FREE ALGEBRAS OVER FIRST ORDER AXIOM SYSTEMS

by
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1. Introduction

The concepts of free semigroups, groups, rings and so on are well known and have several applications. The construction of free semigroups etc. is, roughly speaking, the following (see [1] and [2]): we take the polynomials over the generating system given, and we identify some polynomials in order to make the algebra of polynomials satisfy the given axiom system Σ defining the class of algebras considered. The axioms in Σ are open sentences, i.e. in a normal prenex form they contain no existential quantifier. If it does (e.g. in case of groups) then we introduce further operations (e.g. the operation x^{-1} in groups) so that Σ can be transformed into one containing only open sentences.

Of course, the existential quantifiers cannot always be eliminated by introducing new operations. It is my aim to show that even in this case free algebras can be defined. However, in such a situation one should begin by considering the notion of subalgebra and homomorphism, since, as can be shown by examples, the classical notions do not work well. The modified notions, called Σ-subalgebra and Σ-homomorphism coincide with the classical notions if Σ contains open sentences only.

§ 2 contains the notation and the basic notions. The concepts of Σ-homomorphism, Σ-subalgebra and free Σ-algebra are given in § 3. The results on free Σ-algebras are given in § 4, while the existence theorem is contained in § 5. The notion of free **K**-algebra can be defined over an arbitrary class **K** of algebras; how this concept is connected with free Σ-algebras is shown in § 6.

In the Appendix a necessary and sufficient condition is given for the existence of free algebras in the classical case, when Σ contains open sentences only.

The proofs of the results are not given. These will be published in subsequent publications.

2. Notions and notations

A universal algebra (briefly: algebra) $\mathfrak{A}$ is a sequence $\langle A, f_0, f_1, \ldots, f_\gamma, \ldots\rangle_{\gamma<\alpha}$ where A is a set and f_γ is an n_γ-ary operation on A i.e. $f_\gamma \in A^{A^{n_\gamma}}$; in this notation α, γ denote ordinal numbers, $0 \leq n_\gamma < \omega$. The sequence $\langle n_\gamma\rangle_{\gamma<\alpha}$ is the type of $\mathfrak{A}$, α is the order of $\mathfrak{A}$.

Gothic capital letters $\mathfrak{A}, \mathfrak{B}, \ldots, \mathfrak{F}$ denote algebras while the corresponding latin capital letters $A, B, \ldots, F$ denote the set upon which they are defined.

All algebras considered are of the same type which is kept fixed throughout the paper. We can associate with this type a first order logic with equality sign as follows: it contains operation symbols P_γ for every $\gamma < \alpha$ and P_γ is the symbol of an n_γ-ary operation; the individual variables are $v, x, y, z, \ldots$; it contains further the usual logical connectives $\wedge$, $\vee$, $\neg$, $\to$, (x), $(\exists x)$ ("or", "and", "not", "imply", "for every x", "there exists an x"), and the equality sign $=$.

Formulae are defined as usual. First we define terms: any individual variable x is a term; if $T_1, \ldots, T_{n_\gamma}$ are terms then so is $P_\gamma(T_1, \ldots, T_{n_\gamma})$. Next we define formulae: if T_1 and T_2 are terms then

$$T_1 = T_2$$

is a formula (this is called a prime formula). If Φ_1 and Φ_2 are formulae then so are $\Phi_1 \vee \Phi_2$, $\Phi_1 \wedge \Phi_2$, $\neg \Phi_1$, $\Phi_1 \to \Phi_2$, $(x)(\Phi_1)$, $(\exists x)(\Phi_1)$.
Every formula Φ can be written in a prenex normal form:

$$(Q_1 x_1) \ldots (Q_n x_n)(\Psi) \tag{2.1}$$

where Ψ is a formula containing no quantifier and the Q_i are quantifiers. If in (2.1) Ψ contains no variable other than $x_1, \ldots, x_n$ then Φ is a closed formula. If Φ is closed and no Q_i is an existential quantifier then Φ is an open formula.

An *axiom system* Σ is a set of closed formulae. A Σ-algebra $\mathfrak{A}$ is a model of Σ, i.e. every $\Phi \in \Sigma$ is satisfied on $\mathfrak{A}$.

