ON THE CIRCUITS OF FINITE GRAPHS

by

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§ 1.

The following three statements were proved by G. A. DIRAC: ([2] Theorem 2, 3 and 4.)

(A) *If every vertex of the graph* G[1] *is of valency* $\geqq k$ ($\geqq 2$), *then* G *contains a circuit having at least* $k+1$ *edges.*

(B) *Let* $G^{(n)}$ *be a graph of* n ($\geqq 3$) *vertices, and let us assume that every vertex of* $G^{(n)}$ *is of valency* $\geqq n/2$. *Then* G *is Hamiltonian (i.e.* $G^{(n)}$ *has a circuit containing all vertices).*

(C) *Assume that* $G^{(n)}$ *is twofold connected*[2] *and every vertex is of valency* $\geqq k$ *where* $n \geqq 2k$. *Then* $G^{(n)}$ *contains a circuit of at least* $2k$ *edges.*

Several recent papers generalize (B) and (C). (S. [3], [4], [5], [6].) This paper contains some further generalizations and a sharpening of (A). In § 2 we show (generalizing (B)) that by a suitable sharpening of the lower bound $n/2$ for the valency of the vertices we can infer the existence of a Hamilton line which passes through certain prescribed paths. In § 3 — generalizing (A) and (C) — we show that circuits of length $\geqq k+1$ resp. $\geqq 2k$ exist even if certain vertices are of valency $< k$ (in [6] we generalized (B) in this direction). In § 4 we show (Generalizing (B)) that certain conditions imply that there are $\leqq j$ (j is a given integer) disjoint circuits and vertices (respectively disjoint circuits, edges and vertices), which contain every vertex of our graph.

Some notations: Vertices will be denoted by small Roman letters. The edge connecting a and b will be denoted by ab (or ba). The valency of a (i.e. the number of edges incident to a) will be denoted by $v(a)$. $a \in G$ resp. $ab \in G$ means that the vertex a resp. the edge ab is in G. The empty graph contains neither vertices nor edges. $G^{(n)}$ denotes always a graph with n vertices. $G_1 \cup G_2$ denotes the graph which consists of the vertices and edges contained in G_1 and G_2. $P = (a_1 \ldots a_n)$ denotes the path consisting of the *distinct* vertices $a_1, \ldots, a_n$ and of the edges $a_1a_2, \ldots, a_{n-1}a_n$. (a_1) denotes the degenerate path which contains only the vertex a_1. $(a_i P a_j)$, $1 \leqq i \leqq j \leqq n$ denotes the section of P

[1] In this paper we only consider graphs which do not contain loops or double edges.

[2] A graph $G^{(n)}$, $n \geqq 3$ is called twofold connected if it is connected and has no cut point. A vertex x is called a cut point of the graph G if the omission of x and all the edges incident to x increases the number of the components of G.

between a_i and a_j. $C = (a_1 \ldots a_n a_1), n \geqq 3$ denotes the circuit which contains the distinct vertices $a_1, \ldots, a_n$ and the edges $a_1a_2, \ldots, a_{n-1}a_n, a_na_1$. The vertex a will also be considered as a circuit and will be denoted by (a). The length of a path or a circuit will denote the number of its edges.

§ 2.

Henceforth a graph will be called a *path-system* if its components are non-degenerate paths. The length of a path-system is the sum of the length of its paths.

Theorem 1. *Let $n \geqq 3$, $1 \leqq l \leqq n - 2$ and $k = [(n + l + 1)/2]$. Let further $G^{(n)}$ be a graph every vertex of which has valency not less than k and let S be any path-system of length l in G. Then G has an H-line which passes through S (i.e. all the edges of S are edges of our Hamilton line).*

Proof. Assume that the theorem is not true and let $G^{(n)}$ be a graph which satisfies the requirements of the theorem and which has a path-system S of length l through which there does not pass an H-line of $G^{(n)}$. Let G^* be a graph having the same vertices as $G^{(n)}$ and containing all the edges of $G^{(n)}$, and which does not contain an H-line passing through S, but if two unconnected vertices of G^* are connected by an edge then there is an H-line passing through S. (In other words G^* contradicts to our theorem and is maximal with respect to this property. We obviously obtain G^* by connecting unconnected vertices of $G^{(n)}$, since the complete graph spanned by the vertices of $G^{(n)}$ has an H-line passing through S. G^* clearly exists.)

