ON BI-ORTHOGONAL SYSTEMS OF TRIGONOMETRIC POLYNOMIALS[1]

by

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To Karl Loewner
in friendship

§ 1. Introduction

1. Let $w(\theta)$ be a non-negative function of the class $L(0,2\pi)$ which is not a zero function. It is convenient to interpret it as the weight function of a distribution $w(\theta)d\theta$ on the unit circle $z = e^{i\theta}$. More generally we may consider an arbitrary distribution (measure) $d\alpha(\theta)$ on the unit circle; in what follows, however, we restrict ourselves, for the sake of simplicity, to the previously defined case, i.e., to the case when $\alpha(\theta)$ is absolutely continuous.

It is well known ([1], chapter 2; [2], chapter 11)[3] that a uniquely determined system of polynomials $\{\varphi_n(z)\}$ can be formed which is orthonormal on the unit circle with respect to the given distribution; more precisely,

(a) $\varphi_n(z) = k_n z^n + \ldots + l_n$ is a polynomial of the precise degree n; $k_n > 0$;

(b) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(z)\,\overline{\varphi_m(z)}\, w(\theta)\, d\theta = \delta_{nm}, \quad z = e^{i\theta}; \quad n, m = 0, 1, 2, \ldots .$$

In the following we shall use the standard notation in (a).

In an analogous manner we may consider a weight function $W(x)$ (not a zero function) on the real interval $-1 \leq x \leq 1$, and form the uniquely determined system of orthonormal polynomials $\{p_n(x)\}$ defined by the following conditions:

(a) $p_n(x) = k_n' x^n + \ldots$ is a polynomial of the precise degree n; $k_n' > 0$;

(b) $$\int_{-1}^{1} p_n(x)\, p_m(x)\, W(x)\, dx = \delta_{nm}, \quad n, m = 0, 1, 2, \ldots .$$

A simple and useful relation exists between these two classes of orthonormal polynomials ([2], 11.5). We assume that the weight function $w(\theta)$ of the unit circle is *even*, $w(-\theta) = w(\theta)$. We further assume the following relation between the weight functions $w(\theta)$ and $W(x)$:

$$w(\theta) = W(\cos\theta)\,|\sin\theta|, \tag{1.1}$$

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[3] Numbers refer to the Bibliography.

or in another form, $w(\theta)\,d\theta = W(x)\,|\,dx\,|$ where $x = \cos\theta$. The meaning ot the last condition is obvious; it expresses the invariancy of the mass elemenf in the transition from the upper (or lower) semi-circle to the interval.

Under the condition (1.1) the following identities (1.2) hold:

$$p_n(x) = (2\pi)^{-1/2}\left(1 + \frac{l_{2n}}{k_{2n}}\right)^{-1/2} (z^{-n}\,\varphi_{2n}(z) + z^n\,\varphi_{2n}(z^{-1})) =$$

$$= (2\pi)^{-1/2}\left(1 - \frac{l_{2n}}{k_{2n}}\right)^{-1/2} (z^{-n+1}\,\varphi_{2n-1}(z) + z^{n-1}\,\varphi_{2n-1}(z^{-1}))\,;$$

(1.2)

$$q_{n-1}(x) = \left(\frac{2}{\pi}\right)^{1/2}\left(1 - \frac{l_{2n}}{k_{2n}}\right)^{-1/2} \frac{z^{-n}\,\varphi_{2n}(z) - z^n\,\varphi_{2n}(z^{-1})}{z - z^{-1}} =$$

$$= \left(\frac{2}{\pi}\right)^{1/2}\left(1 + \frac{l_{2n}}{k_{2n}}\right)^{-1/2} \frac{z^{-n+1}\,\varphi_{2n-1}(z) - z^{n-1}\,\varphi_{2n-1}(z^{-1})}{z - z^{-1}}\,; \quad x = \frac{z + z^{-1}}{2}\,.$$

Here $\{\varphi_n(z)\}$ and $\{p_n(x)\}$ have the same meaning as above and $\{q_n(x)\}$ designates the orthonormal system associated with the weight function $(1 - x^2)W(x)$ on $-1 \leq x \leq 1$. These relations can be used for the calculation of the systems $\{p_n(x)\}$ and $\{q_n(x)\}$ provided the system $\{\varphi_n(z)\}$ is known, as well as for the solution of the inverse problem. Indeed, $\varphi_n(z)$ can be expressed as a linear combination of two appropriate polynomials of the systems $\{p_n(x)\}$ and $\{q_n(x)\}$; we observe also that each function $(1 - x^2)\,q_{n-1}(x)$ is a linear combination of $p_{n-1}(x)$, $p_n(x)$ and $p_{n+1}(x)$ [cf. 2, 2.5].

2. The purpose of the present investigation is to extend these relations to the case when the weight $w(\theta)$ is not necessarily even. In this more general case it is convenient to introduce a certain bi-orthogonal system of *trigonometric polynomials* which are orthogonal with respect to the given weight $w(\theta)$. They represent natural generalizations of the simplest bi-orthogonal trigonometric system, namely $\{\cos n\theta, \sin n\theta\}$, corresponding to the weight $w(\theta) = 1$. The trigonometric polynomials thus defined depend only on $w(\theta)$. They can easily be expressed in terms of $\{p_n(x)\}$ and $\{q_n(x)\}$ in the case when $w(\theta)$ is even. From this point of view they appear as certain generalizations of the orthogonal polynomials on a finite interval.

We shall study the principal properties of these trigonometric polynomials systematically; some of these properties are of algebraic (formal) character, some others are of the transcendental (asymptotic) nature. One instance of the latter kind is the question of the asymptotic behavior for large values of the degree and the connected expansion problem. This expansion of an arbitrary function in terms of the bi-orthogonal trigonometric polynomials, represents a very natural generalization of the classical Fourier series.

3. A trigonometric polynomial of degree n with the highest term $a\cos n\theta + a'\sin n\theta$ is called of the precise degree n if the constants a and a' are not both zero. Two trigonometric polynomials $A(\theta) = a\cos n\theta + a'\sin n\theta + \ldots$, $B(\theta) = b\cos n\theta + b'\sin n\theta + \ldots$ of degree n are called linearly independent if the determinant $\begin{vmatrix} a & a' \\ b & b' \end{vmatrix}$ is not zero. As a consequence, $A(\theta)$ and $B(\theta)$ must be of the precise degree n.

Given the weight function $w(\theta)$, we define the corresponding biorthogonal system of trigonometric polynomials $\{A_n(\theta), B_n(\theta)\}$ by the following conditions:

(a) $A_n(\theta)$ and $B_n(\theta)$ are linearly independent trigonometric polynomials of degree n;

(b) they are orthonormal in the following sense:

$$
(1.3)\quad \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} A_n(\theta)\, A_m(\theta)\, w(\theta)\, d\theta = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} B_n(\theta)\, B_m(\theta)\, w(\theta)\, d\theta = \delta_{nm}, \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} A_n(\theta)\, B_m(\theta)\, w(\theta)\, d\theta = 0, \qquad\qquad n, m = 0, 1, 2, \ldots . \end{cases}
$$

The system $\{A_n(\theta), B_n(\theta)\}$ is of course not unique; the most general system of this sort arises by multiplying each vector (matrix) $(A_n(\theta), B_n(\theta))$ with an arbitrary 2×2 orthogonal matrix O_n with constant elements depending on n:

$$
(A_n(\theta), B_n(\theta))\cdot O_n .
$$

As mentioned above we may replace $w(\theta)d\theta$ by an arbitrary distribution $d\alpha(\theta)$.