$\langle A, f_\gamma \rangle_{\gamma<\alpha}$ is a subalgebra of $\langle B, g_\gamma \rangle_{\gamma<\alpha}$ if A is a subset of B, A is closed under g_γ and f_γ is the restriction of g_γ to A, for every $\gamma < \alpha$.

The mapping $\varphi : A \to B$ $(a \to a\varphi)$ is called a homomorphism if

$$f_\gamma(a_1, \ldots, a_{n_\gamma})\varphi = g_\gamma(a_1 \varphi_1, \ldots, a_{n_\gamma}\varphi) \qquad (\gamma < \alpha).$$

All these notions are well-known; for more detailed (and more precise) treatement the reader should consult [3], [5], [6], [7] and [8].

3. The notion of inverse

Let an axiom system Σ be fixed and we suppose every $\Phi \in \Sigma$ is in prenex normal form, e.g.

$$(x)(\exists y)(\Psi(x, y)), \tag{3.1}$$

or

$$(x)(\exists y)(z)(\exists u)(\Psi(x, y, z, u)). \tag{3.2}$$

In (3.1) and (3.2) the formulae $\Psi(x, y)$ and $\Psi(x, y, z, u)$, respectively, contain no quantifier.

Let $\Phi \in \Sigma$, Φ be of the form (3.1), and let $\mathfrak{A}$ be a Σ-algebra, $a, b \in A$. We say that b is an inverse (Φ-inverse) of a if $\Psi(a, b)$ holds. Let Φ be of the form (3.2). Then b is an inverse of a if

$$(z)(\exists u)(\Psi(a, b, z, u))$$

holds true. And d is an inverse of a and c if there exists an inverse b of a such that $\Psi(a, b, c, d)$ holds.

The general definition of inverse is now obvious; if b is a Φ-inverse of $a_1, \ldots, a_n$ then Φ contains an existential quantifier preceded by n universal quantifiers and by some, say k, existential quantifiers; the precise definition can be given by induction on k as in the example above. The details are left to the reader.

To say that b is a Φ-inverse of $a_1, \ldots, a_n$ may sometimes be ambiguous. E.g. if Φ is of the form: $(x)\,(\exists y)\,(\exists z)\,(\Psi(x, y, z))$, then b is a Φ-inverse of a either if $(\exists z)\,(\Psi(a, b, z))$ or if there exists a c with $(\exists z)\,(\Psi(a, c, z))$ and $\Psi(a, c, b)$. Therefore, if we have two existential quantifiers between which there is no universal quantifier than we have to distinguish between the inverses with respect to the first and second existential quantifier.

Since in the discussion we consider as examples axioms of type (3.1) and (3.2) this problem will not arise.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras and $\mathfrak{B}$ be a Σ-algebra. We say that $\mathfrak{A}$ is a Σ-subalgebra of $\mathfrak{B}$ if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, further if $a_1, \ldots, a_n \in A$, $b \in B$, $\Phi \in \Sigma$ and b is a Φ-inverse of $a_1, \ldots, a_n$ in $\mathfrak{B}$ then $b \in A$. It is easy to see that the following results hold:

3.3. A Σ-subalgebra of a Σ-algebra is again a Σ-algebra.

3.4. Let $\mathfrak{A}$ be a Σ-algebra, $H \subseteq A$. Then there exists a smallest Σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $H \subseteq B$.

This $\mathfrak{B}$ is said to be Σ-generated by H, and H called a Σ-generating system of $\mathfrak{B}$.

If Σ is the usual axiom system for groups, i.e. the type is $\langle 2,0 \rangle$ the operations being denoted by $\cdot$ and 1 and the axioms are

$$(x)\,(y)\,(z)\,(x \cdot (y \cdot z) = (x \cdot y) \cdot z)\,,$$

$$(x)\,(x \cdot 1 = x \wedge 1 \cdot x = x)\,,$$

$$(x)\,(\exists y)\,(x \cdot y = 1 \wedge y \cdot x = 1)$$

then a Σ-subalgebra is a subgroup. (Note that a subalgebra is a subsemigroup containing 1.)

If Σ contains open formulae only, then every subalgebra is a Σ-subalgebra.

