ON SOME DISTRIBUTIONS CONNECTED WITH THE ARCSINE LAW

by

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Introduction

1. In the following we shall consider both cases of the finite arcsine law, namely the original form due to CHUNG and FELLER [2] and the general case of SPARRE ANDERSEN [1] as well. In connection with these theorems we shall determine some distributions and shall give for the discrete case combinatorial proofs based on one-to-one correspondences. In formulating our results we shall make use of the model of CHUNG and FELLER:

Two gamblers A and B play a coin tossing game in which player A wins or loses a unit according to whether the result of the coin tossing is „head" or „tail". Denoting his winnings in the i-th trial by ξ_i we have $\mathbf{P}(\xi_i = +1) = \mathbf{P}(\xi_i = -1) = \frac{1}{2}$ and the total amount of his winnings after i trials by $s_i = \xi_1 + \xi_2 + \ldots + \xi_i$, $(s_0 = 0)$; we shall say that A leads over B at the i-th trial, if either $s_i > 0$ or $s_i = 0$ but $s_{i-1} = +1$. Among $2n$ trials A may lead in $0, 2, 4, \ldots, 2n$ trials and we shall denote by $2\gamma_{2n}$ the number of leading steps. As CHUNG and FELLER [2] have shown, γ_{2n} follows the finite arcsine law; i.e. if $\xi_1, \xi_2, \ldots, \xi_{2n}$ are totally independent, then

$$\mathbf{P}(\gamma_{2n} = g) = \frac{1}{2^{2n}} \binom{2g}{g} \binom{2n-2g}{n-g}, \qquad g = 0, 1, 2, \ldots n. \tag{1}$$

The limiting distribution, obtained by P. LÉVY in [6] is

$$\lim_{n\to\infty} \mathbf{P}(\gamma_{2n} < n\alpha) = \frac{1}{\pi} \int_0^{\alpha} \frac{dx}{\sqrt{x(1-x)}} = \frac{2}{\pi} \arc\sin \sqrt{\alpha}\,. \tag{2}$$

In §1 we shall determine the distribution of the number of trials at which the winnings of player A makes at least $2k$. As given in Theorem 1.1, the following very simple modified form of the finite arcsine law is obtained:

$$\mathbf{P}(\gamma_{2n}^{(2k)} = g) = \frac{1}{2^{2n}} \binom{2g}{g} \binom{2n-2g}{n-g+k} \tag{3}$$

for $g = 1, 2, \ldots, n$ and

$$\mathbf{P}(\gamma_{2n}^{(2n)} = 0) = \frac{1}{2^{2n}} \sum_{j=-k}^{k} \binom{2n}{n+j}. \tag{4}$$

In order to obtain a simple combinatorial proof for this relation, we shall give a new proof for the finite arcsine law. (It is to be remarked, that (3) and (4) can be derived from the finite arcsine law with the aid of generating functions too.) In this § we shall give a numerical example and consider the limiting distribution.

In §2 we shall determine the joint distribution of two random variables: the number of leading steps and the number of winning series of player A. A winning series is a sequence $\{s_\nu, s_{\nu+1}, \ldots, s_\mu\}$ for which $s_j \geqq 0$, $j = \nu, \nu+1, \ldots, \mu$, but $s_{\nu-1} = s_{\mu+1} = -1$. (We make the agreement, that $s_{-1} = -s_1$.)

In §3 we shall consider the case $s_{2n} = 0$ for which CHUNG and FELLER in their cited paper found the well known result

$$\mathbf{P}(\gamma_{2n} = g \mid s_{2n} = 0) = \frac{1}{n+1}, \qquad g = 0, 1, \ldots n,$$

i.e. in this case γ_{2n} has a uniform distribution.

