ON SEQUENCES OF QUASI-EQUIVALENT EVENTS, I

by

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Introduction

Let $\{\Omega, \mathcal{S}, \mathbf{P}\}$ be a probability space, $A_1, A_2, \ldots$ be a sequence of events (i.e. A_i $(i = 1, 2, \ldots)$ is an element of the σ-algebra $\mathcal{S}$) and $\xi_1, \xi_2, \ldots$ be a sequence of random variables (i.e. ξ_i $(i = 1, 2, \ldots)$ is a real-valued measurable function defined on Ω). We use the following notations: $\mathcal{B}(A_1, A_2, \ldots)$ is the smallest σ-algebra which includes the events $A_1, A_2, \ldots$. $\mathcal{B}(\xi_1, \xi_2, \ldots)$ is the smallest σ-algebra with respect to which $\xi_1, \xi_2, \ldots$ are measurable. The σ-algebra $\prod_{n=1}^{\infty} \mathcal{B}(A_n, A_{n+1}, \ldots)$ is called the tail of the sequence $A_1, A_2, \ldots$; analogously the σ-algebra $\prod_{n=1}^{\infty} \mathcal{B}(\xi_n, \xi_{n+1}, \ldots)$ is called the tail of the sequence $\xi_1, \xi_2, \ldots$. The σ-algebra $\mathcal{F}$ is called trivial if for each set $A \in \mathcal{F}$ $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$. Especially if the σ-algebra $\mathcal{F}$ is the tail of a sequence $A_1, A_2, \ldots$ or $\xi_1, \xi_2, \ldots$ and $\mathcal{F}$ is trivial then we say that the tail of $A_1, A_2, \ldots$ is trivial resp. the tail of $\xi_1, \xi_2, \ldots$ is trivial. We say that the σ-algebras $\mathcal{F}$ and $\mathcal{G}$ are equivalent $(\mathcal{F} \sim \mathcal{G})$ if for every $F \in \mathcal{F}$ there exists a $G \in \mathcal{G}$ such that[1] $\mathbf{P}(F \circ G) = 0$ and conversely for every $G \in \mathcal{G}$ there exists an $F \in \mathcal{F}$ such that $\mathbf{P}(F \circ G) = 0$.

An important question of the theory of probability is the following: how can be characterized of the sequence of events (random variables) having trivial tail. A classical result in this direction is the zero-one law of KOLMOGOROV [1]:

Zero-one law. *Let $A_1, A_2 \ldots$ $(\xi_1, \xi_2, \ldots)$ be a sequence of mutually independent events (random variables). Then the tail of the sequence $A_1, A_2, \ldots$ $(\xi_1, \xi_2, \ldots)$ is trivial.*

In his paper [2] SUCHESTON obtains a characterization of the sequence of events having trivial tail.

Another direction of the generalization of the zero-one law is the following: we have a given sequence of events having the tail $\mathcal{F}$, how can $\mathcal{F}$ be characterized. In this paper we characterize the tail of a special type of sequences of events, namely we will consider the sequence of equivalent events and further a more general class of sequences which will be called sequences of quasi-equivalent events. The characterization of quasi-equivalent events from other points

[1] Here and in what follows $A \circ B$ denotes the symmetric difference of the events A and B.

of view will be given too. Namely we will obtain the generalization of the well-known properties of equivalent events for quasi-equivalent events.

In the present paper we use some concepts and results of papers [3], [4] and [5]. For the convenience of the reader we recall these concepts and results.

Definition 1 (see [3]). The sequence of events $A_1, A_2, \ldots$ is called mixing if

$$\lim_{n\to\infty} \mathbf{P}(A_n|B) = \lambda$$

where $0 < \lambda < 1$ and B is any event such that $\mathbf{P}(B) > 0$ ($\mathbf{P}(A|B)$ denotes the conditional probability of the event A under the condition B).

Definition 2 (see [4]). The sequence of events $A_1, A_2, \ldots$ is called stable if for every $B \in \mathcal{S}$ the limit

$$\lim_{n\to\infty} \mathbf{P}(A_n|B) = \mathbf{Q}(B)$$

exists. It is easy to see that $\mathbf{Q}(B)$ is a measure defined on the space $\{\Omega, \mathcal{S}\}$ which is absolutely continuous with respect to the measure $\mathbf{P}$. Let the Radon-Nikodym derivative of $\mathbf{Q}$ (with respect to $\mathbf{P}$) be $\lambda(\omega)$, i.e.

$$\mathbf{Q}(B) = \int_B \lambda(\omega)\, d\mathbf{P}\,.$$

The random variable $\lambda(\omega)$ is called the local density of the sequence $A_1, A_2, \ldots$.