4. The trigonometric polynomials under consideration can be obtained by a straight-forward application of the Gram-E. Schmidt process. Another way of generating them is a simple relationship which permits us to derive the bi-orthogonal system from the polynomials $\{\varphi_n(z)\}$ defined in 1. The resulting formulas are generalizations of the identities (1.2) to which they reduce when $w(\theta)$ is an even function, $w(-\theta) = w(\theta)$. In this case the bi-orthogonal trigonometric polynomials can be expressed in terms of the polynomials $\{p_n(x)\}$ and $\{q_n(x)\}$ which are orthogonal on the interval $-1 \leq x \leq 1$. Another interesting specialization appears when $w(\theta)$ is the reciprocal of a positive trigonometric polynomial.

There is a simple recurrence relation satisfied by the bi-orthogonal trigonometric polynomials, generalizing the classical difference equation satisfied by the polynomials orthogonal on the real interval $-1 \leq x \leq 1$. Also the location of the zeros of the bi-orthogonal trigonometric polynomials is studied together with a formula for a mechanical quadrature.

In the further course the finite kernel function of the bi-orthogonal system is introduced and its relation to the finite kernel function of the system $\{\varphi_n(z)\}$ is discussed.

This terminates the part dealing with algebraic properties. So far as asymptotic theorems for the trigonometric polynomials are concerned, they can be easily deduced from the corresponding results on $\varphi_n(z)$; the same holds for the equiconvergence theorem of the ,,Fourier expansion''.

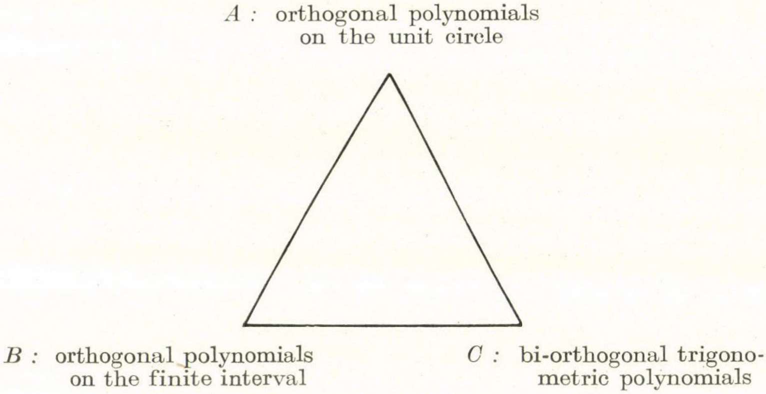
In a short closing section we deal with a corresponding system of surface harmonics which are orthogonal on the unit sphere with respect to a given distribution on this sphere.

5. The relationship between the three orthogonal systems defined above can be described as follows:

The transition from A to B is given by the formulas (1.2); the weight function is even.

The transition from A to C is given by Theorems 1 and 2 of Section 3; the weight function is arbitrary.

The transition from C to B is obtained when the weight in C is even; the suitably normalized bi-orthogonal trigonometric polynomials are in this case pure cosine and pure sine polynomials, respectively, and for $\cos\theta = x$ they yield the polynomials of the type B.



6. Section 2 contains certain preliminaries on trigonometric polynomials in general and on the polynomials orthogonal on the unit circle (cf. A). In Section 3 we describe the generation of the bi-orthogonal system and the transition from A to C. Section 4 deals with a generalization of the recurrence formula well known in the case B. In Section 5 we discuss the location of the zeros of the bi-orthogonal trigonometric polynomials and the associated mechanical quadrature. In Section 6 we define the finite kernel function. In Section 7 we deal with some special cases and in some generality with the asymptotic behavior of the bi-orthogonal trigonometric polynomials and of the kernel function. In Section 8 we prove an equiconvergence theorem. Finally, in Section 9 we define the space analog of C, namely the linear combinations of surface harmonics orthogonal on the unit sphere with respect to a given distribution.

§ 2. Preliminaries

1. An expression of the form

$$a_0 + 2(a_1\cos\theta + b_1\sin\theta) + 2(a_2\cos 2\theta + b_2\sin 2\theta) + \ldots + 2(a_n\cos n\theta + b_n\sin n\theta$$

with real coefficients a_k, b_k is called a trigonometric polynomial of degree n. It is of the precise degree n if a_n and b_n are not both zero. The well known special cases are the cosine and sine polynomials.

Let $A(\theta)$ and $B(\theta)$ be two trigonometric polynomials of degree n. They are linearly independent, if and only if each element of the linear manifold $\lambda\, A(\theta) + \mu\, B(\theta)$ is of the precise degree n, unless λ and μ are both zero. This is equivalent to the definition in 1.3.

In what follows we shall consider certain systems of the form

$$A_0(\theta)\,;\ \{A_n(\theta), B_n(\theta)\} \qquad n = 1, 2, 3, \ldots,$$

where $A_n(\theta)$ and $B_n(\theta)$ are of degree n and linearly independent for each n. Every trigonometric polynomial $T(\theta)$ of degree n can be written in the form

$$T(\theta) = \lambda_0 A_0(\theta) + \lambda_1 A_1(\theta) + \mu_1 B_1(\theta) + \ldots + \lambda_n A_n(\theta) + \mu_n B_n(\theta)$$

where the constants $\lambda_0, \lambda_1, \mu_1, \ldots, \lambda_n, \mu_n$ are uniquely determined. Indeed we can find unique λ_n, μ_n such that

$$T(\theta) - (\lambda_n A_n(\theta) + \mu_n B_n(\theta))$$

will be of degree $n - 1$.

2. A trigonometric polynomial $T(\theta)$ of the precise degree n has exactly $2n$ real or complex zeros $\theta_1, \theta_2, \ldots, \theta_{2n}$ provided we count the zeros as usual with their multiplicity and we restrict ourselves to the strip $-\pi < \mathrm{Re}(\theta) \leqq \pi$.

Let α and β be arbitrary constants; then

$$\sin\frac{\theta-\alpha}{2}\cdot\sin\frac{\theta-\beta}{2}$$

represents a trigonometric polynomial of the first degree. The trigonometric polynomial $T(\theta)$ considered above can be written in the following form:

$$T(\theta) = c\prod_{\nu=1}^{n}\sin\frac{\theta-\theta_{2\nu-1}}{2}\sin\frac{\theta-\theta_{2\nu}}{2}, \qquad c \neq 0 .$$

This representation is of course not unique.

Let $T(\theta)$ vanish for given values $\theta = \alpha_1, \alpha_2, \ldots, \alpha_{2m}$, $m \leqq n$; we form the trigonometric polynomial

$$U(\theta) = \prod_{\nu=1}^{m}\sin\frac{\theta-\alpha_{2\nu-1}}{2}\sin\frac{\theta-\alpha_{2\nu}}{2}$$

of degree m; then $T(\theta)$ is „divisible" by $U(\theta)$, i.e., a trigonometric polynomial $V(\theta)$ of degree $n - m$ can be found such that $T(\theta) = U(\theta)\, V(\theta)$.

