HERMITE EXPANSION AND DISTRIBUTION OF ZEROS OF POLYNOMIALS

by

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To Paul Erdős on his 50th birthday.

1. Some time ago one of us (see [1]) realised that in problems when reality of zeros are concerned the Hermite-expansion seems to be a more appropriate tool than power-series expansion. This fact lends interest to the general problem to study the functional algebra of the Hermite-expansion of polynomials, i.e., the influence of the coefficients of its Hermite-expansion upon the distribution of its zeros. Various results have been reached in this direction (see [2], [3], [4]) but to one problem, raised in [1], no progress was made. This refers to the interesting question, does there exist a theory of the Hermite-expansion corresponding to the LANDAU—FEJÉR—MONTEL-theory of polynomials (see [5]). In this paper we are going to give the first theorem in this direction. The Hermite-polynomials $H_\nu(z)$ are always meant with the normalisation

$$e^{-z^2} H_\nu(z) = (-1)^\nu (e^{-z^2})^{(\nu)} . \tag{1.1}$$

Writing $z = x + iy$ we assert the following

Theorem I. *The „Hermite-trinomial" equation (ζ arbitrary complex)*

$$f(x) \overset{\text{def.}}{=} 1 + H_1(z) + \zeta H_n(z) = 0 \tag{1.2}$$

has always at least one of its zeros in the strip $|y| \leq A$ with a positive numerical constant A.

This will be a simple consequence of the following

Theorem II. *The equation* (1.2) *has for $n \geq 36$ at least one zero in the strip $|y| \leq e^3$.*

The proof of this theorem will not be very delightful; but a first proof can be as ugly as it wants to be.

This paper seems to us fit for dedication to P. ERDŐS who enriched the theory of polynomials by so many ingenious contributions.

2. For the proof we shall need some facts, more or less known from the theory of Hermite-polynomials. They satisfy the recurrence-formula

$$H_m(z) = 2zH_{m-1}(z) - 2(m-1)H_{m-2}(z) , \tag{2.1}$$

further the formula

$$H'_m(z) = 2m\, H_{m-1}(z) . \tag{2.2}$$

We shall need also the deeper fact, due to G. SZEGŐ (see [6] p. 128) (which we use in a slightly weaker form) according to which, if ξ_m stands for the maximal zero of $H_m(x) = 0$, then

$$\xi_m < (2m+1)^{1/2} - 1,8\,(2m+1)^{-1/6} \tag{2.3}$$

and also the inequality of Van VEEN (see [9])

$$\xi_m^2 \geqq 2\,(m+1) - 6,1\,(m+1)^{1/3}$$

$$m \geqq 2\,.$$

This last inequality we use in the slightly weaker form

$$\xi_m > \sqrt{2m+2} - \frac{4,4}{(m+1)^{1/6}}\,. \tag{2.4}$$

$$m \geqq 5\,.$$

Applying (2.3) and (2.4) with $m = n$ resp. $n - 1$ we obtain for $n \geqq 6$

$$\xi_n - \xi_{n-1} < (2n+1)^{1/2} - (2n)^{1/2} - \frac{1,8}{(2n+1)^{1/6}} +$$

$$+ \frac{4,4}{n^{1/6}} = \frac{1}{\sqrt{2n+1} + \sqrt{2n}} + \left(\frac{4,4}{n^{1/6}} - \frac{1,8}{(2n+1)^{1/6}}\right) <$$

$$< \frac{\sqrt{2}}{4\sqrt{n}} + n^{-1/6}\left(4,4 - \frac{1,8}{3^{1/6}}\right) < \tag{2.5}$$

$$< \frac{\sqrt{2}}{4\sqrt{n}} + \frac{3}{n^{1/6}} = \frac{1}{n^{1/6}}\left(3 + \frac{\sqrt{2}}{4n^{1/3}}\right) < 3,2n^{-1/6}$$

We shall further need two inequalities due to one of us (see [7]). The first of these asserts the inequality

$$|H_{m-1}(\xi_m)| \geqq \frac{(m-1)!\, e^{\frac{1}{2}\xi_m^2}}{2\left(\left[\frac{m+1}{2}\right]\right)!} \sqrt{2m+1-\xi_m^2} \tag{2.6}$$

and the second, that for

$$-\sqrt{2m+1} \leqq x \leqq \sqrt{2m+1}$$

the inequality

$$|H_m(x)| \leqq \frac{m!}{\left[\frac{m}{2}\right]!}(2m+1)^{1/4} \frac{e^{\frac{1}{2}x^2}}{(2m+1-x^2)^{1/4}} \tag{2.7}$$

holds. (We remark that the last one could have been replaced by a somewhat weaker inequality due to J. BALÁZS (see [8]) which can be deduced much simpler.) Since from (2.3) we have evidently for $m \geqq 1$ the inequality