Now let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-algebras $\varphi : A \to B$. Then φ is said to be a Σ-homomorphism if it is a homomorphism, further it carries inverse into inverse, i.e. if $a_1, \ldots, a_n, b \in A$ and b is a Φ-inverse of $a_1, \ldots, a_n$ ($\Phi \in \Sigma$) then $b\varphi$ is a Φ-inverse of $a_1\varphi, \ldots, a_n\varphi$ and, conversely, if b is a Φ-inverse of $b_1, \ldots, b_n$ ($b, b_1, \ldots, b_n \in B$), $a_1, \ldots, a_n \in A$ such that $a_1\varphi = b_1, \ldots, a_n\varphi = b_n$ then there exists a Φ-inverse a of $a_1, \ldots, a_n$ such that $a\varphi = b$.

In case of groups any homomorphism is a Σ-homomorphism. The same holds if Σ contains open formulae only.

Let a class of algebras of type $\langle 2 \rangle$ be defined by the following axioms (the binary operation is denoted by $\cup$):

$$(x)\,(x \cup x = x)\,,$$

$$(x)\,(y)\,(x \cup y = y \cup x)\,,$$

$$(x)\,(y)\,(z)\,((x \cup y) \cup z = x \cup (y \cup z))\,,$$

$$(x)\,(y)\,(\exists z)\,(u)\,(x = z \cup x \wedge y = z \cup y \wedge ((x = u \cup x \wedge y = u \cup y) \to z = z \cup u))\,.$$

Then the class of Σ-algebras coincides with the class of lattices, a Σ-subalgebra is a sublattice, a Σ-homomorphism is a lattice-homomorphism. A subalgebra is a subset closed under union, which is a lattice, and the usual notion of homomorphism too does not function well, e.g. the homomorphic image of a distributive lattice (as a Σ-algebra) can be non-distributive. Of course, this cannot happen if we consider Σ-homomorphisms.

3.5. Let $\mathfrak{A}$, $\mathfrak{B}$ be Σ-algebras, φ a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then the image of $\mathfrak{A}$ under φ is a Σ-subalgebra of $\mathfrak{B}$.

3.6. The product of Σ-homomorphisms, if exists, is again a Σ-homomorphism.

Now we define the notion of free Σ-algebra with β Σ-generators, where β is any cardinal number.

The algebra $\mathfrak{F}(\beta) = \langle F(\beta), f_0, f_1, \ldots, f_\gamma, \ldots \rangle_{\gamma<\alpha}$ is said to be the free Σ-algebra with β Σ-generators if

(*a*) $\mathfrak{F}(\beta)$ is a Σ-algebra;

(*b*) $F(\beta)$ contains a sequence of elements $\langle x_\gamma \rangle_{\gamma<\beta}$ Σ-generating $\mathfrak{F}(\beta)$;

(*c*) let $\mathfrak{A}$ be a Σ-algebra, $\langle a_\gamma \rangle_{\gamma<\beta}$ a sequence of elements of A, then the mapping $\varphi : x_\gamma \to a_\gamma$ $(\gamma < \beta)$ can be extended to a Σ-homomorphism $\overline{\varphi}$ of $\mathfrak{F}(\beta)$ into $\mathfrak{A}$.

It is easy to see that the $\overline{\varphi}$ in (*c*) need not be unique.

4. Free Σ-algebras

First we formulate the unicity theorem:

4.1. If $\mathfrak{F}(\beta)$ exists for some β then it is unique up to isomorphism.

We can prove somewhat more, namely

4.2. Let $\mathfrak{F}(\beta)$ and $\mathfrak{F}'(\beta)$ be free Σ-algebras with the Σ-generating systems $\langle x_\gamma \rangle_{\gamma<\beta}$ and $\langle x'_\gamma \rangle_{\gamma<\beta}$. Let φ be the mapping $x_\gamma \to x'_\gamma$ $(\gamma < \beta)$ and $\overline{\varphi}$ any extension of φ into Σ-homomorphism. Then $\overline{\varphi}$ is an isomorphism between $\mathfrak{F}(\beta)$ and $\mathfrak{F}'(\beta)$.

The following theorems are analogues of well-known theorems for the classical case. However, it seems to me that the triviality of the classical results does not imply the existence of an easy proof in this situation.