3. Let

$$g(z) = c_0 + c_1 z + \ldots + c_n z^n$$

be a rational polynomial of degree n in z with arbitrary complex coefficients. We say that

$$g^*(z) = z^n \bar{g}(z^{-1}) = \bar{c}_n + \bar{c}_{n-1} z + \ldots + \bar{c}_0 z^n$$

is reciprocal to $g(z)$. If $\{z_\nu\}$ are the zeros of $g(z)$, those of $g^*(z)$ will be $\{\bar{z}_\nu^{-1}\}$. (The modification necessary for $z_\nu = 0$ or ∞ is obvious.) A polynomial $g(z)$ is called self-reciprocal, or briefly reciprocal if $g(z) = g^*(z)$, i.e., $c_\nu = \bar{c}_{n-\nu}$. Let $T(\theta)$ be a trigonometric polynomial of degree n; then $T(\theta) = z^{-n}\, G(z)$ where $G(z) = G^*(z)$ is a reciprocal polynomial of degree $2n$, so that $T(\theta) = z^{-n}\, G(z) = z^n\, \overline{G}(z^{-1})$; $z = e^{i\theta}$.

4. **Theorem** of L. Fejér and F. Riesz. *Any trigonometric polynomial $T(\theta)$ which is non-negative for all real θ, can be written in the form $|g(z)|^2$, $z = e^{i\theta}$, where $g(z)$ is a polynomial of the same degree as $T(\theta)$. This representation of $T(\theta)$ will be unique if we subject $g(z)$ to one of the following two conditions:*

(a) $g(z) \neq 0$ *in* $|z| < 1$; $g(0)$ *is real and positive;*

(b) $g(z) \neq 0$ *in* $|z| > 1$; *the leading coefficient of* $g(z)$ *is real and positive.*

Conversely, if $g(z)$ is any rational polynomial, the expression $|g(z)|^2$, $z = e^{i\theta}$, represents a trigonometric polynomial which is non-negative for all real θ.

5. Finally, we note a few basic properties of the orthogonal polynomials $\{\varphi_n(z)\}$ defined in **1.1** (cf. [1], chapter 2; [2], chapter 11).

(a) The polynomial $\varphi_n(z)$ is determined (except for a constant factor) by the following property:

$$\int_{-\pi}^{\pi} \varphi_n(z)\, \overline{q(z)}\, w(\theta)\, d\theta = 0 \qquad z = e^{i\theta},$$

where $q(z)$ is an arbitrary polynomial of degree $n-1$.

(b) The polynomial

$$\sum_{\nu=0}^{n} \overline{\varphi_\nu(a)}\, \varphi_\nu(z) = s_n(a, z) \tag{2.1}$$

is called the kernel polynomial of degree n. It has the reproducing property:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} s_n(a, z)\, \overline{q(z)}\, w(\theta)\, d\theta = \overline{q(a)}\ , \qquad z = e^{i\theta},$$

where $q(z)$ is any polynomial of degree n. This kernel can be represented as follows:

$$s_n(a, z) = \frac{\overline{\varphi^*_{n+1}(a)}\, \varphi^*_{n+1}(z) - \overline{\varphi_{n+1}(a)}\, \varphi_{n+1}(z)}{1 - \bar{a}z}\ . \tag{2.2}$$

(c) From (2.2) we conclude easily the identities where k_n and l_n are defined as in **1.1**:

$$\begin{cases} k_n z \varphi_n(z) = k_{n+1}\, \varphi_{n+1}(z) - l_{n+1}\, \varphi^*_{n+1}(z)\,, \\ k_n\, \varphi_{n+1}(z) = k_{n+1}\, z \varphi_n(z) \ + l_{n+1}\, \varphi^*_n(z)\,. \end{cases} \tag{2.3}$$

(d) We have

$$\sum_{\nu=0}^{n} |l_\nu|^2 = k_n^2\,. \tag{2.4}$$

(e) All zeros of $\varphi_n(z)$ are in the open unit circle $|z| < 1$.

(f) Let $w(\theta) = 1/h(\theta)$ where $h(\theta)$ is a positive trigonometric polynomial of the precise degree h. Representing $h(\theta)$ in the form $|g(z)|^2$, $z = e^{i\theta}$, where $g(z)$ is the (uniquely determined) polynomial of degree h with all its zeros in $|z| < 1$ and such that its leading coefficient is real and positive, we have

$$\varphi_n(z) = z^{n-h} g(z)\,, \qquad n \geq h\,. \tag{2.5}$$

§ 3. Construction of the bi-orthogonal system

1. Let $w(\theta)$ be a given weight function characterizing a distribution on the unit circle. We define a linear function space by the following scalar product and norm:

$$(3.1)\qquad \begin{cases} (f,g) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\, g(\theta)\, w(\theta)\, d\theta\,, \\ \|f\|^2 = (f,f)\,. \end{cases}$$

Here w, f, and g are such that the integrals occurring exist in the Lebesgue sense and w is not a zero function.

With this metric we orthogonalize the elementary functions

$$1,\ \cos\theta,\ \sin\theta,\ \cos 2\theta,\ \sin 2\theta,\ \ldots,\ \cos n\theta,\ \sin n\theta,\ \ldots,$$

arranged in a linear order, according to the Gram—E. Schmidt process. This leads at once to certain trigonometric polynomials $A_n(\theta)$, $B_n(\theta)$ having the property described in 1.3. The matrix of the leading coefficients in $A_n(\theta)$, $B_n(\theta)$ has the form

$$\begin{pmatrix} a & 0 \\ b & b' \end{pmatrix}$$

where a and b' are different from zero. The most general system of this kind arises by the formula

$$A_n(\theta)\cos\delta - B_n(\theta)\sin\delta,\ \pm\left(A_n(\theta)\sin\delta + B_n(\theta)\cos\delta\right),$$

where $\delta = \delta_n$ are arbitrary real constants. In what follows we shall describe another way of generating the same bi-orthogonal systems, based on the polynomials $\{\varphi_n(z)\}$.

In the special case when $w(\theta)$ is even, the functions defined by the Gram—E. Schmidt process are obviously cosine and sine polynomials, respectively.

2. We prove the following

Theorem 1. *Let $\{\varphi_n(z)\}$ be the orthonormal system of polynomials associated with the weight function $w(\theta)$, $\varphi_n(0) = l_n$. Let the angle γ_{2n} be chosen such that* $\exp(-2i\gamma_{2n})\cdot l_{2n}$ *is real. The trigonometric polynomials $f_n(\theta)$ and $g_n(\theta)$ defined by*

$$(3.2)\qquad \exp(-i\gamma_{2n})\cdot z^{-n}\varphi_{2n}(z) = f_n(\theta) + i g_n(\theta), \qquad z = e^{i\theta},\ n = 0, 1, 2, \ldots,$$

satisfy the orthogonality (but not the normalization) conditions (1.3).

Thus multiplying $f_n(\theta)$ and $g_n(\theta)$ by appropriate constants, we obtain another generation of the bi-orthogonal system. We have

$$(3.3)\qquad \begin{aligned} 2f_n(\theta) &= \exp(-i\gamma_{2n})\cdot z^{-n}\varphi_{2n}(z) + \exp(i\gamma_{2n})\cdot z^{n}\overline{\varphi}_{2n}(z^{-1})\,, \\ 2ig_n(\theta) &= \exp(-i\gamma_{2n})\cdot z^{-n}\varphi_{2n}(z) - \exp(i\gamma_{2n})\cdot z^{-n}\overline{\varphi}_{2n}(z^{-1})\,, \qquad z = e^{i\theta}. \end{aligned}$$

We note that γ_{2n} is determined $\bmod(\pi/2)$ provided $l_{2n} \neq 0$. If $l_{2n} = 0$, γ_{2n} is arbitrary. The different choices of γ_{2n} ($\gamma_{2n} \pm \pi/2$ and $\gamma_{2n} + \pi$) cause only unessential changes of the vector $(f_n(\theta), g_n(\theta))$.