Definition 3 (see [5], [6]). The events A_n $(n = 1, 2, \ldots)$ are called equivalent if the probability of the event $A_{i_1}A_{i_2}\ldots A_{i_k}$ $(i_j \neq i_l$ if $j \neq l)$ depends only on k and it does not depend on the indices $i_1, i_2, \ldots, i_k$. The numbers

$$\alpha_k = \mathbf{P}(A_{i_1}A_{i_2}\ldots A_{i_k}) \qquad (k = 1, 2, \ldots)$$

are called the moments of the sequence $A_1, A_2, \ldots$.

It is easy to see that a sequence of equivalent events is a stable sequence.

The following five theorems are proved in [3], [4] and [7].

Theorem A ([3]). *If $\{A_n\}$ is a sequence of events such that*

$$\lim_{n\to\infty} \mathbf{P}(A_n|A_k) = \lambda \qquad (k = 1, 2, \ldots)$$

where $0 < \lambda < 1$ and $A_1 = \Omega$, then the sequence $\{A_n\}$ is mixing.

Theorem B ([4]). *If $\{A_n\}$ is a sequence of events such that*

$$\lim_{n\to\infty} \mathbf{P}(A_n|A_k) = \lambda_k \qquad (k = 1, 2, \ldots)$$

where $A_1 = \Omega$ and λ_k is a sequence of real numbers $(0 < \lambda_k \leq 1)$ then the sequence A_n is stable.

Theorem C ([4]). *If H is a Hilbert space and f_n is a sequence of elements of H such that*

$$\lim_{n\to\infty} (f_n, f_k) = \lambda_k \qquad (k = 1, 2, \ldots)$$

and

$$||f_n|| \leq C$$

where C is a positive constant and λ_k is a sequence of real numbers, then f_n converges weakly to an element f of the Hilbert space H, i.e.

$$(f_n, g) \to (f, g) \qquad (n \to \infty)$$

for every element g of H.

Theorem D (see [5], [6] and [7]). *The real numbers* $\alpha_1, \alpha_2, \ldots$ *are the moments of a sequence of equivalent events if and only if there exists a distribution function* $F(x)$ *defined on the interval* $[0, 1]$ *such that*

$$\alpha_k = \int_0^1 x^k \, dF(x) \, .$$

Theorem E ([7]). *Let* $\{A_n\}$ *be a sequence of equivalent events. Let* $\lambda = \lambda(\omega)$ *be the local density of the sequence* $\{A_n\}$, *considered as a stable sequence. Then we have*

$$\mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k} \mid \lambda) = \lambda^k \qquad (\textit{with probability } 1)$$

for $k = 1, 2, \ldots$ *and* $i_1 < i_2 < \ldots < i_k$. *I.e. the events* A_n *are independent under the condition that* λ *takes on a fixed value.*

In this paper we introduce the following two concepts.

Definition 4. The events A_n $(n = 1, 2, \ldots)$ are called quasi-equivalent if the value of the ratio

$$\frac{\mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k})}{\mathbf{P}(A_{i_1}) \mathbf{P}(A_{i_2}) \ldots \mathbf{P}(A_{i_k})} = \alpha_k \qquad (i_j \neq i_l \quad \text{if} \quad j \neq l)$$

depends only on k and it does not depend on the indices $i_1, i_2, \ldots, i_k$ $(k = 1, 2, \ldots)$. The numbers $\alpha_1, \alpha_2, \ldots$ are called the moments of the quasi-equivalent events $A_1, A_2, \ldots$.

It is clear that any sequence of equivalent events and any sequence of independent events is a sequence of quasi-equivalent events.

Another example for quasi-equivalent events is the following:

Let us consider two urns one of them containing R_1 red balls and W_1 white balls, the other one containing R_2 red balls and W_2 white balls. We suppose that $R_1 + W_1 = R_2 + W_2 = N$. We choose at random one of the urns, with probability p $(0 < p < 1)$ and with probability $q = 1 - p$ we choose the other one. From the chosen urn we choose at random a ball (we choose every ball with the same probability). We put back the ball to the urn and we put in the first urn a red ball with probability p_1 $(p_1 < R_1/R_2)$ and a white ball with probability $q_1 = 1 - p_1$; in the second urn we put a red ball with probability $p_1^* = \lambda\, p_1$ (where $\lambda = R_2/R_1$) and a white ball with probability $q_1^* = 1 - p_1^*$. In the next step we choose a ball at random from the urn from which we have already chosen the first ball. We put back this ball to this urn and we put in the first urn a red ball with probability p_2 $(p_2 < R_2/R_1)$ and a white ball with probability $q_2 = 1 - p_2$; in the second urn we put a red ball with probability $p_2^* = \lambda\, p_2$ and a white ball with probability $q_2^* = 1 - p_2^*$. We continue this process, so that in the k-th step we choose a ball from the urn from which we have chosen the first ball and we put back this ball to this urn and we put in the first urn a red ball with probability p_k $(p_k < R_2/R_1)$ and a white ball with probability $q_k = 1 - p_k$; in the second urn we put a red ball with probability $p_k^* = \lambda p_k$ and a white ball with probability $q_k^* = 1 - p_k^*$.