4.3. Suppose that the free Σ-algebra $\mathfrak{F}(\beta)$ exists. Then $\mathfrak{F}(\delta)$ exists for every $\delta < \beta$.

4.4. Suppose $\mathfrak{F}(n)$ exists for every $n < \omega$. Then $\mathfrak{F}(\omega)$ exists too.

4.5. If $\mathfrak{F}(\omega)$ exists then so does $\mathfrak{F}(\beta)$ for every β.

5. The existence theorem

We define multi-polynomials as follows:

(*a*) $P(x_1, \ldots, x_n) = \{x_i\}$ is a multi-polynomial;

(*b*) if f is an operation then $P(x_1, \ldots, x_n) = \{f(x_1, \ldots, x_n)\}$ is a multi-polynomial;

(*c*) let $\exists y$ be preceeded by n universal quantifiers in the axiom Φ; then $P(x_1, \ldots, x_n)$, the set of all Φ-inverses of $x_1, \ldots, x_n$ is a multi-polynomial;

(*d*) if $P_1(x_1, \ldots, x_n), \ldots, P_k(x_1, \ldots, x_n)$, $P(x_1, \ldots, x_k)$ are multi-polynomials then so is $P(P_1, \ldots, P_k)$, where

$x \in P(P_1(x_1, \ldots, x_n), \ldots, P_k(x_1, \ldots, x_n))$ if and only if there exist $y_1, \ldots, y_k$, such that $x \in P(y_1, \ldots, y_k)$ and $y_i \in P_i(x_1, \ldots, x_n)$ for every $1 \leqq i \leqq k$.

(*e*) the multi-polynomials are those and only those which are given by (*a*)—(*d*).

Consider the condition:

(5.1) Every multi-polynomial in k variable is bounded, i.e. there exists an integer n, such that a multi-polynomial $P(x_1, \ldots, x_k)$ cannot contain more than n elements.

Then we can prove

5.2. If $\mathfrak{F}(k)$ exists then (5.1) holds.

The next condition is necessary, too, for the existence of $\mathfrak{F}(k)$:

(5.3) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-algebras, $\mathfrak{A}$ is Σ-generated by $\langle x_i \rangle_{i<k}$ and $\langle y_i \rangle_{i<k}$ are elements of B. Then there exists a Σ-algebra $\mathfrak{C}$ Σ-generated by $\langle z_i \rangle_{i<k}$ such that the mappings $\varphi_1 : z_i \rightarrow x_i$ and $\varphi_2 : z_i \rightarrow y_i$ can be extended to Σ-homomorphisms.

5.4. The conditions (5.1) and (5.3) are necessary and sufficient for the existence of $\mathfrak{F}(k)$.

We formulated 5.4 for $k < \omega$. It is true for ω as well. (5.1) then says that every multi-polynomial is bounded.

A special case of 5.4 is the following:

5.5. Suppose Σ contains open sentences only. Then $\mathfrak{F}(\beta)$ exists if and only if whenever $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-algebras, $\mathfrak{A}$ is generated by $x_0, \ldots, x_\gamma, \ldots$ $(\gamma < \beta)$ and $y_0, \ldots, y_\gamma, \ldots$ $(\gamma < \beta)$ are elements of B then there exists a Σ-algebra $\mathfrak{C}$ generated by the elements $z_0, \ldots, z_\gamma, \ldots$ $(\gamma < \beta)$ such that $\varphi_1 : z_\gamma \rightarrow x_\gamma$ $(\gamma < \beta)$ and $\varphi_2 : z_\gamma \rightarrow y_\gamma$ $(\gamma < \beta)$ can be extended to homomorphisms.

E.g. if Σ contains only equations then such a $\mathfrak{C}$ is the subalgebra of the direct product $\mathfrak{A} \times \mathfrak{B}$, generated by the $\langle x_\gamma, y_\gamma \rangle$ $(\gamma < \alpha)$.

One can obviously infer a characterization of equational classes from 5.5.