In § 4 we shall turn to the general case. Let $(\xi_1, \xi_2, \ldots, \xi_n)$ be a sequence of independent, identically distributed random variables with continuous and symmetric distributions. Let us denote the partial sums by $S_0 = 0$, $S_i = \xi_1 + \xi_2 + \ldots + \xi_i$, $i = 1, 2, \ldots, n$ and by $\Gamma_n^{(k)}$ the number of S_i' s $(i = 0, 1, \ldots, n)$ exceeding the value of the partial sum at the k-th ladder index (see [5] p. 82). We obtain the following simple result, corresponding to (3) and (4):

$$(3') \qquad \mathbf{P}(\Gamma_n^{(k)} = g) = \frac{1}{2^{2n-k}} \binom{2g}{g} \binom{2n-2g-k}{n-g}$$

for $g = 1, 2, \ldots, n$ and

$$(4') \qquad \mathbf{P}(\Gamma_n^{(k)} = 0) = \sum_{j=0}^{k} \frac{1}{2^{2n-j}} \binom{2n-j}{n}.$$

Both (3) and (3′) give the respective arcsine law for $k = 0$.

Our formulas in §1—3 are derived for the special random variables $\xi_i = \pm 1$; according to known invariance principles our limiting distributions are however valid for more general random variables as well.

§ 1. The number of trials with an accumulated gain exceeding $2k$

1. In the following we shall make use of the geometrical description of the game. Let us consider in a coordinate-system the polygonal line whose vertices have abscissae i and ordinates s_i $(i = 0, 1, 2, \ldots, n)$. This figure will be called a path.

Obviously n trials may occur in 2^n different ways each with the common probability 2^{-n}.

Let $2\gamma_{2n}^{(2k)}$ denote the number of indices i $(i = 1, 2, \ldots, 2n)$ for which either $s_i > 2k$ or $s_i = 2k$ but $s_{i-1} = 2k + 1$, where k is a non-negative integer. There holds the following

Theorem 1.1.

(3) $$\mathbf{P}(\gamma_{2n}^{(2k)} = g) = \frac{1}{2^{2n}} \binom{2g}{g} \binom{2n-2g}{n-g+k} \quad g = 1, 2, \ldots n-k,$$

(4) $$\mathbf{P}(\gamma_{2n}^{(2k)} = 0) = \frac{1}{2^{2n}} \sum_{j=-k}^{k} \binom{2n}{n+j}.$$

Let us now denote by $A_{2n,2g}^{(2k)}$ the set of paths $(s_0, s_1, \ldots, s_{2n})$ for which $s_{2g} = s_{2n} = 2k$ and by $B_{2n,2g}^{(2k)}$ the set of paths $(s_0, s_1, \ldots, s_{2n})$ for which $\gamma_{2n}^{(2k)} = g$.

It is trivial that

$$\mathbf{P}(s_{2(n-g)} = s_{2n} = 2k) = \frac{1}{2^{2n}} \binom{2g}{g} \binom{2n-2g}{n-g+k}$$

and therefore by establishing a 1—1 correspondence between the sets $A_{2n,2(n-g)}^{(2k)}$ and $B_{2n,2g}^{(2k)}$ formula (3) will be proved.

To begin with, we shall give the 1—1 correspondence for $k = 0$ i.e. for the classical arcsine law.

Let $(s_0, s_1, \ldots, s_{2n})$ be an element of $B_{2n,2g}^{(0)}$. For the cases $g = 0$ and $g = n$ we refer to lemma 2 in [3], for $1 \leq g < n$ we shall distinguish 4 cases:

a) $s_1 = +1, \quad s_{2n-1} > 0$

b) $s_1 = -1, \quad s_{2n-1} > 0$

c) $s_1 = +1, \quad s_{2n-1} < 0$

d) $s_1 = -1, \quad s_{2n-1} < 0$.

First we consider the case a).

Let us denote by $2\alpha_1 = j_1$, $2(\alpha_1 + \alpha_2) = j_2, \ldots, 2(\alpha_1 + \ldots + \alpha_t) = j_t$ the points where in the sequence $(s_0, s_1, \ldots, s_{2n})$ a change of sign takes place, i.e. $(s_{j_i}, s_{j_i+1}, \ldots, s_{j_{i+1}})$ $i = 0, 1, \ldots, t$ are either winning or losing series; let be $2\alpha_{t+1} = 2n - j_t$. The winning series are of length $2\alpha_1, 2\alpha_3, \ldots$ for which $\alpha_1 + \alpha_3 + \ldots = g$. The last series has the property $(s_{j_t} \geqq 0, s_{j_t+1} \geqq 0, \ldots, s_{2n} \geqq 0)$.