Let A_k denote the event that we choose in the k-th step a red ball.

It is easy to see that the events A_n are neither independent nor equivalent if $\lambda \neq 1$. We prove that they are quasi-equivalent events. Let B_{I} denote the event that the first ball was chosen from the first urn and B_{II} denote the event that the first ball was chosen from the second urn. It is clear that the events $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are independent under the condition B_I, therefore we have

$$\mathbf{P}(A_{i_1}A_{i_2}\ldots A_{i_k} \mid B_I) = \mathbf{P}(A_{i_1} \mid B_I)\,\mathbf{P}(A_{i_2} \mid B_I)\ldots\mathbf{P}(A_{i_k} \mid B_I)$$

and

$$\mathbf{P}(A_{k+1} \mid B_I) = \frac{1}{N+k}\sum_{j=0}^{k}(R_1 + j)\,\mathbf{P}(\nu_k = j)$$

where ν_k denotes the number of red balls which was put in the first urn at the first k steps. So we have

$$\mathbf{P}(A_{k+1} \mid B_I) = \frac{R_1 + \mathbf{M}(\nu_k)}{N+k} = \frac{R_1 + \alpha_k}{N+k}$$

where $\alpha_k = p_1 + p_2 + \ldots + p_k$. A simple calculation gives

$$\mathbf{P}(A_k) = \mathbf{P}(A_k \mid B_I)\,(p + \lambda q) \qquad (k = 1, 2, \ldots)$$

and similarly

$$\mathbf{P}(A_k) = \mathbf{P}(A_k \mid B_{II})\,(p/\lambda + q) \qquad (k = 1, 2, \ldots)$$

so we have

$$\frac{\mathbf{P}(A_{i_1}A_{i_2}\ldots A_{i_k})}{\mathbf{P}(A_{i_1})\,\mathbf{P}(A_{i_2})\ldots\mathbf{P}(A_{i_k})} = \frac{p}{(p+\lambda q)^k} + \frac{q}{(p/\lambda + q)^k}$$

which proves our statement.

Definition 5. The sequence of events $A_1, A_2, \ldots$ is called quasi-stable if for every $B \in \mathscr{S}$ the limit

$$\lim_{n\to\infty}\frac{\mathbf{P}(A_nB)}{\mathbf{P}(A_n)} = \mu(B)$$

exists.

It is easy to see that a stable sequence is a fortiori a quasi-stable sequence, and the set function μ is a probability measure on $\{\Omega, \mathscr{S}\}$ which is absolutely continuous with respect to $\mathbf{P}$.

In § 1 we give the generalization of Theorem B for quasi-stable sequences and the generalization of Theorems D and E for quasi-equivalent events. § 2 contains a strong law of large numbers for quasi-equivalent events and the characterization of the tail of quasi-equivalent events.

§ 1. The generalizations of Theorems B, D and E

In this § we formulate and prove Theorems 1, 2 and 3 which are the generalizations of Theorems B, D and E resp. The proofs of these theorems are very similar to the original proofs. We can only obtain the generalizations of the mentioned theorems under a restriction. Namely we have to assume that

$$\liminf_{n\to\infty} \mathbf{P}(A_n) > 0.$$

Theorem 1. *Let* $A_1, A_2, \ldots$ *be a sequence of events for which*

$$\liminf_{n\to\infty} \mathbf{P}(A_n) > 0$$

and the limit

$$\alpha_k = \lim_{n\to\infty} \frac{\mathbf{P}(A_n \mid A_k)}{\mathbf{P}(A_n)} \qquad (k = 1, 2, \ldots)$$

exists. Let the random variables $a_k(\omega)$ *and* $\eta_k(\omega)$ $(k = 1, 2, \ldots)$ *be defined as follows*

$$a_k(\omega) = \begin{cases} 1 & \text{if } \omega \in A_k \\ 0 & \text{if } \omega \notin A_k \end{cases}$$

and

$$\eta_k(\omega) = \frac{a_k(\omega)}{\mathbf{P}(A_k)}.$$

Then the events $A_n (n = 1, 2, \ldots)$ *are quasi-stable and the sequence* $\eta_n(\omega)$ *converges weakly to a random variable* $\lambda(\omega)$ *which will be called the relative-density of the sequence* $\{A_n\}$.