6. Free K-algebras

Let **K** be a class of algebras. An algebra $\mathfrak{F}(\beta)$ is a free **K**-algebra with β generators if

(*a*) $\mathfrak{F}(\beta) \in \mathbf{K}$;

(*b*) $F(\beta)$ contains a sequence $\langle x_\gamma \rangle_{\gamma<\beta}$ generating $\mathfrak{F}(\beta)$;

(*c*) let $\mathfrak{A} \in \mathbf{K}$ and $\langle a_\gamma \rangle_{\gamma<\beta}$ be a sequence of elements of A, then $\varphi : x_\gamma \rightarrow a_\gamma$ $(\gamma < \beta)$ can be extended to a homomorphism $\overline{\varphi}$ of $\mathfrak{F}(\beta)$ into $\mathfrak{A}$.

The problem is the following: what is the connection between free **K**-algebras and free Σ-algebras.

To settle this problem we define two properties.

We say that the axiom system Σ has the Inverse Preserving Property (IPP) if whenever $\mathfrak{A}$ is a Σ-algebra, $\mathfrak{B}$ a Σ-subalgebra of $\mathfrak{A}$, $a_1, \ldots, a_n$, $b \in B$, $\Phi \in \Sigma$, b is a Φ-inverse of $a_1, \ldots, a_n$ in $\mathfrak{B}$ then the same holds in $\mathfrak{A}$ as well. (The converse of this statement holds always.)

The free Σ-algebra $\mathfrak{F}(\beta)$ is called free in the stronger sense if, in the definition, the Σ-homomorphism $\overline{\varphi}$ is always uniquely determined. If we write "$\mathfrak{F}^0(\beta)$ exists" it means that the free Σ-algebra $\mathfrak{F}(\beta)$ exists and it is free in the stronger sense.

6.1. If $\mathfrak{F}^0(\beta)$ exists and $\delta < \beta$ then $\mathfrak{F}^0(\delta)$ exists too.

6.2. If $\mathfrak{F}^0(n)$ exists for every $n < \omega$ then so does $\mathfrak{F}^0(\omega)$.

6.3. If $\mathfrak{F}^0(\omega)$ exists then so does $\mathfrak{F}^0(\beta)$ for every β.

Now we turn to the main problem:

6.4. Let $\mathbf{K}$ be the class of Σ-algebras $\langle A, f_\gamma\rangle_{\gamma<\alpha}$; we suppose Σ has IPP and the free Σ-algebra $\mathfrak{F}^0(\omega)$ exists. Then we can define new operations $f_\alpha, f_{\alpha+1}, \ldots, f_\gamma, \ldots$ $(\alpha \leqq \gamma < \beta)$ on the Σ-algebras such that the arising class $\overline{\mathbf{K}}$ of algebras and the mapping

$$\langle A, f_\gamma\rangle_{\gamma<\alpha} \longleftrightarrow \langle A, f_\gamma\rangle_{\gamma<\beta}$$

have the following properties:

(*a*) $\langle A, f_\gamma\rangle_{\gamma<\alpha}$ is a Σ-subalgebra of $\langle B, g_\gamma\rangle_{\gamma<\alpha}$ if and only if $\langle A, f_\gamma\rangle_{\gamma<\beta}$ is a subalgebra of $\langle B, g_\gamma\rangle_{\gamma<\beta}$;

(*b*) let φ map A into B; φ is a Σ-homomorphism of $\langle A, f_\gamma\rangle_{\gamma<\alpha}$ into $\langle B, g_\gamma\rangle_{\gamma<\alpha}$ if and only if it is a homomorphism of $\langle A, f_\gamma\rangle_{\gamma<\alpha}$ into $\langle B, g_\gamma\rangle_{\gamma<\beta}$;

(*c*) the free $\overline{K}$-algebras exist.

Theorem 6.4 is the best possible in the following sense:

6.5. Suppose that the conclusion of 6.4 hold for the axiom system Σ. Then Σ has IPP and $\mathfrak{F}^0(\omega)$ exists.

The simplest illustration of 6.4 is given by an axiom system Σ in which all existential quantifiers are of bound one, i.e. the inverses are unique. Then the new operations $f_\alpha, f_{\alpha+1}, \ldots$ are simply the Skolem-functions.

7. Appendix

Now we suppose that every axiom in Σ is an open formula, i.e. of the form

$$(x_1) \ldots (x_n) (\Psi(x_1, \ldots, x_n)). \tag{7.1}$$

Instead of (7.1.) we will write simply

$$\Psi(x_1, \ldots, x_n), \tag{7.2}$$

which contains no quantifier.

