ON THE ISOPERIMETRIC PROPERTY OF THE REGULAR HYPERBOLIC TETRAHEDRA

by

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The isoperimetric property of the regular Euclidean tetrahedron, according to which of all Euclidean tetrahedra of equal surface-area the regular one has the greatest volume, is equivalent to the easily shown fact that of all tetrahedra circumscribed about the unit sphere the regular one has the least volume. Another simple way of proving the isoperimetric property of the regular tetrahedron in Euclidean space is Steiner's symmetrization.

None of these methods can be applied in non-Euclidean spaces. Since, on the other hand, the measurement of volume in non-Euclidean spaces is rather complicated, every information concerning the volume of a non-Euclidean tetrahedron must be considered as a valuable contribution to this subject. Thus it will be of some interest to give a simple direct proof of the following

Theorem 1. *In the hyperbolic space of all tetrahedra of given surface-area the regular tetrahedron has the greatest possible volume.*

Unfortunately, our proof fails to hold in the elliptic space.

The following lemmas operate in a Euclidean or non-Euclidean plane. They touch upon the *momentum*

$$M(D) = \int_D f(OP)\, dp$$

of a domain D with respect to a fixed origin O, where dp is the element of area at the point P and $f(x)$ is a strictly decreasing function. We shall denote a domain and its area with the same letter.

Lemma 1. *Let c be a circle centred at O and s a segment of c. Then the function $M(s)$ is convex for $0 < s < c/2$.*

In order to distinguish this particular function of one variable from our general symbol of the momentum we shall denote it with $\omega(s)$.

Lemma 2. *Let σ be a straight segment contained in the circle c but not containing the point O, σ' the central projection of σ on the boundary of c and t the convex hull of σ and σ'. Then*

$$M(t) \geqq \omega(t)\,.$$

As to the proofs of Lemmas 1 and 2 we refer to [1] and [2].

Lemma 3. *Of all convex n-gons of equal area the regular n-gon with centre O has the greatest possible value of M.*

Let π be a convex n-gon. We may suppose that π contains the point O. Otherwise we could displace π in such a way that each point of it would get nearer to O. By this means M obviously increases.

Let $\bar{\pi}$ be a regular n-gon centred at O and having the same area as π. Consider the intersections $\sigma_1, \ldots, \sigma_n$ of the sides of π with the circumcircle c of $\bar{\pi}$, as well as the corresponding domains $t_1, \ldots, t_n$ (some of which may be empty) defined in Lemma 2. Then, denoting the part of π outside c by π^*, we have

$$M(\pi) = M(c) - M(t_1) - \ldots - M(t_n) + M(\pi^*)$$

whence, in view of Lemma 1 and 2 and Jensen's inequality,

$$M(\pi) \leqq M(c) - \omega(t_1) - \ldots - \omega(t_n) + M(\pi^*) \leqq$$

$$\leqq M(c) - n\,\omega\left(\frac{t_1 + \ldots + t_n}{n}\right) + M(\pi^*)\,.$$

On the other hand,

$$\pi = c - t_1 - \ldots - t_n + \pi^* = c - ns = \bar{\pi}\,,$$

where s is a segment of c cut off by a side of $\bar{\pi}$. Hence

$$\frac{t_1 + \ldots + t_n}{n} = s + \frac{\pi^*}{n}$$

which enables us to write

$$\omega\left(\frac{t_1 + \ldots + t_n}{n}\right) = \omega(s) + M(u)$$

u denoting a domain which completes the segment s to a segment of area $s + \pi^*/n$. Bearing in mind that u lies within c whilst π^* lies outside of it, we have

$$nM(u) \geqq M(\pi^*)$$

on account of which

$$M(\pi) \leqq M(c) - n\,\omega(s) - nM(u) + M(\pi^*) \leqq M(c) - n\,\omega(s) = M(\bar{\pi})\,.$$

The case of equality is obvious.

Note that Lemma 3 remains true if the function $f(x)$, instead of being strictly decreasing, is non-increasing and satisfies the additional condition $f(x_1) > f(x_2)$, whenever $x_1 < r < R < x_2$, where r and R are the inradius and circumradius of $\bar{\pi}$. In this form Lemma 3 involves the well-known facts that of all n-gons of given area the regular n-gon has the greatest incircle and the smallest circumcircle. Thus Lemma 3 seems to be interesting in itself. Since it turns out to be useful also in other problems see ([3] and [4]) we shall refer to it as to the *momentum lemma*.

Lemma 4. *$M(\bar{\pi})$ is a concave function of the area $\bar{\pi}$.*

Let $\pi_1, \pi_2, \pi_3, \pi_4$ be regular n-gons each centred at O in such a way that a ray issuing from O and containing a vertex of one polygon contains a vertex of each of the remaining three polygons as well. Suppose furthermore that $\pi_1 < \pi_2 < \pi_3 < \pi_4$ and that $\pi_2 - \pi_1 = \pi_4 - \pi_3$. It is easy to give an area-preserving transformation of the polygonal ring R_1 enclosed by the boundaries of π_1 and π_2 onto the ring R_3 defined by π_3 and π_4 in such a way, that each point gets farther from O. It follows that $M(R_1) < M(R_3)$ showing that the slope of the function in question is at π_1 greater than at π_3. The slope being decreasing, the function is concave.