With respect to lemma 2 in [3] this last series corresponds to a path, $(s'_{2(g-\alpha_{t+1})}, \ldots s'_{2g})$ for which $s'_{2(g-\alpha_{t+1})} = s'_{2g} = 0$. Now we define $(s'_{2(g-\alpha_{t+1}-\alpha_{t-1})}, \ldots, s_{2(g-\alpha_{t+1})})$ as either $(s_{j_{t-2}}, \ldots, s_{j_{t-1}})$ or $(-s_{j_{t-2}}, \ldots, -s_{j_{t-1}})$ according to whether $s'_{2(g-\alpha_{t+1})+1} = -1$ or $+1$. Further let $(s'_{2(g-\alpha_{t+1}-\ldots-\alpha_{t-2i+1})}, \ldots, s'_{2(g-\alpha_{t+1}-\ldots-\alpha_{t-2i-1})})$ be either $(s_{j_{t-2i}}, \ldots, s_{j_{t-2i+1}})$ or $(-s_{j_{t-2i}}, \ldots -s_{j_{t-2i+1}})$ according to $s'_{2(g-\alpha_{t-1}-\ldots-\alpha_{t-2i+1})+1} = -1$ or $+1$. Similar construction is made for $(s'_{2g}, \ldots, s'_{2n})$, i.e. we join the losing series one after the other, reflecting every second to obtain intersection between two consecutive ones. The first $(s'_{2g}, \ldots)$ will be defined by $s'_1 s'_{2g+1} = +1$.

The resulting path $(s'_0, s'_1, \ldots, s'_{2g}, \ldots, s'_{2n})$ is of type $A_{2n,2g}^{(0)}$.

In the other cases, similar constructions can be performed. The winning series in $(s_0, s_1, \ldots, s_{2n})$ will be transformed into $(s'_0, s'_1, \ldots, s'_{2g})$ while the losing series into $(s'_{2g}, s'_{2g+1}, \ldots, s'_{2g})$ in such a way that in case b) and d) $s'_1 s'_{2g+1} = -1$ and in case c) $s'_1 s'_{2g+1} = +1$.

In the reversed procedure, i.e. if we start from a path $(s'_0, s'_1, \ldots, s'_{2n})$ of $A^{(0)}_{2n,2g}$ the number of series occurring in section $(s'_0, s'_1, \ldots, s'_{2g})$ and in section $(s'_{2g}, \ldots, s'_{2n})$ are separately considered. If this number is greater in $(s'_0, \ldots, s'_{2g})$ than in $(s'_{2g}, \ldots, s'_{2n})$ we obtain case a) or b), according to whether $s'_1 s'_{2g+1} = +1$ or -1. If more intersections take place in the last section, we are led to case c) or d), according to whether $s'_1 s'_{2g+1} = +1$ or -1. If the number of series equals in both sections, we obtain case b) or c), according to whether $s'_1 s'_{2g+1} = -1$ or $+1$.

This argument shows that the correspondence between the sets $A^{(0)}_{2n,2g}$ and $B^{(0)}_{2n,2g}$ is one-to-one.

Turning to the case $k \geq 1$ we remark that a one-to-one correspondence holds between $A^{(0)}_{2n,2(n-g)}$ and $B^{(0)}_{2n,2g}$ as well.

If $2r$ denotes the first index for which $s_i = 2k$ then — similarly to the foregoing — the section $(s_{2r}, s_{2r+1}, \ldots, s_{2n})$ can be corresponded to a section $(s'_{2r}, s'_{2r+1}, \ldots, s'_{2n})$ for which $s'_{2r} = s'_{2(n-g)} = s'_{2n} = 2k$. Thus the path $(s_0, s_1, \ldots, s_{2r-1}, s'_{2r}, s'_{2r+1}, \ldots, s'_{2n})$ belongs to $A^{(2k)}_{2n,2g}$. This correspondence can be reversed, which proves formula (3).

In order to prove formula (4) we have to substitute in (3) the value $g = 0$ and instead of k the variable index j. In this case

$$\mathbf{P}(\max_{0 \leq i \leq 2n} s_i = 2j) = \mathbf{P}(\max_{0 \leq i \leq 2n} s_i = 2j - 1) = \frac{1}{2^{2n}} \binom{2n}{n+j}$$

for $j = 1, 2, \ldots, n$ and

$$\mathbf{P}(\max_{0 \leq i \leq 2n} s_i = 0) = \frac{1}{2^{2n}} \binom{2n}{n}.$$

Summation over j leads to our formula (4).