The proof is immediate. Indeed, any trigonometric polynomial of degree $n-1$ is a linear combination of the functions $z^\nu = e^{i\nu\theta}$, $-n+1 \leq \nu \leq \leq n-1$. Now

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} z^{-n}\,\varphi_{2n}(z)\,z^\nu w(\theta)\,d\theta =$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_{2n}(z)\,\bar{z}^{n-\nu}\,w(\theta)\,d\theta = 0\,, \qquad z = e^{i\theta}.$$

Here $f_n(\theta)$ and $g_n(\theta)$ are orthogonal to any trigonometric polynomial of degree $n-1$. Further we form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\exp(-i\gamma_{2n})\,z^{-n}\,\varphi_{2n}(z))^2\,w(\theta)\,d\theta = \|f_n\|^2 - \|g_n\|^2 + 2i(f_n, g_n), \quad z = e^{i\theta}.$$

In view of the orthogonality this integral is

$$= \exp(-2i\gamma_{2n})\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_{2n}(z)\,z^{-2n}\,\varphi_{2n}(z)\,w(\theta)\,d\theta =$$

$$= \exp(-2i\gamma_{2n})\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_{2n}(z)\,l_{2n}\,z^{-2n}\,w(\theta)\,d\theta =$$

$$= \exp(-2i\gamma_{2n})\cdot\frac{l_{2n}}{k_{2n}} = \pm\frac{|l_{2n}|}{k_{2n}},$$

so that $(f_n, g_n) = 0$ and $\|f_n\|^2 - \|g_n\|^2 = \pm |l_{2n}|/k_{2n}$. Also, $\|f_n\|^2 + \|g_n\|^2 = 1$, so that

$$\|f_n\|^2 = \frac{1}{2}\left(1 \pm \frac{|l_{2n}|}{k_{2n}}\right), \qquad \|g_n\|^2 = \frac{1}{2}\left(1 \mp \frac{|l_{2n}|}{k_{2n}}\right).$$

Let us choose γ_{2n} such that the upper signs hold. The trigonometric polynomials

$$(3.4)\begin{cases} A_n(\theta) = 2^{1/2}\left(1+\dfrac{|l_{2n}|}{k_{2n}}\right)^{-1/2} f_n(\theta) = \\ = 2^{-1/2}\left(1+\dfrac{|l_{2n}|}{k_{2n}}\right)^{-1/2}(\exp(-i\gamma_{2n})\cdot z^{-n}\,\varphi_{2n}(z)+\exp(i\gamma_{2n})\cdot z^{n}\,\bar{\varphi}_{2n}(z^{-1})), \\ B_n(\theta) = 2^{1/2}\left(1-\dfrac{|l_{2n}|}{k_{2n}}\right)^{-1/2} g_n(\theta) \\ = 2^{-1/2}\left(1-\dfrac{|l_{2n}|}{k_{2n}}\right)^{-1/2}\cdot -i(\exp(-i\gamma_{2n})\cdot z^{-n}\,\varphi_{2n}(z)-\exp(i\gamma_{2n})\cdot z^{n}\,\bar{\varphi}_{2n}(z^{-1})), \end{cases}$$

$$z = e^{i\theta},$$

form then a bi-orthogonal and normalized set. We note that

$$2^{1/2}\exp(-i\gamma_{2n})\cdot z^{-n}\varphi_{2n}(z) = \tag{3.5}$$
$$= \left(1+\frac{|l_{2n}|}{k_{2n}}\right)^{1/2} A_n(\theta) + i\left(1-\frac{|l_{2n}|}{k_{2n}}\right)^{1/2} B_n(\theta), \qquad z=e^{i\theta}.$$

If $\exp(-2i\,\gamma_{2n})\cdot l_{2n} = -|l_{2n}|$, the normalizing factors in (3.4) and (3.5) have to be modified replacing $|l_{2n}|$ by $-|l_{2n}|$.

3. In the first identity (2.3) we replace n by $2n-1$ and we obtain the following formulas in which the same symbols are used as in 2.

Theorem 2. *For each* n

$$\exp(-i\gamma_{2n})\cdot z^{1-n}\varphi_{2n-1}(z) =$$
$$= \left(1-\frac{|l_{2n}|}{k_{2n}}\right)^{1/2}\left(1+\frac{|l_{2n}|}{k_{2n}}\right)^{-1/2} f_n(\theta) + i\left(1+\frac{|l_{2n}|}{k_{2n}}\right)^{1/2}\left(1-\frac{|l_{2n}|}{k_{2n}}\right)^{-1/2} g_n(\theta),$$

$$2\left(1-\frac{|l_{2n}|}{k_{2n}}\right)^{1/2}\left(1+\frac{|l_{2n}|}{k_{2n}}\right)^{-1/2} f_n(\theta) =$$
$$= \exp(-i\gamma_{2n})\cdot z^{1-n}\varphi_{2n-1}(z) + \exp(i\gamma_{2n})\cdot z^{n-1}\bar{\varphi}_{2n-1}(z^{-1}),$$

$$2\left(1+\frac{|l_{2n}|}{k_{2n}}\right)^{1/2}\left(1-\frac{|l_{2n}|}{k_{2n}}\right)^{-1/2} i g_n(\theta) =$$
$$= \exp(-i\gamma_{2n})\cdot z^{1-n}\varphi_{2n-1}(z) - \exp(i\gamma_{2n})\cdot z^{n-1}\bar{\varphi}_{2n-1}(z^{-1}),$$

$$2^{1/2}\exp(-i\gamma_{2n})\cdot z^{1-n}\varphi_{2n-1}(z) =$$
$$= \left(1-\frac{|l_{2n}|}{k_{2n}}\right)^{1/2} A_n(\theta) + i\left(1+\frac{|l_{2n}|}{k_{2n}}\right)^{1/2} B_n(\theta), \qquad z=e^{i\theta}.$$

Indeed, $z=e^{i\theta}$,

$$k_{2n-1} z^{1-n}\varphi_{2n-1}(z) =$$
$$= k_{2n} z^{-n}\varphi_{2n}(z) - l_{2n} z^{n}\bar{\varphi}_{2n}(z^{-1}) =$$
$$= k_{2n}\exp(i\gamma_{2n})\cdot(f_n(\theta)+ig_n(\theta)) - l_{2n}\exp(-i\gamma_{2n})\cdot(f_n(\theta)-ig_n(\theta))$$

and

$$\frac{k_{2n}-\exp(-2i\gamma_{2n})\cdot l_{2n}}{k_{2n-1}} = \frac{k_{2n}-|l_{2n}|}{(k_{2n}^2-|l_{2n}|^2)^{1/2}},$$
$$\frac{k_{2n}+\exp(-2i\gamma_{2n})\cdot l_{2n}}{k_{2n-1}} = \frac{k_{2n}+|l_{2n}|}{(k_{2n}^2-|l_{2n}|^2)^{1/2}}.$$

Here we used (2.4). Again we assumed that $\exp(-2i\,\gamma_{2n})\cdot l_{2n} = |l_{2n}|$. If $\exp(-2i\,\gamma_{2n})\cdot l_{2n} = -|l_{2n}|$, the normalizing factors in Theorem 2 must be modified as in 2.

