ON A "BIG DEVIATIONS" PROBLEM

by

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A result of Yu V. LINNIK [1] states that if $\xi_1, \ldots, \xi_n$ are independent random variables with the same distribution function $F(x)$ of the form

$$
(1)\qquad
\begin{aligned}
F(x) &= \frac{A_a}{|x|^a} + \ldots + \frac{A_{4a+5}}{|x|^{4a+5}} + O\left(\frac{1}{|x|^{4a+5+\varepsilon}}\right); \text{ for } x \to -\infty \\
1 - F(x) &= \frac{A_a}{x^a} + \ldots + \frac{A_{4a+5}}{x^{4a+5}} + O\left(\frac{1}{x^{4a+5+\varepsilon}}\right); \text{ for } x \to \infty
\end{aligned}
$$

$$(a \text{ integer}, \ a > 3, 0 < \varepsilon < 1)$$

and the density function exists, then, denoting by $F_n(x)$ the distribution function of the variable

$$
(2)\qquad \frac{\sum_{i=1}^{n} \xi i - n\mathbf{E}(\xi_1)}{\sqrt{n}\, \mathbf{D}(\xi_1)}
$$

we have

$$
(3)\qquad
\begin{aligned}
&\frac{F_n(x)}{\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{u^2}{2}} du + r(x, \sqrt{n})} \to 1 \quad (x \leqq -1); \\
&\frac{1 - F_n(x)}{\frac{1}{\sqrt{2\pi}} \int\limits_{x}^{\infty} e^{-\frac{u^2}{2}} du + r(x, \sqrt{n})} \to 1 \quad (x \geqq 1) \text{ for } n \to \infty
\end{aligned}
$$

uniformly in x where $r(x, \sqrt{n})$ is a rational function of both variables. Furthermore, if $x > n^{\frac{3}{2} + \frac{1}{a} + \varepsilon}$, then

$$
(4)\qquad 1 - F_n(x) \sim n(1 - F(x\sqrt{n}))\,.
$$

Yu. V. LINNIK raised also the problem of finding an analogue of this theorem, for the case when $F(x)$ is of the form:

$$(5)\qquad \begin{aligned} 1 - F(x) &= \int_a^{a+5} \frac{dG(\nu)}{x^\nu} + O\left(\frac{1}{x^{4a+5+\varepsilon}}\right), \text{ for } x \to \infty \\ F(x) &= \int_a^{a+5} \frac{dG(\nu)}{|x|^\nu} + O\left(\frac{1}{|x|^{4a+5+\varepsilon}}\right), \text{ for } x \to -\infty, \end{aligned}$$

where $G(\nu)$ is a functions of bounded variation. In the present note this problem is solved. We assume that $\int_a^{a+\varepsilon} d\,|\,G(\nu)\,| > 0$ for every $\varepsilon > 0$.

We state and prove the following theorem.

Theorem. *Let $\xi_1, \ldots, \xi_n$ be independent random variables with a common distribution function $F(x)$ of the form* (5) *and let us suppose that the density function exists. Then* (3) *and* (4) *are valid, but $r(x, \sqrt{n})$ is replaced by a function of the form*

$$(6)\qquad R(x, n) = \int_m^M \frac{1}{n^\nu x^{\nu-1}} q(x, \nu)\, dK(\nu)$$

where $q(x, \nu)$ is a rational function of x bounded as $x \to \infty$, and a continuous function of ν as $K(\nu)$ is a function of bounded variation.

Proof. For sake of simplicity, let us suppose, that the variables ξ_i are symmetrical, i.e. $\mathbf{P}(\xi_i > x) = \mathbf{P}(\xi_i < -x)$ for $x > 0$ and that $\mathbf{D}^2(\xi_i) = 1$. It is easy to see, as in [1] that though the supposition of symmetricity simplifies the calculations, it does not play a siginificant role.