Proof. It is easy to see that the conditions of Theorem C are fulfilled (if we substitute f_k by η_k) because η_k is an element of the Hilbert-space $L^2\{\Omega, \mathbf{P}\}$ for which

$$\|\eta_k\| = \frac{1}{\sqrt{\mathbf{P}(A_k)}} \leqq \frac{1+\varepsilon}{\sqrt{\liminf\limits_{n\to\infty} \mathbf{P}(A_n)}} = C$$

if $k \geqq k_0(\varepsilon)$, and

$$\lim_{n\to\infty} (\eta_n, \eta_k) = \lim_{n\to\infty} \frac{1}{\mathbf{P}(A_n)\,\mathbf{P}(A_k)} \int\limits_{\Omega} a_n(\omega)\, a_k(\omega)\, d\mathbf{P} = \lim_{n\to\infty} \frac{\mathbf{P}(A_n A_k)}{\mathbf{P}(A_n)\,\mathbf{P}(A_k)} = \alpha_k .$$

If B is an arbitrary event and

$$\beta(\omega) = \begin{cases} 1 & \text{if } \omega \in B \\ 0 & \text{if } \omega \notin B \end{cases}$$

then by Theorem C we have

$$\lim_{n\to\infty} \frac{\mathbf{P}(A_n B)}{\mathbf{P}(A_n)} = \lim_{n\to\infty} (\eta_n, \beta) = (\lambda, \beta) = \int\limits_B \lambda\, d\mathbf{P}$$

where λ is the weak limit of η_n. So we have proved Theorem 1.

Remark. A simple example shows that Theorem 1 is not valid without the condition $\liminf\limits_{n\to\infty} \mathbf{P}(A_n) > 0$.

The generalization of Theorem D will be given in Theorems 2a and 2b.

Theorem 2a. *If* $A_1, A_2, \ldots$ *is a sequence of quasi-equivalent events with the moments* $\alpha_1, \alpha_2, \ldots$ *such that*

$$\liminf_{n\to\infty} \mathbf{P}(A_n) = K > 0$$

then there exists a distribution function $F(x)$ defined on the interval $[0, 1/K]$ such that

$$\alpha_k = \int_0^{1/K} x^k \, dF(x) .$$

Proof. Let the quasi-density of the sequence $A_1, A_2, \ldots$ of quasi-equivalent events be $\lambda(\omega)$ and denote the indicator function of A_n by $a_n(\omega)$. Let us put

$$\eta_n(\omega) = \frac{a_n(\omega)}{\mathbf{P}(A_n)} \qquad (n = 1, 2, \ldots)$$

and

$$\alpha_k = \frac{\mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k})}{\mathbf{P}(A_{i_1}) \mathbf{P}(A_{i_2}) \ldots \mathbf{P}(A_{i_k})} = \int_\Omega \eta_{i_1} \eta_{i_2} \ldots \eta_{i_k} d\mathbf{P} = (\eta_{i_1} \eta_{i_2} \ldots \eta_{i_{k-1}}, \eta_{i_k}) .$$

Thus by Theorem 1 we have

$$\alpha_k = \lim_{i_k \to \infty} (\eta_{i_1} \eta_{i_2} \ldots \eta_{i_{k-1}}, \eta_{i_k}) = (\eta_{i_1} \eta_{i_2} \ldots \eta_{i_{k-1}}, \lambda) = (\eta_{i_1} \eta_{i_2} \ldots \eta_{i_{k-2}} \lambda, \eta_{i_{k-1}}) .$$

Applying the same argument again we obtain

$$\alpha_k = \lim_{i_{k-1} \to \infty} (\eta_{i_1} \eta_{i_2} \ldots \eta_{i_{k-2}} \lambda, \eta_{i_{k-1}}) = (\eta_{i_1} \eta_{i_2} \ldots \eta_{i_{k-2}} \lambda, \lambda) .$$

Applying the same argument again $k - 2$ times we obtain that

$$\alpha_k = \mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k}) = \int_\Omega \lambda^k(\omega) \, d\mathbf{P} = \int_0^{1/K} x^k \, dF_\lambda(x)$$

where $F_\lambda(x)$ is the distribution function of $\lambda(\omega)$. (It is clear that $\mathbf{P}(0 \leq \lambda(\omega) \leq \leq 1/K) = 1$). Thus Theorem 2a is proved.