§ 4. Recurrence relations

1. In the second identity (2.3) we replace n by $2n$ and obtain

$$k_{2n}z^{-n}\varphi_{2n+1}(z) = k_{2n+1}z\cdot z^{-n}\varphi_{2n}(z) + l_{2n+1}z^n\bar{\varphi}_{2n}(z^{-1}),$$

so that in view of Theorems 1 and 2:

$$\begin{aligned} k_{2n}\exp(i\gamma_{2n+2})\cdot(p_n f_{n+1}(\theta) + iq_n g_{n+1}(\theta)) &= \\ = k_{2n+1}\,z\exp(i\gamma_{2n})\cdot(f_n(\theta)+ig_n(\theta)) &+ \\ + l_{2n+1}\exp(-i\gamma_{2n})\cdot(f_n(\theta)-ig_n(\theta)) &= \\ = k_{2n+1}\,z\exp(i\gamma_{2n})\cdot(f_n(\theta)+ig_n(\theta)) &+ \\ + |l_{2n+1}|\exp[i(2\gamma_{2n+1}-\gamma_{2n})]\cdot(f_n(\theta)-ig_n(\theta)), \end{aligned}$$

where

$$p_n = q_n^{-1} = \left(\frac{1-|l_{2n+2}|/k_{2n+2}}{1+|l_{2n+2}|/k_{2n+2}}\right)^{1/2} \tag{4.1}$$

and

$$\exp(-2i\gamma_{2n+1})\cdot l_{2n+1} = |l_{2n+1}|, \qquad z = e^{i\theta}.$$

Hence we conclude the relations

$$\begin{cases} k_{2n}p_n f_{n+1}(\theta) = (k_{2n+1}\cos(\theta+\delta_n) + |l_{2n+1}|\cos\delta_n')f_n(\theta) + \\ \qquad + (-k_{2n+1}\sin(\theta+\delta_n) + |l_{2n+1}|\sin\delta_n')g_n(\theta); \\ k_{2n}q_n g_{n+1}(\theta) = (k_{2n+1}\sin(\theta+\delta_n) + |l_{2n+1}|\sin\delta_n')f_n(\theta) + \\ \qquad + (k_{2n+1}\cos(\theta+\delta_n) - |l_{2n+1}|\cos\delta_n')g_n(\theta), \end{cases} \tag{4.2}$$

where

$$\delta_n = \gamma_{2n} - \gamma_{2n+2},\ \delta_n' = 2\gamma_{2n+1} - \gamma_{2n} - \gamma_{2n+2}. \tag{4.3}$$

It is easy to transcribe these relations into some others between the normalized functions $A_n(\theta)$, $B_n(\theta)$. Here we have chosen $\exp(-2i\gamma_{2n+2})\cdot l_{2n+2}$ and $\exp(-2i\gamma_{2n+1})\cdot l_{2n+1}$ to be positive ($\geqq 0$) and $\exp(-2i\gamma_{2n})\cdot l_{2n}$ real. In the case when the first of these three quantities is negative, p_n and q_n must be interchanged; when the second quantity is negative, $|l_{2n+1}|$ must be replaced by $-|l_{2n+1}|$.

The relations (4.2) can be written in the matrix form

$$(f_{n+1}(\theta), g_{n+1}(\theta)) = (f_n(\theta), g_n(\theta))\begin{pmatrix} a_n(\theta) & b_n(\theta) \\ c_n(\theta) & d_n(\theta)\end{pmatrix} \tag{4.4}$$

where the elements of the 2×2 matrix are trigonometric polynomials of the first order.

A typical instance is the trivial case $w(\theta) = 1$; the recurrences assume the form

$$\begin{cases} \cos(n+1)\theta = \cos\theta\cos n\theta - \sin\theta\sin n\theta, \\ \sin(n+1)\theta = \sin\theta\cos n\theta + \cos\theta\sin n\theta. \end{cases} \tag{4.5}$$

2. The special case when $w(\theta)$ is even, $w(-\theta) = w(\theta)$, is of particular interest; the recurrences (4.2) become then the classical recurrence relations for orthogonal polynomials of a finite real interval. Indeed, in this case all coefficients of the polynomials $\varphi_n(z)$ are real, thus l_n is real. We choose $\gamma_{2n} = 0$ for all n, so that $f_n(\theta)$ will be a cosine and $g_n(\theta)$ a sine polynomial. Also we choose $\gamma_{2n+1} = 0$ or $\pi/2$ according as l_{2n+1} is positive or negative. We have then the relations

$$\text{(4.6)}\qquad \begin{cases} k_{2n}p_n f_{n+1}(\theta) = (k_{2n+1}\cos\theta + l_{2n+1})\, f_n(\theta) - k_{2n+1}\sin\theta\cdot g_n(\theta)\,, \\ k_{2n}q_n g_{n+1}(\theta) = k_{2n+1}\sin\theta\cdot f_n(\theta) + (k_{2n+1}\cos\theta - l_{2n+1})\, g_n(\theta)\,, \end{cases}$$

$$p_n = q_n^{-1} = \left(\frac{1 - l_{2n+2}/k_{2n+2}}{1 + l_{2n+2}/k_{2n+2}}\right)^{1/2}.$$

From the first equation we derive

$$-k_{2n+1}\sin\theta\cdot g_n(\theta) = k_{2n}p_n f_{n+1}(\theta) - (k_{2n+1}\cos\theta + l_{2n+1})\, f_n(\theta)\,,$$

$$-k_{2n+3}\sin\theta\cdot g_{n+1}(\theta) = k_{2n+2}p_{n+1} f_{n+2}(\theta) - (k_{2n+3}\cos\theta + l_{2n+3})\, f_{n+1}(\theta)\,.$$

Combining this with the second equation (4.6) and taking the identity

$$k_{2n+1}\sin^2\theta + (k_{2n+1}\cos\theta - l_{2n+1})\,\frac{k_{2n+1}\cos\theta + l_{2n+1}}{k_{2n+1}} = k_{2n+1} - \frac{l_{2n+1}^2}{k_{2n+1}}$$

into account, we obtain a recurrence of the classical type

$$f_{n+2}(\theta) = (r_n\cos\theta + s_n)\, f_{n+1}(\theta) + t_n f_n(\theta)$$

for $f_n(\theta)$. Similarly, we can derive a recurrence for $g_n(\theta)$.

§ 5. Zeros. Mechanical quadrature

1. We use the previous notation and prove

Theorem 3. *Let a and b be real constants, not both zero. The trigonometric polynomial $af_n(\theta) + bg_n(\theta)$ has real and distinct zeros. The zeros of $f_n(\theta)$ and $g_n(\theta)$ are interlacing each other.*

More generally, the zeros of

$$af_n(\theta) + bg_n(\theta), \quad -bf_n(\theta) + ag_n(\theta)$$

interlace. The assertion is an immediate consequence of the argument principle applied to the rational function

$$(a - ib)\exp(-i\gamma_{2n})\cdot z^{-n}\,\varphi_{2n}(z) = w\,,$$

which has a pole of order n at the origin and $2n$ zeros in $|z| < 1$. (The modification is obvious if $z = 0$ is a zero of $\varphi_{2n}(z)$.) Hence the index number of the curve described by w as $|z| = 1$, is $2n - n = n$. Consequently every half ray issued from the origin $w = 0$, will be intersected by this curve at least n times in such a manner that the argument of w at the point of intersection is increasing. Choosing for these rays the real and imaginary semi-axes, we obtain at least $2n$, hence exactly $2n$ zeros for both trigonometric polynomials n question. The interlacing property will be obvious also.