Let $\varphi(t)$ be the characteristic function of ξ_i. Then

$$(7)\qquad \varphi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) = h_1(t) + 2\,\mathrm{Re} \int_1^{\infty} e^{itx}\, dF(x)$$

where $h_1(t)$ is an analytical function. As to the second term of (7) we have

$$(8)\qquad \int_1^\infty e^{itx}\, dF(x) = \int_1^\infty e^{itx}\, d \int_a^{4a+5} \frac{dG(\nu)}{x^\nu} + h_2(t),$$

here $h_2(t)$ is $4a + 5$ times differentiable. Now

$$(9)\qquad \int_1^\infty e^{itx}\, d \int_a^{4a+5} \frac{dG(\nu)}{x^\nu} = \int_a^{4a+5} \left(\int_1^\infty e^{itx}\, d\left(\frac{1}{x^\nu}\right)\right) dG(\nu)$$

and

$$(10)\qquad \int_1^\infty e^{itx}\, d\left(\frac{1}{x^\nu}\right) = -\nu \int_1^\infty \frac{e^{itx}}{x^{\nu+1}}\, dx = h_{1\nu}(t) + h_2(t) \int_1^\infty \frac{e^{itx}}{x^{\{\nu\}}}\, dx$$

where $h_{j\nu}(t)$ $(j = 1, 2)$ is a polynomial and $\{\nu\}$ is the fractional part of ν.

$$\int_1^\infty \frac{e^{itx}}{x^{\{\nu\}}}\,dx = h_{3\nu}(t) + t^{\{\nu\}-1}\int_0^\infty \frac{e^{iy}}{y^{\{\nu\}}}\,dy \tag{11}$$

as $h_{2\nu}(t) = c(\nu)t^{[\nu]}$ we have

$$\int_1^\infty \frac{e^{itx}}{x^{\nu+1}}\,dx = h_{4\nu}(t) + c(\nu)\,t^\nu\,. \tag{12}$$

Finally we obtain the following formula for the characteristic function $\varphi(t)$.

$$\varphi(t) = \int_b^B t^\nu dG_2(\nu) + O(t^{4a+5})\,, \qquad (t > 0, t \to 0)\,. \tag{13}$$

As it is easy to show, there exist b_1, B_1 and $G_3(\nu)$ such, that

$$\Big(\int_b^B t^\nu\,dG_2(\nu)\Big)^n = \int_{b_1}^{B_1} t^\nu\,dG_3(\nu)\,. \tag{14}$$

According to our assumptions $\mathbf{M}(\xi_1) = 0$, $\mathbf{D}(\xi_1) = 1$, that is

$$\varphi(t) = 1 - \frac{t^2}{2} + O(t^3) \qquad (t > 0\,,\ t \to 0) \tag{15}$$

and thus it follows from (12) and (13), that there exists $\varepsilon_0 > 0$ such, that for $0 \leqq t \leqq \varepsilon_0$

$$\log\varphi(t) = -\frac{t^2}{2} + \int_3^{B_3} t^\nu\,dG_4(\nu) + O(t^{4a+5})\,. \tag{16}$$

For $n \geqq 2$ we have for the density $f_n(x)$ of (2)

$$f_n(x) = \frac{\sqrt{n}}{2\pi}\int_{-\infty}^{\infty} (\varphi(t))^n\,e^{-\sqrt{n}\,itx}\,dt = \frac{\sqrt{n}}{\pi}\,\mathrm{Re}\int_0^\infty (\varphi(t))^n\,e^{-\sqrt{n}\,itx}\,dt \tag{17}$$

as ξ_1 and hence $f_n(x)$ too, are symmetrical. In the same way, as in [1] it can be shown, that

$$f_n(x) = \frac{\sqrt{n}}{\pi}\,\mathrm{Re}\int_0^{\frac{\log n}{\sqrt{n}}} (\varphi(t))^n\,e^{-\sqrt{n}\,itx}\,dt + O(e^{-\varepsilon_1(\log n)^2})\,. \tag{18}$$

From (16) we have

$$(\varphi(t))^n = e^{n \log \varphi(t)} = e^{-n\frac{t^2}{2}} \sum_{k=0}^{\infty} \frac{\left(n \int_3^{B_3} t^\nu \, dH_k^*(\nu)\right)^k}{k!} = \tag{19}$$
$$= e^{-n\frac{t^2}{2}} + e^{-n\frac{t^2}{2}} \sum_{0<k\leq N} n^k \int_{3k}^{D_k} t^\nu \, dH_k(\nu) + O(t^{4a+5})$$

where $H_k^*(\nu)$ and $H_k(\nu)$ are functions of bounded variation. Substituting the expression in (19) for $(\varphi(t))^n$ into (18) we get

$$f_n(x) = \frac{\sqrt{n}}{\pi} \int_0^{\frac{\log n}{\sqrt{n}}} e^{-n\frac{t^2}{2}} \sum_{0<k\leq N} n^k \int_{3k}^{D_k} t^\nu \, dH_k(\nu) \, e^{-ixt} \, dt + \tag{20}$$
$$+ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + O(n^{-2a-2}) ;$$

changing the order of integration in (20) we obtian

$$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sum_{0<k\leq N} n^k \int_{3k}^{D_k} \left(\int_0^{\frac{\log n}{\sqrt{n}}} e^{-\frac{t^2}{2}} t^\nu e^{-ixt} \, dt \right) dH_k(\nu) + O(n^{-2a-2}) . \tag{21}$$