Theorem 2b. *If $\lambda(\omega)$ is a random variable such that*

$$\mathbf{P}(0 \leq \lambda(\omega) \leq 1/K) = 1 \quad \textit{and} \quad \int_\Omega \lambda(\omega) \, d\mathbf{P} = 1 ,$$

K is a positive number in the interval $[0, 1]$ and a_k is a sequence of the real numbers for which $0 < a_k < K$ then there exists a sequence of quasi-equivalent events $A_1, A_2, \ldots$ such that

$$\mathbf{P}(A_k) = a_k \tag{1}$$

and

$$\frac{\mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k})}{\mathbf{P}(A_{i_1}) \mathbf{P}(A_{i_2}) \ldots \mathbf{P}(A_{i_k})} = \mathbf{M}(\lambda^k) = \alpha_k . \tag{2}$$

Proof. Let us define a probability space Ω as follows:

$$\Omega = I_1 \times I_2$$

where I_1 is the interval $[0,1/K]$ and I_2 is the interval $[0, 1]$. Let the probability measure $\mathbf{P}$ on Ω be the product measure

$$\mathbf{P} = \mu_1 \times \mu_2$$

where μ_1 is the Lebesgue—Stieltjes measure on I_1 defined by the distribution function $F_\lambda(x) = \mathbf{P}\{\lambda < x\}$ and μ_2 is the ordinary Lebesgue measure on I_2.

To define the events A_n in Ω we need to define a set of polynomials

$$p_k^{(n)}(x) \qquad (k = 0, 1, 2, \ldots, 2^n;\ n = 1, 2, \ldots)$$

as follows $p_0^{(n)}(x) \equiv 0$ and

$$p_{k+1}^{(n)}(x) = \sum_{j=0}^{k} (a_1x)^{\varepsilon_1^{(j)}} (1-a_1x)^{1-\varepsilon_1^{(j)}} (a_2x)^{\varepsilon_2^{(j)}} (1-a_2x)^{1-\varepsilon_2^{(j)}} \ldots (a_nx)^{\varepsilon_n^{(j)}} (1-a_nx)^{1-\varepsilon_n^{(j)}}$$

if $k \geqq 0$, where $\varepsilon_l^{(j)}$ denotes the lth digit in the dyadic expansion of $1 - \dfrac{j+1}{2^n}$ more exactly

$$1 - \frac{j+1}{2^n} = \sum_{i=1}^{n} \frac{\varepsilon_i^{(j)}}{2^i} \qquad (\varepsilon_i \text{ is } 0 \text{ or } 1).$$

Thus for instance

$$p_0^{(3)}(x) \equiv 0$$

$$p_1^{(3)}(x) = a_1a_2a_3x^3$$

$$p_2^{(3)}(x) = a_1a_2a_3x^3 + a_1a_2x^2(1-a_3x) = a_1a_2x^2$$

$$p_4^{(3)}(x) = a_1a_2x^2 + a_1x(1-a_2x)a_3x + a_1x(1-a_2x)(1-a_3x) = a_1x$$

$$p_5^{(3)}(x) = a_1x + (1-a_1x)a_2a_3x^2$$

$$p_6^{(3)}(x) = a_1x + (1-a_1x)a_2a_3x^2 + (1-a_1x)a_2x(1-a_3x) = a_1x + (1-a_1x)a_2x$$

$$p_7^{(3)}(x) = a_1x + (1-a_1x)a_2x + (1-a_1x)(1-a_2x)a_3x$$

$$p_8^{(3)}(x) = a_1x + (1-a_1x)a_2x + (1-a_1x)(1-a_2x)a_3x + (1-a_1x)(1-a_2x)(1-a_3x) \equiv 1.$$

The condition $0 < a_k < K$ implies that $p_0^{(n)}(x) \leqq p_1^{(n)}(x) \leqq \ldots \leqq p_{2^n}^{(n)}(x)$ in the interval $[0, 1/K]$.

Now, let $B_k^{(n)}$ be the set of all points (x, y) of Ω for which $p_{2k}^{(n)}(x) \leqq y < p_{2k+1}^{(n)}(x)$ and let A_n be the union of the sets $B_k^{(n)}$ $(k = 0, 1, 2, \ldots, 2^{n-1}-1)$ i.e.

$$A_n = \sum_{k=0}^{2^{n-1}-1} B_k^{(n)}.$$

It is easy to verify that the events A_n are quasi-equivalent and (1) and (2) hold.