$$2m + 1 - \xi_m^2 \geqq (2m+1)^{1/3}$$

and hence from (2.6) and (2.4) for $m \geqq 5$

$$|H_{m-1}(\xi_m)| \geqq \frac{(m-1)!}{2\left(\left[\frac{m+1}{2}\right]\right)!}(2m+1)^{1/6}\cdot e^{\frac{1}{2}\{2m+2-8,8\sqrt{2}\,(m+1)^{1/3}\}} >$$

$$> \frac{(m-1)!}{2\left(\left[\frac{m+1}{2}\right]\right)!}(2m+1)^{1/6}\cdot e^{m+1-6,3(m+1)^{1/3}}.$$

Thus with $m = n - 1$, $n \geqq 6$ this gives

$$|H_{n-2}(\xi_{n-1})| \geqq \frac{(n-2)!}{2\left[\frac{n}{2}\right]!}(2n-1)^{1/6}\cdot e^{n-6,3n^{1/3}}$$

and taking in account (2.1) with $m = n$

$$\begin{aligned}|H_n(\xi_{n-1})| &= 2(n-1)\,|H_{n-2}(\xi_{n-1})| \geqq \\ (2.8)\qquad &\geqq \frac{(n-1)!}{\left[\frac{n}{2}\right]!}(2n-1)^{1/6}\cdot e^{n-6,3n^{1/3}}.\end{aligned}$$

As it is well-known we have $\xi_{n-1} < \xi_n$; let us consider $|H_n(z)|$ on the circle $|z - \xi_n| = \xi_n - \xi_{n-1}$. Let us observe that owing to the fact that all zeros of $H_n(z)$ are real, the minimum of $|H_n(z)|$ on our circle is attained only at $z = \xi_{n-1}$. Hence from (2.8) we get the

Lemma. *On the circle* $|z - \xi_n| = \xi_n - \xi_{n-1}$ *the inequality*

$$|H_n(z)| \geqq \frac{(n-1)!}{\left[\frac{n}{2}\right]!}(2n-1)^{1/6}\, e^{n-6,3n^{1/3}}$$

holds, if only $n \geqq 6$.

3. We shall need another inequality too, which is an easy deduction from (2.7) and (2.2). Let $K(\delta)$ stand for the circle $\left|z + \frac{1}{2}\right| = \delta$, $\delta > 0$ and we want to have an upper bound for $|H_n(z)|$ on $K(\delta)$. Repeated use of (2.2) gives

$$H_n^{(k)}(z) = 2^k \frac{n!}{(n-k)!} H_{n-k}(z)$$

and hence on $K(\delta)$

$$|H_n(z)| = \left|\sum_{k=0}^{n} \frac{1}{k!} H_n^{(k)}\left(-\frac{1}{2}\right)\left(z + \frac{1}{2}\right)^k\right| \leqq \sum_{k=0}^{n} \binom{n}{k} (2\delta)^k \left|H_{n-k}\left(-\frac{1}{2}\right)\right|.$$

Using (2.7) this gives on $K(\delta)$

$$|H_n(z)| \leq n! \sum_{k=0}^{n} \frac{(2\delta)^k}{k! \left[\frac{n-k}{2}\right]!} \left(\frac{2n-2k+1}{2n-2k+\frac{3}{4}}\right)^{1/4} \cdot e^{1/8} <$$

(3.1)

$$< \left(\frac{4}{3}\right)^{1/4} e^{1/8}(1+2\delta)^n \cdot \max_k \cdot \frac{(n-k)!}{\left[\frac{n-k}{2}\right]!} < \left(\frac{4}{3}\right)^{1/4} e^{1/8} \frac{n!}{\left[\frac{n}{2}\right]!} \cdot e^{2\delta n}.$$

This is what we shall need.

4. Now we turn to the proof of our theorem. On the circle $|z - \xi_n| = \xi_n - \xi_{n-1}$ we have from (2.3) and (2.5) for $n \geq 6$ the inequality

$$|1 + H_1(z)| = |1 + 2z| \leq 1 + 2(2\xi_n - \xi_{n-1}) <$$

(4.1) $$< 1 + 2\{(2n+1)^{1/2} - 1{,}8(2n+1)^{-1/6} + 3{,}2n^{-1/6}\} < 3\sqrt{2n+1}$$

and hence, if only

(4.2) $$|\zeta| \frac{n!}{\left[\frac{n}{2}\right]!} (2n-1)^{1/6} e^{n-6{,}3n^{1/3}} > 3n\sqrt{2n+1}$$