Turning now to the proof of Theorem 1, let T be a tetrahedron in the hyperbolic space, O the centre of its insphere and t one of the four partial tetrahedra determined by O and a face Δ of T. We shall perform successive transformations on these tetrahedra. Terms like "increases" will be used as abbreviations instead of "increases, unless the transformation is an isometry".

Let O' be the foot of the perpendicular drawn from O to the plane p of Δ. Let dp be the "element of the plane p" at the point P (as well as its area), dv the volume of the "cone" with basis dp and apex O and $d\alpha$ the solid angle subtended by dp at O. Writing $OP = \varrho$, $\sphericalangle OPO' = \beta$ and denoting the surface-area and the volume of a sphere of radius ϱ with $S(\varrho)$ and $V(\varrho)$ we have

$$d\alpha = 4\pi \frac{\sin\beta}{S(\varrho)}\, dp \quad \text{and} \quad dv = \frac{V(\varrho)\sin\beta}{S(\varrho)}\, dp\,.$$

In view of

$$S(\varrho) = 4\pi \operatorname{sh}^2 \varrho\,, \quad V(\varrho) = \int_0^{\varrho} S(x)\,dx = \pi(\operatorname{sh} 2\varrho - 2\varrho)$$

and

$$\sin\beta = \frac{\operatorname{sh} r}{\operatorname{sh}\varrho}\,, \quad r = OO'$$

we deduce

$$d\alpha = \operatorname{sh} r \operatorname{sh}^{-3}\varrho\, dp \quad \text{and} \quad dv = \frac{1}{4} \operatorname{sh} r \operatorname{sh}^{-3}\varrho(\operatorname{sh} 2\varrho - 2\varrho)\, dp\,.$$

Note that $\operatorname{sh}^{-3}\varrho$ is a decreasing function of ϱ. In order to show the same for

$$g(\varrho) = \operatorname{sh}^{-3}\varrho(\operatorname{sh} 2\varrho - 2\varrho)$$

we introduce the function

$$h(\varrho) = \frac{1}{2}\operatorname{sh}^4\varrho\, g'(\varrho) = 3\varrho \operatorname{ch}\varrho - \operatorname{sh}\varrho(3 + \operatorname{sh}^2\varrho)$$

and observe that $h(0) = 0$ and

$$h'(\varrho) = 3\operatorname{sh}\varrho\left(\varrho - \frac{1}{2}\operatorname{sh} 2\varrho\right) < 0\,, \quad \varrho > 0\,.$$

This involves, for $\varrho > 0$, $h(\varrho) < 0$.

Obviously, the considered functions are at the same time decreasing functions of the distance $O'P$. Thus we can consider the volume v of the tetrahedron t, as well as its solid angle α at O as momenta of Δ with respect to the point O' formed with strictly decreasing functions. Therefore, replacing Δ by a regular triangle of the same area centred at O', both v and α will increase, according to the momentum lemma.

The considered transformation of Δ involves a transformation of the tetrahedron t into a "straight tatrahedron". In a quite similar way, we transform into straight tetrahedra the remaining tetrahedra based on the faces of T and having O as common apex. Again, we transform the obtained four tetrahedra into new straight tetrahedra leaving their altitudes invariant, replacing, on the other hand, their bases by equal (regular) triangles having the same total area as before. In view of Lemma 4 and Jensen's inequality, the total volume of the tetrahedra as well as the sum of their solid angles at O will increase also in this step. Now we subject the tetrahedra to a last transformation: we leave their bases invariant but increase their altitude. By means of this the volumes of the tetrahedra continue to increase, but their solid angles at the apex obviously decrease. Since before the last transformation these angles were greater than π, we can perform this transformation so as to obtain tetrahedra with solid angles equal to π. These tetrahedra can be put together so as to form one regular tetrahedron, having the same surface-area as the original one. But since the total volume of the partial tetrahedra has increased in each of the above steps, the volume of the regular tetrahedron is greater than that of T, unless T was originally regular.

This completes the proof of Theorem 1.

Using the expression *tangent polyhedron* for a polyhedron circumscribed about a sphere, we can state the following more general

Theorem 2. *In the hyperbolic space, let Π be a tangent polyhedron bounded by n ν-gons. Let S be the surface area of Π, V its volume and v the volume of a straight ν-gonal pyramid having a regular basis of area S/n and a solid angle at his apex equal to $4\pi/n$. Then*

$$V \leqq nv.$$

Equality holds only for the Platonic solids.

For instance, in the hyperbolic space, of all isoperimetric tangent dodecahedra bounded by pentagons the regular dodecahedron has the greatest volume.

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ОБ ИЗОПЕРИМЕТРИЧЕСКИХ СВОЙСТВАХ РЕГУЛЯРНОГО ГИПЕРБОЛИЧЕСКОГО ТЕТРАЭДРА

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Резюме

Основным содержанием статьи является доказательство изопериметрического свойства экстремальной величины, относящейся к регулярному тетраэдру в гиперболическом пространстве: Среди тетраэдров заданного объема в гиперболическом пространстве наименьшую площадь поверхности имеет регулярный тетраэдр.