The limiting distribution is given by

Theorem 1.2. *In case* $k \sim y\sqrt{2n}$

$$\lim_{n\to\infty} \mathbf{P}(\gamma^{(2k)}_{2n} < \alpha n) = \sqrt{\frac{8}{\pi}} \int_0^y e^{-2t^2}\, dt + \frac{1}{\pi} \int_0^\alpha \frac{e^{-\frac{2y^2}{1-x}}}{\sqrt{x(1-x)}}\, dx, \qquad 0 \leq \alpha \leq 1. \tag{5}$$

The proof can easily be derived by means of well known asymptotic formulae.

2. Let us now consider the shape of the distributions obtained in Theorems 1.1 and 1.2. These distributions are of course, not symmetric, namely the most likely value of $\gamma^{(2n)}_{2n}$ is 0 and the values close to $n - k$ have small probabilities, if $k \neq 0$. The density function $\frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} e^{-\frac{2y^2}{1-x}}$ of the limiting distribution is infinite at $x = 0$ and it is 0 at $x = 1$, if $y \neq 0$. It could be expected that the probabilities are monotonically decreasing from $x = 0$, but this is not always the case.

Let us consider the limiting density function,

$$f_y(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} e^{-\frac{2y^2}{1-x}}$$

and its derivative

$$f'_y(x) = -\frac{1}{\pi}\frac{e^{-\frac{2y^2}{1-x}}}{2x^{3/2}(1-x)^{5/2}}[2x^2+(4y^2-3)x+1].$$

We can see that there is a critical value of y, namely $y_0 = \frac{\sqrt{2}-1}{2}$, for which $f'(x) = 0$ holds for a single value of x, namely $x = \frac{y_0}{\sqrt{2}}$.

If $y > y_0$, then $f'_y(x) < 0$ for $0 \leq x < 1$; for $0 < y < y_0$, $f'_y(x) = 0$ has two roots, $x_1(y)$ and $x_2(y)$, for which $\frac{1}{2} < x_1(y) < \frac{1}{\sqrt{2}} < x_2(y) < 1$; in this case

$$f'_y(x)\begin{cases} < 0\,, & \text{if} \quad 0 < x < x_1(y) \\ = 0\,, & \text{if} \quad x = x_1(y) \\ > 0\,, & \text{if} \quad x_1(y) < x < x_2(y) \\ = 0\,, & \text{if} \quad x = x_2(y) \\ < 0\,, & \text{if} \quad x_2(y) < x < 1\,. \end{cases}$$

Taking into account, that $\lim\limits_{x\to 1} f'_y(x) = 0$, we obtain the shape of the density function.

The value $f_y(x_2(y))$ of the local maximum tends to infinity as $y \to 0$.

The same property holds also for finite n; for $y \ll \frac{\sqrt{2}-1}{2}\sqrt{2n}$ there is a wave in the sequence of probabilities, for $y \gg \frac{\sqrt{2}-1}{2}\sqrt{2n}$, however they are monotonically decreasing. We give the probabilities for $n = 15$ with $k = 1$ and $k = 2$.

g	$k = 1$	$k = 2$
0	0,4153	0,6384
1	0,0697	0,0567
2	0,0540	0,0432
3	0,0465	0,0365
4	0,0421	0,0324
5	0,0394	0,0296
6	0,0377	0,0274
7	0,0366	0,0256
8	0,0360	0,0240
9	0,0259	0,0224
10	0,0361	0,0206
11	0,0368	0,0184
12	0,0378	0,0151
13	0,0387	0,0097
14	0,0374	—

§ 2. Joint distribution of the number of leading steps and the number of winning series

According to the definition of a winning series given in our introduction, we formulate the following