2. **Theorem 4.** *Let us denote by $\theta_1, \theta_2, \ldots, \theta_{2n}$ the zeros of the trigonometric polynomial $af_n(\theta) + bg_n(\theta)$ defined in Theorem 3. There exist certain positive constants $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ such that for any trigonometric polynomial $T(\theta)$ of degree $2n - 1$ the following identity holds:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} T(\theta)\, w(\theta)\, d\theta = \sum_{\nu=1}^{2n} \lambda_\nu T(\theta_\nu). \tag{5.1}$$

The proof follows the classical pattern. We write $u(\theta) = af_n(\theta) + bg_n(\theta)$ and form the expression

$$h(\theta) = \sum_{\nu=1}^{2n} T(\theta_\nu)\left(\frac{u(\theta)}{2u'(\theta_\nu)\sin\frac{\theta-\theta_\nu}{2}}\right)^2.$$

Here $u'(\theta_\nu) \neq 0$. Let μ be any value different from ν; there exists a trigonometric polynomial $v(\theta)$ of degree $n - 1$ such that

$$u(\theta) = \sin\frac{\theta-\theta_\nu}{2}\sin\frac{\theta-\theta_\mu}{2}\cdot v(\theta),$$

so that

$$\left(\frac{u(\theta)}{2u'(\theta_\nu)\sin\frac{\theta-\theta_\nu}{2}}\right)^2 = \left(\frac{\sin\frac{\theta-\theta_\mu}{2}}{2u'(\theta_\nu)}\right)^2 (v(\theta))^2.$$

Hence $h(\theta)$ is a trigonometric polynomial of degree $2n - 1$, and obviously $h(\theta_\nu) = T(\theta_\nu)$. Consequently, $T(\theta) - h(\theta) = u(\theta)\, v_1(\theta)$ where $v_1(\theta)$ is of degree $n - 1$, so that

$$\int_{-\pi}^{\pi} (T(\theta) - h(\theta))\, w(\theta)\, d\theta = 0.$$

Writing

$$\lambda_\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{u(\theta)}{2u'(\theta_\nu)\sin\frac{\theta-\theta_\nu}{2}}\right)^2 w(\theta)\, d\theta$$

the assertion follows immediately.

§ 6. Kernel function

1. We define the finite reproducing kernel function of the bi-orthogonal system by

$$A_0(\alpha)\, A_0(\theta) + \sum_{\nu=1}^{n} (A_\nu(\alpha)\, A_\nu(\theta) + B_\nu(\alpha)\, B_\nu(\theta)) = K_n(\alpha, \theta) \tag{6.1}$$

where α and θ are real variables. Obviously, each term $A_\nu(\alpha)\, A_\nu(\theta) + B_\nu(\alpha)\, B_\nu(\theta)$ will be invariant if we multiply the vector $(A_\nu(\theta), B_\nu(\theta))$ by an arbitrary orthogonal matrix with real constant elements. Thus, $K_n(\alpha, \theta)$ depends only on the weight function $w(\theta)$ (and, of course, on α, θ, n).

The kernel function possesses the reproducing property

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\alpha, \theta) \cdot t(\theta)\, w(\theta)\, d\theta = t(\alpha), \tag{6.2}$$

where $t(\theta)$ is an arbitrary trigonometric polynomial of degree n.

2. **Theorem 5.** *Let $s_n(a, z)$ be the kernel function associated with the system $\{\varphi_n(z)\}$; let $a = e^{i\alpha}$, $z = e^{i\theta}$. We have the identity*

$$K_n(\alpha, \theta) = (a\bar{z})^n\, s_{2n}(a, z). \tag{6.3}$$

Thus, the right-hand side is real. This follows also from the identity $s_{2n}(a, z) = (\bar{a}z)^{2n}\, s_{2n}(\bar{z}^{-1}, \bar{a}^{-1})$ [cf. 2, (11.3.4)] for $|a| = |z| = 1$.

For the proof of (6.3) we verify that the right-hand expression has the reproducing property. We choose $t(\theta) = z^\nu = e^{i\nu\theta}$ where ν is an integer, $-n \leqq \nu \leqq n$. We have, using the reproducing property of $s_{2n}(a, z)$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (a\bar{z})^n\, s_{2n}(a, z) \cdot t(\theta)\, w(\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} a^n\, s_{2n}(a, z) \cdot \bar{z}^{n-\nu}\, w(\theta)\, d\theta =$$

$$= a^n \cdot \bar{a}^{n-\nu} = a^\nu = t(\alpha).$$

3. Combining Theorem 5 with (2.2) and with the last formula in Theorem 2, we obtain a closed form for the kernel function (6.1). Indeed,

$$K_{n-1}(\alpha, \theta) = (a\bar{z})^{n-1}\, \frac{\overline{\varphi_{2n-1}^*(a)}\, \varphi_{2n-1}^*(z) - \overline{\varphi_{2n-1}(a)}\, \varphi_{2n-1}(z)}{1 - \bar{a}z} =$$

$$= \frac{(\bar{a}z)^{n-1/2}\, \varphi_{2n-1}(a)\, \overline{\varphi_{2n-1}(z)} - (a\bar{z})^{n-1/2}\, \overline{\varphi_{2n-1}(a)}\, \varphi_{2n-1}(z)}{(a\bar{z})^{1/2} - (\bar{a}z)^{1/2}} =$$

$$= \frac{\operatorname{Im} \{(\bar{a}z)^{n-1/2}\, \varphi_{2n-1}(a)\, \overline{\varphi_{2n-1}(z)}\}}{\operatorname{Im} \{(a\bar{z})^{1/2}\}}.$$

Writing $2r_n = 1 - |l_{2n}|/k_{2n}$, $2s_n = 1 + |l_{2n}|/k_{2n}$, we find

$$\operatorname{Im}\left\{(\bar{a}z)^{n-1/2}\varphi_{2n-1}(a)\,\overline{\varphi_{2n-1}(z)}\right\} =$$

$$= \operatorname{Im}\left\{(\bar{a}z)^{1/2}\left(r_n^{1/2}A_n(\alpha) + is_n^{1/2}B_n(\alpha)\right)\left(r_n^{1/2}A_n(\theta) - is_n^{1/2}B_n(\theta)\right)\right\} =$$

$$= (r_n s_n)^{1/2}\cos\frac{\theta-\alpha}{2}\left(A_n(\theta)B_n(\alpha) - A_n(\alpha)B_n(\theta)\right) +$$

$$+ \sin\frac{\theta-\alpha}{2}\left(r_n A_n(\alpha)A_n(\theta) + s_n B_n(\alpha)B_n(\theta)\right),$$

so that

$$K_{n-1}(\alpha,\theta) = \frac{1}{2}\frac{k_{2n-1}}{k_{2n}}\operatorname{ctg}\frac{\theta-\alpha}{2}\left(A_n(\alpha)B_n(\theta) - A_n(\theta)B_n(\alpha)\right) - \tag{6.4}$$

$$- \left(r_n A_n(\alpha)A_n(\theta) + s_n B_n(\alpha)B_n(\theta)\right).$$

Here we used the formula

$$2\,r_n \cdot 2\,s_n = 1 - \frac{|l_{2n}|^2}{k_{2n}^2} = \left(\frac{k_{2n-1}}{k_{2n}}\right)^2, \tag{6.5}$$

which is a consequence of (2.4).

§ 7. Special cases; asymptotic behavior

1. Let us consider the special case defined in Section 2.5 (f): $w(\theta) = 1/h(\theta)$ where $h(\theta) = |g(z)|^2$, $z = e^{i\theta}$, is a trigonometric polynomial of the precise degree h; $g(z)$ is a polynomial of degree h, all zeros of which are in $|z| < 1$, and $g(z)$ has a positive leading coefficient.