Let a and b be two unconnected vertices of G^*, if we connect them by an edge we obtain a graph which has an H-line passing through S. This H-line clearly passes through the edge ab and hence G^* has an open H-line passing through S (an open H-line is a path which passes through all vertices of the graph) whose endpoints are a and b. Let

$$P = (a_1 \ldots a_n), \, a_1 = a, \, a_n = b$$

be such an open H-line and denote by $a_{i_1}, a_{i_2}, \ldots, a_{i_p}$ $(2 = i_1 < \ldots < i_p < n)$ the vertices of G^* which are connected with a_1 by an edge in G^*. By our assumption $p \geqq k$. If $a_{i_\alpha - 1} a_{i_\alpha} \notin S$ $(2 \leqq \alpha \leqq p)$ then $a_{i_\alpha - 1} a_n \notin G^*$ (for if not then $(a_1 a_{i_\alpha} a_{i_\alpha + 1} \ldots a_n a_{i_\alpha - 1} a_{i_\alpha - 2} \ldots a_1)$ would be an H-line of G^* which contains S). At most l of the edges $a_{i_\alpha - 1} a_{i_\alpha}$ can belong to S and therefore (counting a_1 too) there are at least $p - l$ vertices in G^* with which a_n is not connected by an edge. Hence

$$v(a_n) \leqq n - 1 - (p - l) \leqq n - k + l - 1 < k .$$

This contradicts our assumptions and hence Theorem 1 is proved.

Now we show that Theorem 1 is best possible. Let $n \geqq 3$, $1 \leqq l \leqq n - 2$, $k = [(n + l + 1)/2]$. The vertices of $G^{(n)}$ are $a_1, a_2, \ldots, a_n$. The edge a_ia_j $(i < j)$ belongs to $G^{(n)}$ if and only if $i \leqq k - 1$. Clearly every vertex of our $G^{(n)}$ has valency $\geqq k - 1$. Let S be the path $(a_1a_2 \ldots a_{l+1})$. It can be shown by a simple argumentation left to the reader that $G^{(n)}$ does not contain an H-line which passes through S.

§ 3.

First we prove the following sharpening of (A) (mentioned already in § 1):

Theorem 2. *Let $n > 0$ and let us assume that for every $0 \leqq i \leqq k - 1$ ($k \geqq 2$) the number of vertices of valency $\leqq i$ of $G^{(n)}$ is $\leqq i$. Then $G^{(n)}$ contains a circuit of length $\geqq k + 1$.*

Proof. Consider the longest paths of $G^{(n)}$ and let

$$P = (a_1 \dots a_m), \qquad m \geqq 2$$

be such a longest path, for which the sum of the valencies of the endpoints $v(a_1) + v(a_m)$ is maximal. Without loss of generality we can assume $v(a_1) \geqq v(a_m)$. We show $v(a_1) \geqq k$. Assume that this is not true and put $v(a_1) = p < k$. Clearly all the vertices connected with a_1 are in P (for otherwise P would not be a longest path.) Let these points be

$$a_{i_1}, \dots, a_{i_p} \; (2 = i_1 < i_2 < \dots < i_p \leqq m).$$

By our assumption at least one of the $p + 1$ vertices

$$a_1 = a_{i_1 - 1}, a_{i_2 - 1}, \dots, a_{i_p - 1}, a_m$$

has valency $> p$. From $v(a_m) \leqq v(a_1) = p$ it follows that this vertex must differ from a_1 and a_m, hence $p \geqq 2$. Let $v(a_{i_\alpha - 1}) > p$ $(1 < \alpha \leqq p)$. But then $(a_{i_\alpha - 1} \dots a_1 a_{i_\alpha} \dots a_m)$ is a longest path for which

$$v(a_{i_\alpha - 1}) + v(a_m) > v(a_1) + v(a_m)$$

which contradicts the maximality property of P. This contradiction proves $p \geqq k$. But hence the length of the circuit $(a_1 \dots a_{i_p} a_1)$ is at least $k + 1$ which completes the proof of Theorem 2.

The complete k-gon or the complete k-gon with an edge attached to it shows that in a certain sense Theorem 2 is also sharp (our graphs have k vertices of valency $\leqq k - 1$ and no circuits of length $\geqq k + 1$), but the question of its sharpness is not yet completely cleared up.