Theorem 2.1. *Let us denote by* λ_{2n} *the number of winning series, then*

$$\mathbf{P}(\lambda_{2n}=l, \gamma_{2n}=g)=$$

$$=\frac{1}{2^{2n}}\frac{\binom{2g}{g-l}}{g(n-g)}\left\{l(2n-g+l)\binom{2n-2g}{n-g+l}+(l-1)(l+g)\binom{2n-2g}{n-g-l+1}\right\},$$

if $l=1,2,\ldots,n,\ g=l,l+1,\ldots,n-l;$
in the case of $g=0$, *also* $l=0$ *and*

$$\mathbf{P}(\lambda_{2n}=0, \gamma_{2n}=0)=\frac{\binom{2n}{n}}{2^{2n}}.$$

In the proof of Theorem 2.1 we refer to the considerations used in § 1 for the proof of the arcsine law.

In case a), i.e. $s_1=+1,\ s_{2n-1}>0$ there are $l-1$ complete winning series and one incomplete winning series, their total length being $2g$; according to Theorem 1 of [3] p. 283 the number of the possibilities equals

$$2\binom{2g-1}{g-l}=\frac{l+g}{g}\binom{2g}{g-l}.$$

To each of these winning series there belong complete losing series of total length $2n-2g$, the number of the latter being — according to Theorem 1.2 of [4] p. 101 —

$$\frac{l-1}{n-g}\binom{2n-2g}{n-g-l+1}.$$

Hence case a) results in

$$\frac{(l-1)(l+g)}{g(n-g)}\binom{2g}{g-l}\binom{2n-2g}{n-g-l+1}$$

possible paths. By the same argument we obtain in case b)

$$\frac{l(l+g)}{g(n-g)}\binom{2g}{g-l}\binom{2n-2g}{n-g-l},$$

in case c)

$$\frac{l(n-g+l)}{g(n-g)}\binom{2g}{g-l}\binom{2n-2g}{n-g-l},$$

in case d)

$$\frac{l(n-g-l)}{g(n-g)}\binom{2g}{g-l}\binom{2n-2g}{n-g-l}.$$

The sum of the values in a)—d) leads to our formula in Theorem 2.1. For the limiting case we have the following

Theorem 2.2.

$$\lim_{n\to\infty} \mathbf{P}(\lambda_{2n} < y\sqrt{2n}, \gamma_{2n} < zn) = \frac{4}{\pi}\int_0^y\int_0^z \frac{u}{[v(1-v)]^{3/2}}\, e^{-\frac{2u^2}{v(1-v)}}\, du\, dv,$$

for $y \geqq 0,\ 1 \geqq z \geqq 0.$

This is a consequence of simple and known asymptotic relations. Integration with respect to y and z resp. leads to the known relations

$$\lim_{n\to\infty} \mathbf{P}(\lambda_{2n} < y\sqrt{2n}) = \sqrt{\frac{32}{\pi}}\int_0^y e^{-8u^2}\, du,$$

$$\lim_{n\to\infty} \mathbf{P}(\gamma_{2n} < nz) = \frac{1}{\pi}\int_0^z \frac{du}{\sqrt{u(1-u)}}.$$

§3. Some remarks concerning the case $s_{2n} = 0$.

In this § we consider the case, when the game becomes balanced after $2n$ steps, i.e. we assume throughout the condition $s_{2n} = 0$. We shall consider distribution laws of type dealt with in our § 1, i.e. concerning the random variables $2\gamma_{2n}^{(k)}$ and $\lambda_{2n}^{(k)}$. The first relation gives the number of steps in which the cumulative gain of A exceeds k and $\lambda_{2n}^{(k)}$ denotes the number of series of this kind. Authors determined in their paper [4] the following distribution

$$\mathbf{P}(\varkappa_{2n} > k, \gamma_{2n}^{(k)} = g, \lambda_{2n}^{(k)} = l),$$

where $\varkappa_{2n}$ denotes the maximum of the cumulative gain of A in the course of $2n$ games.