then owing to the lemma and (4.1) we have

$$|\zeta H_n(z)| > |1 + H_1(z)|.$$

But then Rouché's theorem gives at once the existence of a zero in the circle

$$|z - \xi_n| \leq \xi_n - \xi_{n-1}$$

i.e., owing to (2.5) in the strip

(4.3) $$|Iz| \leq e^{\frac{3{,}2}{6^{1/6}}} < e^3.$$

On the other hand, if only

(4.4) $$|\zeta| \frac{n!}{\left[\frac{n}{2}\right]!} \left(\frac{4}{3}\right)^{1/4} e^{9/8} < \frac{1}{n}$$

then choosing $\delta = \frac{1}{2n}$ we have on the circle $K\left(\frac{1}{2n}\right)$, using (3.1),

$$|1 + H_1(z)| = 2\left|z + \frac{1}{2}\right| = \frac{1}{n} >$$

(4.5)

$$> |\zeta| \frac{n!}{\left[\frac{n}{2}\right]!} \left(\frac{4}{3}\right)^{1/4} e^{9/8} > |\zeta|\,|H_n(z)|$$

i.e., again Rouché's theorem gives the existence of a zero in the circle $\left|z+\frac{1}{2}\right| \leqq \frac{1}{2n}$, i.e., a fortiori in the strip

$$|Iz| \leqq \frac{1}{12}. \tag{4.6}$$

Obviously (4.2) and (4.4) cover all ζ-values if

$$\left(\frac{3}{4}\right)^{1/4} e^{-9/8}\,\frac{1}{n} > \frac{3n\sqrt{2n+1}}{(2n-1)^{1/6}}\, e^{-n+6,3n^{1/3}} \tag{4.7}$$

5. Since, as easy to see, we have even for $n \geqq 1$

$$\frac{\sqrt{2n+1}}{(2n-1)^{1/6}} < 2n^{1/3},$$

(4.7) is true a fortiori if

$$\left(\frac{3}{4}\right)^{1/4} e^{-9/8} > 6n^{7/3}\, e^{-n+6,3n^{1/3}},$$

or if

$$e^{n} > 6\left(\frac{4}{3}\right)^{1/4} e^{9/8+6,3n^{1/3}}\, n^{7/3}. \tag{5.1}$$

Since

$$6\left(\frac{4}{3}\right)^{1/4} < e^{2+\frac{1}{12}} \quad\text{and}\quad \frac{7}{3}\log n < 2{,}6n^{1/3}$$

(5.1) is in turn certainly true if

$$e^{n} > e^{3,22+9n^{1/3}}$$

resp.

$$n > 3{,}2 + 9n^{1/3}.$$

But this true for $n \geqq 36$ and hence Theorem II is proved.

6. In order to prove Theorem I we have to consider the case

$$n \leqq 35. \tag{6.1}$$

Hence if

$$|\zeta| < \frac{2}{\max\limits_{|z+\frac{1}{2}|=1} |H_n(z)|} \leqq c_1 \tag{6.2}$$

(c_1 and later $c_2, \ldots$ being positive explicitly calculable numerical constants) then the equation (1.2) has owing to Rouché's theorem a zero in the circle

$\left|z+\frac{1}{2}\right| \leqq 1$, i.e., in $|Iz| \leqq 1$. Further if

$$(6.3)\qquad |\zeta| > \frac{\max\limits_{|z-\xi_{n-1}|=\xi_n-\xi_{n-1}} |1+2z|}{|H_n(\xi_{n-1})|} \geqq c_2$$

(in consequence of (6.1.), then the equation (1.2) has at least one zero owing to Rouché's theorem in the circle

$$|z-\xi_{n-1}| \leqq \xi_n - \xi_{n-1} \leqq c_3$$

i.e., in $|Iz| \leqq c_3$. If $c_1 > c_2$, we are ready and our strip is

$$(6.4)\qquad |Iz| \leqq \max(e^3, c_3).$$

If not, then the zeros of the equations (1.2) with

$$c_1 \leqq |\zeta| \leqq c_2, \qquad n \leqq 35$$

are certainly in a finite universal domain, i.e., with suitable c_4 in

$$|Iz| \leqq c_4.$$

In this case the strip

$$|Iz| \leqq \max(e^3, c_3, c_4)$$

fulfills our requirement. Hence Theorem I is proved too.

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РАЗЛОЖЕНИЕ В РЯД ПО МНОГОЧЛЕНАМ ЭРМИТА И РАСПРЕДЕЛЕНИЕ НУЛЕЙ МНОГОЧЛЕНОВ

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Резюме

Если $H_\nu(z)$ является многочленом Эрмита, определенным формулой (1.1), тогда корни многочленов определенных формулой (1.2) размещаются вблизи действительной оси плоскости комплексного переменного. Точнее, существует постоянная величина А, независшая от n и от ζ такая, что мнимые части y корней удовлетворяют неравенству $|y| < A$.