Now we prove the following sharpening of (C):

Theorem 3. *Let $n \geqq 2k$, $k \geqq 2$, $G^{(n)}$ be a twofold-connected graph. Assume further that for $i = 1, 2, \dots, k - 1$ the number of vertices of valency $\leqq i$ is $\leqq i - 1$. Then $G^{(n)}$ contains a circuit of length $\geqq 2k$.*

Proof. I) Assume that the theorem is not true and let $G^{(n)}$ be a graph which satisfies the conditions of the theorem and for which the longest circuit has length $< 2k$. As in the proof of Theorem 1, we construct the graph G^* which does not yet contain a circuit of length $\geqq 2k$, but if we connect two not connected vertices of G^* by an edge we obtain at least one circuit of length $\geqq 2k$. As in the proof of Theorem 1, G^* clearly exists, and satisfies the conditions of Theorem 3, furthermore its longest path has length $\geqq 2k - 1$.

Consider the longest paths of G^* and let

$$P = (a_1 a_2 \dots a_{m-1} a_m), \qquad m \geqq 2k$$

be such a maximal path for which $v(a_1) + v(a_m)$ is maximal. (v here denotes the valency in G^*.)

Since G^* does not contain a circuit of length $\geqq 2k$, $a_1 a_m \notin G^*$. Denote by

$$a_{i_1}, \ldots, a_{i_p} \quad (2 = i_1 < \ldots < i_p < m)$$

and

$$a_{j_1}, \ldots, a_{j_q} \quad (1 < j_1 < \ldots < j_q = m-1)$$

the vertices connected in G^* with a_1 and with a_m, resp.

II) We now prove $p \geqq k$, $q \geqq k$. It will suffice to show $p \geqq k$. a_1 can be connected in G^* only with the vertices of P, for otherwise P would not be a longest path. Thus p is the valency of a_1 in G^*. Assume $p < k$. By our assumption at least one of the p vertices $a_{i_1-1}, a_{i_2-1}, \ldots, a_{i_p-1}$ has valency $> p$ in G^*. Let $a_{i_\alpha-1}$ be such a vertex. Clearly $\alpha \neq 1$ and thus the path $(a_{i_\alpha-1} \ldots a_2 a_1 a_{i_\alpha} \ldots a_m)$ has the same length as P and $v(a_{i_\alpha-1}) + v(a_m) > v(a_1) + v(a_m)$, which contradicts the maximality property of P, hence $p \geqq k$, $q \geqq k$ is proved.

III) Next we show $i_p \leqq j_1$. Assume $i_p > j_1$ and put

$$\min(i_\gamma - j_\delta) = \triangle; \quad i_\gamma > j_\delta, \quad 1 \leqq \gamma \leqq p, \quad 1 \leqq \delta \leqq q.$$

By our assumption $\Delta > 0$, and assume that $i_\alpha - j_\beta = \Delta$. Clearly the inner vertices of $(a_{j_\beta} P a_{i_\alpha})$ can not be connected with a_1 or a_m by an edge. Therefore the circuit (belonging to G^*)

$$C = (a_1 a_2 \ldots a_{j_\beta} a_m a_{m-1} \ldots a_{i_\alpha} a_1)$$

contains all the vertices $a_{j_1}, \ldots, a_{j_q}$ and except $a_{i_\alpha-1}$ also all the vertices $a_{i_1-1}, \ldots, a_{i_p-1}$. These $p + q - 1$ vertices are all distinct, for if $j_\delta = i_\gamma - 1$ then

$$(a_1 \ldots a_{j_\delta} a_m a_{m-1} \ldots a_{i_\gamma} a_1)$$

would be a circuit of lentgh $m \geqq 2k$ of G^*. Thus together with a_m C contains at least $p + q \geqq 2k$ vertices. This contradiction proves $i_p \leqq j_1$.

IV) The following theorem is due to DIRAC ([2] Lemma 2, [1] pp. 196—197).

Let $\tilde{P} = (x_1 \ldots x_s)$, $s \geqq 2$ be a path of the twofold-connected graph $\tilde{G}$. Then there are two paths $\tilde{P}_1$ and $\tilde{P}_2$ connecting x_1 and x_s which are disjoint except for their endpoints x_1 and x_s and the common vertices of $\tilde{P}_i$ and $\tilde{P}$ occur in the same order in both paths $(i = 1, 2)$.