For limiting distribution we obtained

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\varkappa_{2n}}{\sqrt{2n}} > a, \frac{\lambda_{2n}^{(k)}}{\sqrt{2n}} < y, \frac{\gamma_{2n}^{(k)}}{n} < z\right) =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^y\int_0^z \frac{u^2+2au}{[v(1-v)]^{3/2}}\, e^{-\frac{(u+2av)^2}{2v(1-v)}}\, du\, dv.$$

The extension of the distribution

$$\mathbf{P}(\gamma_{2n} = g \,|\, s_{2n} = 0) = \frac{1}{n+1}, \qquad g = 0, 1, \ldots n$$

is obtained by substituting $y = \infty$. For the density function

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\varkappa_{2n}}{\sqrt{2n}} > a,\ z < \frac{\gamma_{2n}^{(2k)}}{n} < z + dz\right) \sim \sqrt{\frac{2}{\pi}} \int\limits_{z}^{1} \frac{a}{[v(1-v)]^{3/2}} e^{-\frac{2a^2}{v(1-v)}} \, dv\, dz$$

holds.

It can be seen that — similarly to the arcsine law — the density is infinite for $z = 0$, it is however zero for $z = 1$ in consequence of the conditions $s_{2n} = 0$ and $a > 0$.

The uniform distribution belonging to the case $a = 0$ can be obtained using the substitution

$$x = \frac{2a^2}{1-v}.$$

We obtain for our above expression

$$\frac{1}{\sqrt{\pi}} \int\limits_{\frac{2a^2}{1-z}}^{\infty} \frac{x}{(x - 2a^2)^{3/2}} e^{-x} \, dx\, dz,$$

which is the analogon of the expression (5) in case $s_{2n} = 0$. Substituting $a = 0$ we obtain

$$\lim_{n\to\infty} \mathbf{P}\left(z < \frac{\gamma_{2n}}{n} < z + dz\right) \sim \frac{1}{\sqrt{\pi}} \int\limits_{0}^{\infty} \frac{1}{\sqrt{x}} e^{-x} \, dx\, dz = dz.$$

§ 4. The case of continuous and symmetric variables

Let $\xi_1, \xi_2, \ldots, \xi_n$ be totally independent random variables with a common continuous symmetric distribution function. Let further be $S_0 = 0$, $S_i = \xi_1 + \ldots + \xi_i$, $i = 1, 2, \ldots, n$. An index i is called a ladder index if $S_j < S_i$, $j = 0, 1, \ldots, i-1$. It may occur that $i = 0$ is the only ladder index, the probability of which is in consequence of the arcsine law (the case $g = 0$)

$$\frac{1}{2^{2n}} \binom{2n}{n}.$$

If there are several ladder indices $i_1 < i_2 < \ldots < i_t$, then $S_{i_1} < S_{i_2} < \ldots$ $\ldots < S_{i_t}$. Using the above assumptions and notations, there holds the following

Theorem 3.1. *Denoting by* $\Gamma_n^{(k)}$ *the number of terms in* $(S_0, S_1, \ldots, S_n)$ *which exceed* S_{i_k} *the following relations hold:*

$$\mathbf{P}(\Gamma_n^{(k)} = g) = \frac{1}{2^{2n-k}} \binom{2g}{g} \binom{2n-2g-k}{n-g} \tag{3'}$$

for $g = 1, 2, \ldots, n$ *and*

$$\mathbf{P}(\Gamma_n^{(k)} = 0) = \sum_{j=0}^{k} \frac{1}{2^{2n-j}} \binom{2n-j}{j}. \tag{4'}$$

Remark: $\Gamma_{n+k}^{(2k)}$ has formally the same distribution as $\gamma_{2n}^{(2k)}$.

Let us denote by $\varphi_r^{(k)}$ the probability that r is the k-th ladder index in $(S_0, S_1, \ldots, S_r)$. Then using the arcsine law

$$\mathbf{P}(\Gamma_n^{(k)} = g) = \sum_{r=k}^{n-g} \varphi_r^{(k)} \frac{1}{2^{2n-r}} \binom{2g}{g} \binom{2n-2r-2g}{n-r-g}$$

holds for $g = 1, 2, \ldots, n$.