We assume that $2n - 1 \geqq h$. In view of (2.5) we have $l_{2n} = 0$ so that $\gamma_{2n} = \gamma$ is arbitrary. We have by the last formula in Theorem 2:

$$2^{1/2}\,e^{-i\gamma}z^{n-h}g(z) = A_n(\theta) + iB_n(\theta), \qquad z = e^{i\theta}. \tag{7.1}$$

Here γ is arbitrary real. Formula (3.5) yields the same result, and in addition also the case $2n = h$; we have then $l_{2n} = l_h = g(0)$ and γ is defined as in Theorem 1.

2. Now let $w(\theta)$ be a positive weight function defined on the unit circle and satisfying the Lipschitz-Dini condition

$$|w(\theta+\delta) - w(\theta)| < L|\log\delta|^{-1-\lambda}, \tag{7.2}$$

where L and λ are positive constants. This case was considered in [2], chapters 10, 12 and 13; we refer here to those results which are relevant for our purposes.

There exists an analytic function $D(z)$ regular for $|z| < 1$ and continuous for $|z| \leqq 1$, such that $w(\theta) = |D(e^{i\theta})|^2$. By the additional conditions $D(z) \neq 0$ in $|z| < 1$ and $D(0) > 0$, this function is uniquely determined.

For each n there exists a polynomial $h(z)$ of degree $n-1$, $h(z) \neq 0$ for $|z| \leq 1$, such that

$$|D(z) - (h(z))^{-1}| < Q(\log n)^{-\lambda} \tag{7.3}$$

uniformly in $|z| \leq 1$ [cf. 2, (10.3.12)]. From this we conclude [cf. 2, (13.7.4)] that

$$|D(e^{i(\theta+\delta)}) - D(e^{i\theta})| < L'|\log \delta|^{-\lambda}. \tag{7.4}$$

Finally we have [2, Theorem 12.1.3]

$$\varphi_n(z) = z^n\{\overline{D(z)}\}^{-1} + \varepsilon_n(z),\ |\varepsilon_n(z)| < C(\log n)^{-\lambda}, \qquad z = e^{i\theta}. \tag{7.5}$$

Here Q and C depend only on L, λ, and on the minimum and maximum of $w(\theta)$. We have, $z = e^{i\theta}$,

$$\begin{aligned} l_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(z)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} z^n\{\overline{D(z)}\}^{-1} d\theta + O[(\log n)^{-\lambda}] = \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} z^n[\{\overline{D(z)}\}^{-1} - \overline{h(z)}]\, d\theta + O[(\log n)^{-\lambda}] = O[(\log n)^{-\lambda}]\,. \end{aligned} \tag{7.6}$$

Thus we find from (3.5) that for $n \to \infty$, $z = e^{i\theta}$,

$$A_n(\theta) + iB_n(\theta) = 2^{1/2}\exp(-i\gamma_{2n})\cdot z^n\{\overline{D(z)}\}^{-1} + O[(\log n)^{-\lambda}]\,. \tag{7.7}$$ [4]

We conclude that $A_n(\theta)$ and $B_n(\theta)$ are uniformly bounded as $n \to \infty$. Another important consequence is that for $a = e^{i\alpha}$, $z = e^{i\theta}$, we have

$$\begin{aligned} &\frac{1}{2}\,\frac{k_{2n-1}}{k_{2n}}\,(A_n(\alpha)\,B_n(\theta) - A_n(\theta)\,B_n(\alpha)) = \\ &= \frac{1}{2}\,\frac{k_{2n-1}}{k_{2n}}\,\mathrm{Im}\left\{(A_n(\alpha) - iB_n(\alpha))\,(A_n(\theta) + iB_n(\theta))\right\} = \\ &= \frac{k_{2n-1}}{k_{2n}}\,\mathrm{Im}\left\{\exp[in(\theta-\alpha)]\,[D(a)]^{-1}\,[\overline{D(z)}]^{-1}\right\} + O\,[(\log n)^{-\lambda}]\,, \\ &\qquad\qquad\qquad a = e^{i\alpha},\ z = e^{i\theta}\,, \end{aligned} \tag{7.8}$$

since k_{2n-1}/k_{2n} is bounded. We note also that in view of (2.4) and (7.6)

$$\frac{k_{2n-1}}{k_{2n}} = 1 + O(|l_{2n}|^2) = 1 + O[(\log n)^{-2\lambda}]\,. \tag{7.9}$$

This formula will be used later.

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[4] It would be possible to make more precise statements about the constants occurring in the various remainder terms. For the sake of brevity we omit these details.

§ 8. Equiconvergence

1. Based on the previous preparations we prove now the following:

Theorem 6. *Let $f(\theta)$ be an arbitrary bounded and measurable function. We denote by*

$$s_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, K_n(\alpha, \theta)\, w(\theta)\, d\theta \tag{8.1}$$

the n^{th} partial sum of the expansion of $f(\theta)$ in the generalized Fourier series proceeding in terms of the bi-orthogonal trigonometric polynomials associated with the weight function $w(\theta)$. Here $w(\theta)$ satisfies the Lipschitz-Dini condition (7.2) with $\lambda > 1$.

We denote by

$$s'_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \frac{\sin (2n+1)\dfrac{\theta-\alpha}{2}}{\sin \dfrac{\theta-\alpha}{2}}\, d\theta \tag{8.2}$$

the n^{th} partial sum of the ordinary Fourier series of $f(\theta)$. Both s_n and s'_n are taken at $\theta = \alpha$. Then

$$\lim_{n\to\infty} (s_n - s'_n) = 0\,. \tag{8.3}$$

2. We make first some preliminary observations.

(a) As remarked in Section 7, $A_n(\theta)$ and $B_n(\theta)$ are uniformly bounded as $n \to \infty$. By Riemann's Lemma

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(\theta)\, w(\theta)\, A_n(\theta)\, d\theta = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(\theta)\, w(\theta)\, B_n(\theta)\, d\theta = 0\,.$$

The assertions

$$\lim_{n\to\infty} (s_{n-1} - s'_n) = \lim_{n\to\infty} \left(s_{n-1} - \frac{k_{2n-1}}{k_{2n}} s'_n \right) = 0 \tag{8.4}$$

are equivalent with (8.3) since $s'_n = O(\log n)$ (Lebesgue constants) so that in view of (7.9), $2\lambda > 1$, we have

$$\left(\frac{k_{2n-1}}{k_{2n}} - 1 \right) s'_n = o(1)\,, \qquad \text{as } n \to \infty\,.$$

We shall use (6.4). The contribution of the second term of (6.4) to s_{n-1} tends to zero.

(b) We assume that $|\,\theta - \alpha\,| < \varepsilon/n$ where $\varepsilon > 0$ is independent of n. The expression (7.8) as a function of θ is uniformly bounded; hence, by S. Bernstein's theorem, its derivative is $O(n)$, so that the corresponding parts of the integrals (8.1) and (8.2) are equal to $\varepsilon \cdot O(1/n) \cdot O(n) = \varepsilon \cdot O(1)$.