Let us now apply this theorem to the path P of our graph G^* and let us choose among the pairs of paths satisfying the theorem a pair P_1 and P_2 such that their union contains a maximal number of vertices of P. Put $P_1 \cup P_2 = C$. We shall now show that C contains all the vertices $a_{i_1}, \ldots, a_{i_p}$ and $a_{j_1}, \ldots, a_{j_q}$. It will suffice to show this for $a_{i_1}, \ldots, a_{i_p}$. Assume $a_{i_\alpha} \notin C$ $(1 \leqq \alpha \leqq p)$. Let g be the greatest of the indices $1, 2, \ldots, i_\alpha - 1$ for which $a_g \in C$ and h the smallest of the indices $i_\alpha + 1, i_\alpha + 2, \ldots, m$ for which $a_h \in C$. a_g and a_h can not belong to the same P_i $(i = 1$ or $2)$. For if let us say both belong to P_1 then the graph

$$P_1' = (a_1 P_1 a_g) \cup (a_g P a_h) \cup (a_h P_1 a_m)$$

is a path and the pair P_1', P_2 also satisfies the requirements of DIRAC's theorem, contains all the vertices of C which are contained in P, and furthermore contains a_{i_α} too. This contradicts the maximality property of the pair P_1, P_2. We can thus assume

$$a_g \in P_1 \quad \text{and} \quad a_h \in P_2 .$$

But then the graphs

$$P_3 = (a_1 P a_g) \cup (a_g P_1 a_m) \quad \text{and} \quad P_4 = (a_1 a_{i_\alpha} a_{i_\alpha+1} \dots a_h) \cup (a_h P_2 a_m)$$

are paths belonging to G^* which satisfy the requirements of DIRAC's theorem. $P_3 \cup P_4$ contains all the vertices of P contained in C and also a_{i_α}, which again contradicts the maximality property of the pair P_1, P_2. This contradiction proves our assertion.

By III) the vertices $a_{i_1}, \dots, a_{i_p}$; $a_{j_1}, \dots, a_{j_q}$ are all distinct except possibly $a_{i_p} = a_{j_1}$. Therefore from $p \geqq k$, $q \geqq k$ and $a_1 \in C$, $a_m \in C$ we obtain that C contains at least $2k + 1$ vertices. This contradicts our assumption that the greatest circuit of G^* has length $< 2k$. This completes the proof of Theorem 3.

We can show that Theorem 3 is not sharp but we have not succeeded in finding the best possible theorem here.

§ 4.

A graph will be called a *system of circuits* if its components are circuits; isolated vertices will be regarded as circuits. If the system of circuits T contains every vertex of G, we say that T covers G.

Theorem 4. *Let $n \geqq 5$, $1 \leqq j \leqq n - 4$, $k = [(n - j + 2)/2]$. Assume that for $i = 1, \dots, k - 1$ the number of vertices of $G^{(n)}$ of valency $\leqq i$ is at most $i - 1$. Then $G^{(n)}$ is covered by a system T containing at most j circuits.*

Remark. If $j = 1$ then Theorem 4 is identical with the sharpening of (B) proved in [6].

The proof of Theorem 4 depends on the following

Lemma. *Let $n \geqq 2k$, $k \geqq 3$. Assume that for every $i = 1, \dots, k - 1$ the number of vertices of $G^{(n)}$ of valency $\leqq i$ is at most $i - 1$. Then either $G^{(n)}$ containts a circuit of length $\geqq 2k$, or it contains two circuits having disjoint vertices, the sum of whose lengths is $\geqq 2k + 1$.*

Proof. Assume that $G^{(n)}$ contradicts the theorem. Then by Theorem 3 $G^{(n)}$ can not be twofold connected and hence must contain two endlobes.[3]

Let G_1 and G_2 be two endlobes. By our assumptions every vertex of G has valency $\geqq 2$, thus G_i $(i = 1, 2)$ has at least three vertices. We define the vertices a_i $(i = 1, 2)$ as follows:

If G_i contains a cutpoint of $G^{(n)}$ then let a_i be this cutpoint. If G_i does not contain a cutpoint of $G^{(n)}$ then a_i is an arbitrary vertex of G_i. The valency of every vertex of G_1 (with the possible exception of a_1) in the graph G_1 is the same as its valency in $G^{(n)}$. Therefore for $i = 0, 1, \dots, k - 1$ the number of vertices in G_1 of valency $\leqq i$ is at most i. By Theorem 2 G_1 contains a circuit C_1 of length $\geqq k + 1$. Let G_2^* be the graph obtained from G_2 by omitting

[3] The lobes of a graph which is not connected are the lobes of its components. The *endlobes* are the lobes which contain at most one cutpoint of the graph (s. [5] p. 88).

the vertex a_2 and all the edges incident to it. Clearly G_2^* is non-empty and the valency of every vertex in G_2^* is by at most one smaller than the valency of the same vertex in $G^{(n)}$. Hence by Theorem 2 G_2^* contains a circuit C_2 of length $\geqq k$. C_1 and C_2 are disjoint and the sum of their lengths is $\geqq 2k+1$, which proves the lemma.