The conjunctive normal form of Ψ be

$$\Psi : \Psi_1 \wedge \ldots \wedge \Psi_K,$$

and we replace Ψ in Σ by $\Psi_1, \ldots, \Psi_k$. Thus we may suppose that every $\Psi \in \Sigma$ is of the form

$$\Psi : \Psi_1 \vee \ldots \vee \Psi_K \tag{7.3}$$

where every Ψ_i is a prime formula or a negation of a prime formula.

If every $\Psi \in \Sigma$ is of the form (7.3) then we say that Σ is in the *normal form*.

Let Σ be in the normal form. Σ is *reduced* if every $\Psi \in \Sigma$ is reduced in the following sense: either $k = 1$ or the sequence $\langle 1, 2, \ldots, k\rangle$ has no proper subsequence $\langle i_1, \ldots, i_n\rangle$ $(n < k)$ such that Ψ is equivalent to Ψ':

$$\Psi' : \Psi_{i_1} \vee \ldots \vee \Psi_{i_n}. \tag{7.4}$$

More precisely to say, Σ is not equivalent to $(\Sigma \setminus \{\Psi\}) \cup \{\Psi'\}$.

If Ψ is not reduced then there may be many sequences $\langle i_1, \ldots, i_n \rangle$ for which Σ is equivalent to $(\Sigma \setminus \{\Psi\}) \cup \{\Psi'\}$. A reduction is the following: we replace Ψ by a Ψ' for which $n = 1$ or n is the least possible.

A *reduction* of Σ is the following: we reduce every $\Psi \in \Sigma$.

7.5. Let Σ' be any reduction of Σ. Then Σ is equivalent to Σ'.

7.6. Any axiom system Σ equivalent to an axiom system Σ' which is in reduced normal form.

7.6 is due to PEREMANS [4].

Now suppose Σ is in reduced normal form, $\Psi \in \Sigma$ is of the form (7.3), where $\Psi_1, \ldots, \Psi_{s(\Psi)}$ are negations of prime formulae, $\Psi_{s(\Psi)+1}, \ldots, \Psi_k$ are prime formulae (it is allowed that $s(\Psi) = 0$ or $s(\Psi) = k$).

A P-specialization of $\Psi(x_1, \ldots, x_m)$ is $\Psi(T_1, \ldots, T_m)$ if $T_1, \ldots, T_m$ are terms and each $\Psi_i(T_1, \ldots, T_m)$, $1 \leq i \leq s(\Psi)$ is identically false (i.e. $\neg \Psi_i(T_1, \ldots, T_m)$ holds always in every Σ-algebra).

An axiom $\Psi \in \Sigma$ said to have property P if either $s(\Psi) = k$ or whenever $\Psi(T_1, \ldots, T_m)$ is a P-specialization then there exists an $s(\Psi) < i \leq k$ such that $\Psi_i(T_1, \ldots, T_m)$ is identically true.

7.7. If Ψ is of the form (7.3) and $k = 1$ then Ψ has property P.

7.8. If Ψ is of the form (7.3) and $s(\Psi) \geq k - 1$ then Ψ has property P.

7.9. Suppose Ψ in (7.3) is reduced, $k > 1$ and $s(\Psi) = 0$. Then Ψ does not have property P.

An axiom system Σ has property P if every axiom $\Psi \in \Sigma$ has it.

The main result of this section is the following:

7.10. The free algebra $\mathfrak{F}(\omega)$ over the class of all Σ-algebras exists if and only if Σ has property P.

7.7 and 7.10 together yield a result of BIRKHOFF [1] and [2] while the combination of 7.8 and 7.9 with 7.10 give the two main results of PEREMANS [4].

One can easily sharpen 7.10 by giving a necessary and sufficient condition for the existence of $\mathfrak{F}(n)$, $n < \omega$.

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СВОБОДНЫЕ АЛГЕБРЫ НАД СИСТЕМАМИ АКСИОМ ПЕРВОГО ПОРЯДКА

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Резюме

Резюмируя результаты нескольких его статьей, которые вскоре будут опубликованы, автор сообщает о них не приводя доказательств.