)^n\sim c^{r_1}\left(\frac{\sqrt[r_1]{s_1}}{\sqrt[r]{s}}\right)^{nr_1}$$

and thus $\mathbf{M}(\eta_1) \to 0$ for $n \to \infty$. Thus we obtained a contradiction, which proves theorem 3.

Let us mention that theorem 3 contains a second proof of the fact, mentioned earlier, that every regular relation is balanced.

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О СЛУЧАЙНЫХ МНОЖЕСТВАХ

KATALIN BOGNÁR

Резюме

В статье автор занимается со следующей проблемой от A. RÉNYI (в частном случае от P. ERDŐS) (см. [1]):

Пусть $\mathscr{H}$ множество из n элементов. Пусть $R(X_1, X_2, \ldots, X_r)$ любая реляция от r величин, определенная на подмножествах $\mathscr{H}$, выражаемая с помощью булевых операций и выполняемая и через r различных подмножеств. Выребем случайно, независимо друг от друга и с равной вероятностью $N(n, R) \overset{\text{def.}}{=} N$ подмножеств множества $\mathscr{H}$. При каких N и которых типах реляций существует, с вероятностью близкой к 1 при достаточно больших n, хотя бы одна r-адка множеств из этих N подмножеств, удовлетворяющая данной реляции?

Указывается класс $\mathscr{R}$ реляций («регулярные реляции», см. опр. 1, (8)) такой, что для $R \in \mathscr{R}$ и подходящего N ($N \sim (C(R))^n \omega(n)$, где $C(R) > 1$ — некоторая постоянная, зависящая только от реляции R; $\omega(n) \to \infty$, если $n \to \infty$) между N выбранными случайно подмножествами $\mathscr{H}$ найдётся с вероятностью, сходящейся к 1 при $n \to \infty$, хотя бы одна r-адка множеств, выполняющая реляцию R. (Теорема 1.)

Далее указывается класс $\mathscr{R}_1 (\subset \mathscr{R})$ реляций («строго регулярные» реляции, см. опр. 1, (9)) такой, что для $R \in \mathscr{R}_1$ и $N \sim c(C(R))^n$ ($c > 0$ — зафиксированное число) вероятность того, что между N случайно выбранными подмножеств $\mathscr{H}$ найдётся хотя бы одна r-адка множеств, удовлетворяющая реляции R, стремится к $1 - e^{-c^r}$ при $n \to \infty$. (Теорема 2.)