The proof of Theorem 4 follows easily from the lemma. Assume that $G^{(n)}$ satisfies the requirements of our theorem. If $j=1$ the theorem follows from the theorem stated in our remark. Assume thus that $j \geqq 2$. Then $n \geqq 2k$, $k \geqq 3$, thus by our lemma $G^{(n)}$ contains either a circuit C of length $\geqq 2k$ or two disjoint circuits C_1 and C_2 the sum of whose lengths is $\geqq 2k+1$. Let $\tilde{T}$ be either C or $C_1 \cup C_2$. Then $\tilde{T}$ together with the vertices of $G^{(n)}$ not belonging to $\tilde{T}$ give a system of $\leqq j$ circuits which covers $G^{(n)}$, hence Theorem 4 is proved.

Finally we prove a covering theorem which differs from Theorem 4 inasmuch as we allow in the covering besides circuits and isolated vertices also "isolated edges". A set $a_1, \ldots, a_m$ $(m \geqq 1)$ of vertices of G is said to be *independent* if no two of them are connected by an edge. The maximal number of independent vertices is denoted by $\varphi(G)$.

Theorem 5. *Let G be a non-empty graph. Then it always contains a covering system of disjoint circuits, edges and vertices having at most $\varphi(G)$ members.*

Proof. We use induction with respect to $\varphi(G)$. If $\varphi(G)=1$, G is complete and can be covered by one circuit or an edge or a vertex. Assume that the theorem holds if $\varphi(G) \leqq j-1$ $(j>1)$ and let $\varphi(G)=j$.

Assume that G has n vertices. Let

$$P=(a_1 \ldots a_p) \qquad (p \geqq 1)$$

be a longest path of G. If $p=1$, G consists of $n=j$ isolated vertices, hence our theorem is trivial. Assume thus $p>1$. As stated previously, a_1 can be connected (by an edge) only with the vertices of P. Denote by

$$a_{i_1}, \ldots, a_{i_q} \quad (2=i_1<\ldots<i_q, q \geqq 1)$$

the vertices connected with a_1. Omit from G the vertices $a_1, a_2, \ldots, a_{i_q}$ and the edges incident to them, and denote the remaining graph by G'. If G' is empty then $q>1$ and the circuit $(a_1 \ldots a_{i_q} a_1)$ covers G. If G' is non-empty then a_1 is not connected with any vertex of G', hence $\varphi(G')<\varphi(G)$. By our induction hypothesis G' can be covered by a covering system having at most $\varphi(G')$ components and together with $(a_1 \ldots a_{i_q} a_1)$ (or if $q=1$ with the "edge" $(a_1 a_2)$) we obtain a covering system having at most $1+\varphi(G') \leqq \varphi(G)$ components, which completes our proof.

A graph whose components are triangles show that Theorem 5 is best possible.

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ОБ ОКРУЖНОСТЯХ КОНЕЧНЫХ ГРАФОВ

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Резюме

Автор доказывает для графов, не содержащих петель и многократных рёбер теоремы, из которых он приводит следующие:

Теорема 1. *Пусть будет* $n \geqq 3$, $1 \leqq l \leqq n-2$, $k = [(n+l+1)/2]$ *а* G *— граф с* n *вершинами, такой что в нём любая вершина имеет степень* $\geqq k$. *Далее, пусть* S *— система путей в графе* G, *не содержающих попарно общие вершины и содержащая* l *рёбер графа* G. *Тогда существует в* G *Гамильтонова линия, содержащая все рёбра от* S.

Теорема 3. *Пусть* $n \geqq 2k$, $k \geqq 2$, *и* G *двухсвязный граф, содержающий* n *вершин, такой, что число вершин степени* $\leqq i$ *в нём не более* $i-1$ *для любого значения* $i = 1, 2, \ldots, k-1$. *Тогда* G *содержит по крайней мере одну окружность, состоящую из* $2k$ *рёбер.*

Теорема 4. *Пусть* $n \geqq 5$, $1 \leqq j \leqq n-4$, $k = [(n-j+2)/2]$ *и* G *граф, содержающий* n *вершин, в котором число вершин степени* $\leqq i$ *не более* $i-1$ *для любого значения* $i = 1, 2, \ldots, k-1$. *Тогда существует в* G *система окружностей попарю не содержащих общих вершин, которая состоит из составляющих не более* j *и которая содержит все вершины графа* G (здесь рассматривается одна единственная вершина также как окружность).