Theorem 3. *Let $\{A_n\}$ be a sequence of quasi-equivalent events for which $\liminf_{n\to\infty} \mathbf{P}(A_n) = K > 0$. Let $\lambda(\omega)$ be the quasi-density of the sequence $\{A_n\}$ considered as a quasi-stable sequence. Then we have*

$$\begin{aligned}(3)\qquad \mathbf{P}(A_{i_1}A_{i_2}\ldots A_{i_k} \mid \lambda) &= \lambda^k \mathbf{P}(A_{i_1})\mathbf{P}(A_{i_2})\ldots\mathbf{P}(A_{i_k}) = \\ &= \mathbf{P}(A_{i_1} \mid \lambda)\ldots\mathbf{P}(A_{i_k} \mid \lambda) \qquad (\textit{with probability } 1)\end{aligned}$$

for $k = 1, 2, \ldots$ and $i_1 < i_2 < \ldots < i_k$.

By other words the events A_n are independent under the condition that the value of $\lambda(\omega)$ is fixed.

Proof. First of all we prove (3) for $k = 1$. Let us put

$$a_k(\omega) = \begin{cases} 1 & \text{if } \omega \in A_k \\ 0 & \text{if } \omega \notin A_k \end{cases}$$

and

$$\eta_k(\omega) = \frac{a_k(\omega)}{\mathbf{P}(A_k)}$$

Let us assume that

$$\mathbf{M}(\eta_n \mid \lambda) = \lambda + \varepsilon_n \,. \tag{4}$$

Here $\varepsilon_n(\omega)$ is a Baire-function of λ by the definition of the conditional probability. Let $\varepsilon_n(\omega) = g_n(\lambda(\omega))$. Then we have

$$\mathbf{M}(\eta_n) = \mathbf{M}(\lambda) = \mathbf{M}(\mathbf{M}(\eta_n \mid \lambda)) = \mathbf{M}(\lambda + \varepsilon_n) = \mathbf{M}(\lambda) + \mathbf{M}(\varepsilon_n)$$

therefore $\mathbf{M}(\varepsilon_n) = 0$ $(n = 1, 2, \ldots)$. Similarly we have

$$\mathbf{M}(\eta_k\eta_l) = \mathbf{M}(\lambda^2) = \mathbf{M}(\lambda\eta_k) = \mathbf{M}[\mathbf{M}(\lambda\eta_k \mid \lambda)] =$$
$$= \mathbf{M}(\lambda(\lambda + \varepsilon_k)) = \mathbf{M}(\lambda^2) + \mathbf{M}(\lambda\varepsilon_k) \,.$$

Therefore $\mathbf{M}(\lambda\varepsilon_k) = 0$. Similarly we obtain

$$\mathbf{M}(\lambda^n\varepsilon_k) = \int_0^{1/K} x^n g_k(x)\, dF_\lambda(x) = 0 \qquad (k = 1, 2, \ldots; n = 1, 2, \ldots)$$

where $F_\lambda(x)$ is the distribution function of $\lambda(\omega)$. (It is clear that $0 \leq \lambda(\omega) \leq 1/K$.) The fact that the sequence $\{x^n\}$ is a complete sequence in the space $L^2_{F\lambda}[0, 1/K]$ (the space of functions in the interval $[0, 1/K]$ which are square integrable with respect to the measure defined by the distribution function $F_\lambda(x)$ (implies that $g_n(x)$ is equal to 0 almost everywhere with respect to the measure defined by $F_\lambda(x)$, so we have

$$\mathbf{P}(\varepsilon_k = 0) = 1 \qquad (k = 1, 2, \ldots)$$

therefore

$$\mathbf{M}(\eta_n \mid \lambda) = \lambda$$

and

$$\mathbf{P}(A_n \mid \lambda) = \lambda\, \mathbf{P}(A_n) \,. \tag{5}$$

The proof for $k = 2$ is completely similar to the above written proof. Let us put

$$\mathbf{M}(\eta_i\eta_k \mid \lambda) = \lambda^2 + \varepsilon_{ik} \,.$$

where ε_{ik} is a Baire-function of λ. With these notations we have

$$\mathbf{M}(\eta_i\eta_k) = \mathbf{M}(\lambda^2) = \mathbf{M}(\mathbf{M}(\eta_i\eta_k \mid \lambda)) = \mathbf{M}(\lambda^2 + \varepsilon_{ik})$$