(c) Let $|\theta - \alpha| \geqq \varepsilon/n$. Taking (8.4) into account, the first term in the last expression of (7.8) will yield:

$$(8.5)\qquad \frac{1}{2\pi}\int f(\theta)\,\mathrm{Im}\left\{\exp\left[i(n+1/2)(\theta-\alpha)\right]\left(\sin\frac{\theta-\alpha}{2}\right)^{-1}\times\right.$$
$$\left.\times\left[\frac{D(z)}{D(a)}\exp\left[-i(\theta-\alpha)/2\right]\cos\frac{\theta-\alpha}{2}-1\right]\right\}d\theta\,,\ a=e^{i\alpha},\ z=e^{i\theta},$$

where we integrate over $|\theta-\alpha| \geqq \varepsilon/n$. Now by (7.4)

$$\frac{D(z)}{D(a)}-1=O\left[\left|\log|\theta-\alpha|\right|^{-\lambda}\right]$$

and

$$|\theta-\alpha|^{-1}\cdot\left|\log|\theta-\alpha|\right|^{-\lambda}$$

is integrable. The same holds for

$$\left(\sin\frac{\theta-\alpha}{2}\right)^{-1}\left[\exp\left[-i(\theta-\alpha)/2\right]\cos\frac{\theta-\alpha}{2}-1\right]$$

(this function is continuous), so that

$$(8.6)\qquad f(\theta)\cdot\left(\sin\frac{\theta-\alpha}{2}\right)^{-1}\left[\frac{D(z)}{D(a)}\exp\left[-i(\theta-\alpha)/2\right]\cos\frac{\theta-\alpha}{2}-1\right]$$

is integrable. Adding now to (8.5) the same integral extended over $|\theta-\alpha| < \varepsilon/n$, the added part will be

$$O(1)\cdot\int\limits_{|\theta-\alpha|<\varepsilon/n}|\theta-\alpha|^{-1}\cdot|\log(\theta-\alpha)|^{-\lambda}\,d\theta=O[(\log n)^{1-\lambda}]=o(1)\ \text{as}\ n\to\infty,$$

since $\lambda > 1$. Using Riemann's Lemma, the total expression (8.5) (extended over the whole period $-\pi \leqq \theta \leqq \pi$) tends to zero as $n \to \infty$.

(d) Finally we deal with the contribution of the remainder term in (7.8). Since

$$\int\limits_{|\theta-\alpha|\geqq\varepsilon/n}\left|\sin\frac{\theta-\alpha}{2}\right|^{-1}d\theta=O(\log n)\,,$$

we obtain $O(\log n)$. $O[(\log n)^{-\lambda}] = o(1)$, taking again $\lambda > 1$ into account.

Thus the theorem on equiconvergence is established.

§ 9. Problems on the sphere

There is an analog of the bi-orthogonal trigonometric polynomials in higher dimensional euclidean spaces. They are linear combinations of surface harmonics orthogonal on the unit sphere with respect to a given weight function. The construction described in 3.1 can be applied. However, the problems about nodal lines, asymptotic behavior and equiconvergence seem to be rather difficult. There is, of course, no analog of the polynomials $\{\varphi_n(z)\}$ of a complex variable z. For the sake of simplicity we restrict ourselves to the three-dimensional case.

1. Let θ and φ be the usual coordinates on the unit sphere, θ the distance from the pole, $0 \leqq \theta \leqq \pi$, $-\pi \leqq \varphi < \pi$. Let $w(\theta, \varphi)$ be a positive and continuous weight function on the unit sphere. We denote the surface harmonics of degree n, namely the functions

$$P_n(\cos\theta); \quad (\sin\theta)^\nu P_n^{(\nu)}(\cos\theta)\exp(i\nu\varphi), \qquad 1 \leqq \nu \leqq n, \tag{9.1}$$

briefly (in any fixed order) by

$$Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_N^{(n)}, \qquad N = 2n+1. \tag{9.2}$$

We apply now the Gram—E. Schmidt process to the functions

$$Y_k^{(m)}, \ 1 \leqq k \leqq 2m+1; \ m = 0, 1, \ldots, n-1; \ Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_h^{(n)}, \tag{9.3}$$

where $1 \leqq h \leqq 2n+1$. The ordering of the systems $\{Y_k^{(m)}\}$, $m < n$, is immaterial, but the ordering of the harmonics of degree n is essential. The scalar product is defined as follows:

$$(f, g) = \frac{1}{4\pi}\int_0^\pi\int_{-\pi}^\pi f(\theta,\varphi)\, g(\theta,\varphi)\, w(\theta,\varphi)\, d\theta\, d\varphi. \tag{9.4}$$

We obtain certain surface harmonics of degree $\leqq n$; the terms of degree n can be described as a linear transformation of $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_{2n+1}^{(n)}$, the matrix of which has main diagonal elements $\neq 0$ and all elements *above* the main diagonal are $= 0$.

The most general orthogonal system of this kind arises by applying to the system of functions of degree n, thus defined, an arbitrary orthogonal transformation of order $2n+1$ with constant coefficients.

2. The „zonal" case, i.e., the case when $w(\theta, \varphi)$ is independent of φ, $w(\theta, \varphi) = w(\theta)$, allows certain simplifications. We have then an orthogonal system of the following kind:

$$p_n(\cos\theta); \quad (\sin\theta)^\nu p_{n\nu}(\cos\theta)\exp(i\nu\varphi), \qquad 1 \leqq \nu \leqq n. \tag{9.5}$$

Here $p_n(x)$ and $p_{n\nu}(x)$ are polynomials of degree n and $n - \nu$, respectively, satisfying the following orthogonality conditions [cf. 1.1]:

$$\begin{cases} \displaystyle\int_{-1}^{1} p_n(x)p_m(x)\, W(x)\, dx = \delta_{nm}, & n, m = 0, 1, 2, \ldots; \\ \displaystyle\frac{1}{2}\int_{-1}^{1} p_{n\nu}(x)p_{m\nu}(x)\cdot(1-x^2)^\nu W(x)\, dx = \delta_{nm}, & n, m = \nu, \nu+1, \nu+2, \ldots. \end{cases} \tag{9.6}$$

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О БИОРТОГОНАЛЬНЫХ СИСТЕМАХ ТРИГОНОМЕТРИЧЕСКИХ МНОГОЧЛЕНОВ

G. SZEGŐ

Резюме

Пусть $w(\theta)$ неотрицательная, L-интегрируемая, не идентично исчезающая весовая функция с периодом 2π. Тогда существует система тригонометрических многочленов $(f_n(\theta), g_n(\theta))$, ортонормированных с весом $w(\theta)$ в смысле (1.3). Функция $w(\theta)$ определяет эти многочлены с точностью до ортогональных преобразований векторов $(f_n(\theta), g_n(\theta))$. В том частном случае, когда $w(\theta)$ является чётной функцией, функции $f_n(\theta)$ и $g_n(\theta)$ можно легко выразить через некоторые многочлены переменной $x = \cos\theta$, являющиеся ортогональными на отрезке $-1 \leqq x \leqq 1$, в отношении подходящих весов на этом отрезке. В этом смысле наши тригонометрические многочлены представляют собой обобщение обыкновенных ортогональных тригонометрических многочленов конечного отрезка. Существует также тесная связь между многчленами $f_n(\theta)$ и $g_n(\theta)$ и многочленами величины $e^{i\theta}$, являющимися ортонормированными относительно подходящей весовой функции на единичной окружности. В статье исследуются разные свойства функции $f_n(\theta)$ и $g_n(\theta)$, например рекурсивные соотношения места нулей, керн-функция, механическая квадратура, и т. д. Кроме того, исследуется для больших индексов n асимптотическое поведение функций $f_n(\theta)$ и $g_n(\theta)$ и устанавливается одна теорема равносходимости. При помощи последней теоремы могут быть легко обобщены некоторые классические свойства обыкновенных рядов Фур'е.

Аналогические системы ортонормированных функций, представляющих собой соответствующие обобщения классических сферических функций Лапласа, могут быть исследованы на шаре.