By substituting $t = \frac{\tau}{\sqrt{n}}$,

$$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sum_{0<k\leq N} \int_{3k}^{D_k} \left(\int_0^{\log n} e^{-\frac{\tau^2}{2}} \left(\frac{\tau}{\sqrt{n}} \right)^\nu e^{-ix\tau} \, d\tau \right) dH_k(\nu) + O(n^{-2a-2}). \tag{22}$$

As

$$\int_{\log n}^{\infty} e^{-\frac{\tau^2}{2}} \tau^\nu \, d\tau = O\left(e^{-\frac{1}{4}(\log n)^2}\right), \tag{23}$$

we have

$$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sum_{0<k\leq N} \int_{3k}^{D_k} \left(\int_0^{\infty} e^{-\frac{\tau^2}{2}} \left(\frac{\tau}{\sqrt{n}} \right)^\nu e^{-ix\tau} \, d\tau \right) dH_k(\nu) + O(n^{-2a-2}) . \tag{24}$$

The following asymptotic expansion holds for the integral

$$\int_0^{\infty} e^{-\frac{\tau^2}{2}} \tau^\nu e^{-ix\tau} \, d\tau = \frac{c_0}{x^{\nu-1}} + \ldots + \frac{c_L}{x^{L+\nu}} + O\left(\frac{c_L}{x^{L+\nu+1}} \right) \tag{25}$$

(see [2] p. 61).

Finally we find for $f_n(x)$

$$
f_n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sum_{0<k\leq N} n^k \int_{3k}^{D_k} \left(\frac{1}{\sqrt{n}}\right)\left(\frac{c_0(\nu)}{x^{\nu-1}} + \ldots + \frac{c_L(\nu)}{x^{L+\nu}}\right) dH_k(\nu) + \tag{26}
$$
$$
+ O(n^{-2a-2}) + O\left(\frac{1}{n} x^{-L}\right)
$$

where L is as large as we want.

It can be shown by the same method as in [1], that for $x < n^{\frac{3}{2}+\frac{1}{a}+\varepsilon}$

$$
1 - F_n(x) \sim n(1 - F(x\sqrt{n})) \tag{27}
$$

holds.

It follows from (26) and (27), as in [1], integrating (26) from x $(1 < x \leq n^2)$ to n^2

$$
1 - F_n(x) = 1 - \varphi(x) + R(x, n) + O(n^{-2a}) + O\left(\frac{1}{n} x^{-L+1}\right) \tag{28}
$$

where $R(x, n)$ is according to our statement of the form

$$
R(x, n) = \int_m^M \frac{1}{n^\nu x^{\nu-1}} q(x, \nu)\, dk(\nu) . \tag{29}
$$

It follows from (28) and (27) that for $n^{\frac{7}{4}} < x < n^2$

$$
R(x, n) \sim 1 - F_n(x) \sim n \int_a^{a+\varepsilon} (x\sqrt{n})^{-\nu}\, dG(\nu) . \tag{30}
$$

For (29) the relation (30) must hold for $x > n^2$ too. As for $x < n^2$ the error terms in (28) may be neglected in comparison with $1 - F_n(x)$ and for $x > n^{\frac{7}{4}}$ $1 - \Phi(x)$ is infinitely small in comparison with $1 - F_n(x)$, the theorem is proved.

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REFERENCES

[1] LINNIK, Yu. V.: "On the probability of large deviations for the sums of independent variables". *Proc. 4 th. Berkeley Symp. Mat. Stat. and Prob. Vol II.* pp. 289—306. Univ. California Press, Berkeley, Calif. 1961.

[2] ERDÉLYI, A.: *Asymptotic expansions.* Dover Publications, Inc.

ОБ ОДНОЙ ПРОБЛЕМЕ „БОЛЬШИХ УКЛОНЕНИЙ"

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Резюме

Пусть $\xi_1, \ldots, \xi_n$ случайные величины с функцией распределения вида (5), то имеет место (3) равномерно на всей оси, где вместо функцией $r(x, \sqrt{n})$ пишется $R(x, n)$ вида (6), а $q(x, v)$ — рациональная функция от $x, K(v)$ — функция ограниченной вариации.