so

$$\mathbf{M}(\varepsilon_{ik}) = 0 \,.$$

Similarly we have

$$\mathbf{M}(\eta_i\eta_j\eta_k) = \mathbf{M}(\lambda^3) = \mathbf{M}(\eta_i\eta_j\lambda) = \mathbf{M}(\mathbf{M}(\eta_i\eta_j\lambda \mid \lambda)) = \mathbf{M}(\lambda(\lambda^2 + \varepsilon_{ij}))\,.$$

so

$$\mathbf{M}(\varepsilon_{ik}\lambda) = 0$$

and in general we obtain

$$\mathbf{M}(\varepsilon_{ik}\,\lambda^n) = 0 \ (n = 1, 2, \ldots) \quad \text{i. e.} \quad \mathbf{P}(\varepsilon_{ik} = 0) = 1\,.$$

Therefore

$$\mathbf{M}(\eta_i\,\eta_k \mid \lambda) = \lambda^2$$

and

$$\mathbf{P}(A_i\,A_k \mid \lambda) = \lambda^2\,\mathbf{P}(A_i)\,\mathbf{P}(A_k)$$

and using (5) we obtain (3) for $k = 2$.

The proof of (3) for any value of k is essentially the same.

Remark. From Theorem 3 easily follows that $\mathbf{P}\left(0 \leq \lambda \leq \dfrac{1}{\sup \mathbf{P}(A_n)}\right) = 1$ and that it is the best possible estimation follows from Theorem 2b.

§ 2. Some further properties of sequences of quasi-equivalent events

In this § we prove a strong law of large numbers for quasi-equivalent events and we give the characterization of the tail of sequences of quasi-equivalent events.

Theorem 4a. *Let $A_1, A_2, \ldots$ be a sequence of quasi-equivalent events such that*

$$\liminf_{n\to\infty} \mathbf{P}(A_n) = K > 0\,.$$

Let us denote the quasi-density of this sequence by $\lambda(\omega)$. Then we have

$$\mathbf{P}\left\{\frac{1}{n}\sum_{k=1}^{n}\frac{\alpha_k(\omega)}{\mathbf{P}(A_k)} \to \lambda(\omega)\right\} = 1$$

where $\alpha_k(\omega)$ is the indicator function of A_k.

Proof. Let us represent the events $A_1, A_2, \ldots$ in the rectangle $\left[0, \dfrac{1}{K}\right] \times [0,1]$ of the plane as we did in the proof of Theorem 2b. Then by the strong law of large numbers we have

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\alpha_k(x_0, y)}{\mathbf{P}(A_k)} \to \lambda(x_0)$$

for every x_0 in the interval $\left[0, \dfrac{1}{K}\right]$ and for almost every y in $[0, 1]$ (with respect to the ordinary Lebesgue measure). So by the Fubini-theorem we have

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\alpha_k(x, y)}{\mathbf{P}(A_k)} \to \lambda(x)$$

almost everywhere in the rectangle $\left[0, \dfrac{1}{K}\right] \times [0, 1]$.

The validity of the strong law of large numbers does not depend on the concrete representation of the random variables, therefore the proof is complete.

By the same method it is possible to prove the following version of Theorem 4a.

Theorem 4b. *Let* $A_1, A_2, \dots$ *be a sequence of quasi-equivalent events such that*

$$\liminf_{n\to\infty} \mathbf{P}(A_n) = K > 0 .$$

Let us denote the quasi-density of this sequence by $\lambda(\omega)$. *Then we have*

$$\mathbf{P}\left\{\frac{1}{n}\sum_{k=1}^{n}[\alpha_k(\omega) - \lambda\, \mathbf{P}(A_k)] \to 0\right\} = 1$$

where $\alpha_k(\omega)$ *is the indicator function of* A_k.

Theorem 5. *Let* $A_1, A_2, \dots$ *be a sequence of quasi-equivalent events for which* $\liminf_{n\to\infty} \mathbf{P}(A_n) > 0$. *Let* $\lambda(\omega)$ *be the quasi-density of the sequence* $\{A_n\}$, *considered as a quasi-stable sequence. Let us denote the tail of the sequence* $A_1, A_2, \dots$ *by* $\mathcal{A}$. *Then* $\mathcal{A} \sim \mathcal{B}(\lambda)$.

In the proof of this theorem we can follow the known method of the proof of the zero-one law.