According to [5] p. 86—87 for symmetric variables

$$\sum_{r=k}^{\infty} \varphi_r^{(k)} z^r = \left(\sum_{r=1}^{\infty} \varphi_r^{(1)} z^r \right)^k = (1 - \sqrt{1-z})^k$$

holds. As

$$\sum_{j=0}^{\infty} \frac{1}{2^{2j}} \binom{2j}{j} z^j = \frac{1}{\sqrt{1-z}},$$

$\mathbf{P}(\Gamma_n^{(k)} = g)$ is the coefficient of z^{n-g} in the generating function

$$2^{-2g} \binom{2g}{g} \frac{1}{\sqrt{1-z}} (1 - \sqrt{1-z})^k.$$

As the known relation

$$\sum_{\alpha=0}^{\infty} \binom{k+2\alpha}{\alpha} \frac{1}{2^{2\alpha}} z^\alpha = \frac{1}{\sqrt{1-z}} \left(\frac{2}{1+\sqrt{1-z}} \right)^k$$

holds, the result is our formula (3′).

Substituting $g = 0$ and j instead of k in (3′) summation over j from 0 to k gives (4′).

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О НЕКОТОРЫХ РАСПРЕДЕЛЕНИЯХ СВЯЗАННЫХ С АРКСИНУС ЗАКОНОМ

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Резюме

1. Пусть $\{\xi_i\}$ $(i=1,2,\ldots,2n)$ последовательность независимых случайных величин, для которых $\mathbf{P}(\xi_i=+1)=\mathbf{P}(\xi_i=-1)=\frac{1}{2}$. Пусть далее $s_0=0$, $s_i=\xi_1+\ldots+\xi_i$ $(i=1,2,\ldots,2n)$. $2\gamma_{2n}^{(2k)}$ обозначает число индексов i, для которых или $s_i>2k$, или $s_i=2k$ и $s_{i-1}=2k+1$. Тогда имеет место соотношение:

$$\mathbf{P}(\gamma_{2n}^{(2k)}=g)=\frac{1}{2^{2n}}\binom{2g}{g}\binom{2n-2g}{n-g+k} \qquad g=1,2,\ldots,n \tag{3}$$

$$\mathbf{P}(\gamma_{2n}^{(2k)}=0)=\frac{1}{2^{2n}}\sum_{j=-k}^{k}\binom{2n}{n+j}. \tag{4}$$

Доказательство этих формул производится элементарным комбинаторным методом.

2. $\lambda-1$ обозначает число индексов i $(0<i<2n)$, для которых $s_i=0$ и $s_{i-1}s_{i+1}=-1$. Имеет место следующее равенство:

$$\mathbf{P}(\lambda_{2n}=l,\gamma_{2n}^{(0)}=g)=\frac{1}{2^{2n}}\frac{\binom{2g}{g-l}}{g(n-g)}\times$$

$$\times\left\{l(2n-g+l)\binom{2n-2g}{n-g+l}+(l-1)(l+g)\binom{2n-2g}{n-g-l+1}\right\},$$

$$l=1,2,\ldots,n,\quad g=l,l+1,\ldots,n-l$$

и

$$\mathbf{P}(\lambda_{2n}=0,\gamma_{2n}^{(0)}=0)=\frac{1}{2^{2n}}\binom{2n}{n}.$$

3. Пусть $\{\xi_i\}$ $(i=1,2,\ldots,n)$ последовательность одинакого распределенных независимых случайных величин с непрерывной функцией распределения $F(x)$, обладающей свойством $F(x)=1-F(-x)$ и пусть $S_0=0$, $S_i=\xi_1+\ldots+\xi_i$ $(i=1,2,\ldots,n)$.

Индекс i называется ступенью, если $S_0 < S_i$, $S_1 < S_i$, ..., $S_{i-1} < S_i$. Пусть $i_1 < i_2 < \ldots < i_l < \ldots$ обозначают все ступени в последовательности $(S_0, S_1, \ldots, S_n)$. $\Gamma_n^{(k)}$ обозначает число индексов j для которых $S_j > S_{i_k}$. Тогда имеет место

$$\mathbf{P}(\Gamma_n^{(k)} = g) = \frac{1}{2^{2n-k}} \binom{2g}{g} \binom{2n-2g-k}{n-g} \qquad g = 1, 2, \ldots, n \tag{3'}$$

$$\mathbf{P}(\Gamma_n^{(k)} = 0) = \sum_{j=0}^{k} \frac{1}{2^{2n-j}} \binom{2n}{n-j}. \tag{4'}$$

В случае $k = 0$ формулы (3) и (3′) переносятся на формулы специального и общего арксинус законов.