Proof. Let A be an element of the σ-algebra $\mathcal{A}$ and let $\mathcal{G}$ be the class of measurable sets F with the property that

$$\mathbf{P}(AF \mid \lambda) = \mathbf{P}(A \mid \lambda)\, \mathbf{P}(F \mid \lambda) \quad \text{(with probability 1)}.$$

Then according to our Theorem 3 $\mathcal{G}$ includes the σ-algebra $\mathcal{B}(A_1, A_2, \dots, A_n)$ $(n = 1, 2, \dots)$. This fact implies that $\mathcal{B}$ includes the σ-algebra $\mathcal{B}(A_1, A_2, \dots)$ and therefore $A \in \mathcal{G}$. So we have

$$\mathbf{P}(A \mid \lambda) = \mathbf{P}(A \mid \lambda)\, \mathbf{P}(A \mid \lambda)$$

i.e. $\mathbf{P}(A|\lambda) = 0$ or $\mathbf{P}(A|\lambda) = 1$ with probability 1. This last fact implies that there is a $B \in \mathcal{B}(\lambda)$ such that $\mathbf{P}(A \circ B) = 0$ and therefore there exists a σ-algebra $\mathcal{B}_1 \subset \mathcal{B}(\lambda)$ for which $\mathcal{B}_1 \sim \mathcal{A}$.

Let us define the random variable $a_i(\omega)$ $(i = 1, 2, \dots)$ as follows:

$$a_k(\omega) = \begin{cases} 1 & \text{if } \omega \in A_k \\ 0 & \text{if } \omega \notin A_k . \end{cases}$$

By Theorem 4 we have

$$\frac{1}{n}\sum_{k=1}^{n} \frac{a_k(\omega)}{\mathbf{P}(A_k)} \to \lambda \qquad \text{(with probability 1)}.$$

It is clear that

$$\mathcal{B}\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{a_k(\omega)}{\mathbf{P}(A_k)}\right) \subset \mathcal{A}$$

i.e.

$$\mathcal{B}(\lambda) \subset \mathcal{A}.$$

So the proof of Theorem 5 is complete.

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О КВАЗИЭКВИВАЛЕНТНЫХ ПОСЛЕДОВАТЕЛЬНОСТЯХ СОБЫТИЙ

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Резюме

Последовательность событий $A_1, A_2, \ldots$ называется квазиэквивалентной, если значение дроби

$$\frac{\mathbf{P}(A_{i_1}A_{i_2}\ldots A_{i_k})}{\mathbf{P}(A_{i_1})\,\mathbf{P}(A_{i_2})\ldots\mathbf{P}(A_{i_k})} \qquad (i_j \neq i_l \text{ если } j \neq l)$$

зависит лишь от k и не зависит от индексов $i_1, i_2, \ldots, i_k$. И вполне независимые события, и эквивалентные события, очевидно, квазиэквивалентны. Цель работы исследовать свойства последовательностей квазиэквивалентных событий.

Основным результатом работы является следующий: Пусть квазиэквивалентные события $A_1, A_2, \ldots$ определены на поле вероятностей $\{\Omega, \mathscr{S}, \mathbf{P}\}$. Предположим, что

$$\liminf_{n\to\infty} \mathbf{P}(A_n) > 0,$$

тогда существует случайная величина $\lambda(\omega)$ такая, что

$$(1) \qquad \mathbf{P}(A_{i_1}A_{i_2}\ldots A_{i_k}\mid\lambda(\omega)) = \lambda^k\,\mathbf{P}(A_{i_1})\,\mathbf{P}(A_{i_2})\ldots\mathbf{P}(A_{i_k}) = \mathbf{P}(A_{i_1}\mid\lambda)\,\mathbf{P}(A_{i_2}\mid\lambda)\ldots\mathbf{P}(A_{i_k}\mid\lambda).$$

$$(2) \qquad \mathbf{P}\left\{\frac{1}{n}\sum_{k=1}^{n}\frac{\alpha_k(\omega)}{\mathbf{P}(A_k)}\to\lambda\right\} = 1$$

где $\alpha_k(\omega)$ индикаторная функция события A_k,

$$(3) \qquad \prod_{n=1}^{\infty}\mathscr{B}(A_n, A_{n+1}, \ldots) = \mathscr{B}(\lambda)$$

где $\mathscr{B}(A_n, A_{n+1}, \ldots)$ обозначает σ-алгебру порожденную событиями A_n, $A_{n+1}, \ldots$, а $\mathscr{B}(\lambda)$ обозначает σ-алгебру порожденную случайной величиной $\lambda(\omega)$. Две σ-алгебры считаются равными, если любой элемент одной из них отличается от некоторого элемента другой лишь на множестве меры нуль и наоборот. Формулу (3) можно рассматривать как обобщение закона нуля и единицы.