FILLING OF A DOMAIN BY DISCS

by

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As it is well known the area a and perimeter p of a plane domain[1] satisfy the so called isoperimetric inequality $p^2 \geqq 4\,\pi\, a$, and equality holds only for a circle. This statement has two meanings: on the one hand, of the domains of given perimeter the circle has the greatest possible area, on the other hand, of the domains of given area the circle has the least possible perimeter. Of the numerous variants of the isoperimetric inequality we mention only the following result of BESICOVITCH [1]:

Let C be a convex domain and $C(r)$ the union of the points of those circles of radius r which can be placed into C. Then $C(r)$ has of all isoperimetric discs lying in C the greatest area, and $C(r)$ has of all equiareal discs lying in C the least perimeter.

In the case when C is a convex polygon, $C(r)$ — the outer parallel domain of radius r of the inner parallel domain of radius r of C — arises from C by rounding off each corner by circular arcs which can be put together to form one circle of radius r (Fig. 1). Such a domain we shall call a *smooth polygon*.

Let R be a regular hexagon of unit area, $a_6(p)$ the upper bound of the areas of the discs of perimeter $\leqq p$ contained in R and $p_6(a)$ the lower bound of the perimeters of the discs of area $\geqq a$ contained in R. It is clear that for small values of p (for small values of a) the bound $a_6(p)$ ($p_6(a)$) is attained by a circle, on the other hand, in view of the theorem of BESICOVITCH, for values of p (of a) graeter than the perimeter (area) of the incircle of R the extremal domain will be a smooth hexagon. An elementary computation shows that

$$a_6(p) = \begin{cases} \dfrac{p^2}{4\pi} & \text{for} \quad \dfrac{p^2}{4\pi} \leqq d \\[2ex] \dfrac{p^2}{4\pi} \cdot \dfrac{d}{1-d} \cdot \left(\dfrac{2}{\sqrt{d \cdot \dfrac{p^2}{4\pi}}} - 1 - \dfrac{1}{\dfrac{p^2}{4\pi}} \right) & \text{for} \quad d \leqq \dfrac{p^2}{4\pi} \leqq \dfrac{1}{d} \\[2ex] 1 & \text{for} \quad \dfrac{1}{d} \leqq \dfrac{p^2}{4\pi} \end{cases}$$

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[1] Both words „domain" and „disc" will be used for a bounded closed set the inner points of which form a simply connected set.

and

$$p_6^2(a) = \begin{cases} 4\pi a & \text{for} \quad a \leqq d \\ 4\pi a \left\{ \sqrt{\frac{1}{d} \cdot \frac{1}{a}} - \sqrt{\left(\frac{1}{d} - 1\right)\left(\frac{1}{a} - 1\right)} \right\}^2 & \text{for } d \leqq a \leqq 1 \end{cases}$$

where the constant $d = \frac{\pi}{\sqrt{12}}$ equals the density of the densest packing of equal circles in the plane[2], i.e. the ratio of the area of a circle and that of the circumscribed regular hexagon. (For $\frac{p^2}{4\pi} \leqq \frac{1}{d}$ $a_6(p)$ and $p_6(a)$ are inverse functions.)

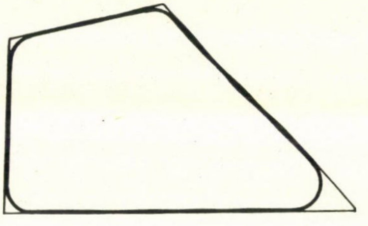

Fig. 1.

FEJES TÓTH extended the investigation of the isoperimetric problem, concerning with a single domain, to the study of a certain set of domains and raised the following two problems:

Problem A. Find the upper bound of the average area a of n non-overlapping discs, each of perimeter $\leqq p$, lying in a given domain[3] D.

Problem P. Find the lower bound of the average perimeter p of n non-overlapping discs, each of area $\geqq a$ $\left(a \leqq \frac{D}{n}\right)$ lying in a given domain D.

As it is quite hopeless to solve these problems in such a general form they were investigated under several restrictions. The problem of the determination of the asymptotic behaviour of the extremal configuration for great values of n, may be consider as the fundamental isoperimetric problems for two-dimensional cellaggregates. For convex discs these questions were investigated by FEJES TÓTH [2] (Problem A) and by FEJES TÓTH and the author [3] (Problem P). Their results are summarised in the following theorems[4]

Theorem A. *The average area a of n convex discs, each of perimeter $\leqq p$ lying in a convex hexagon[5] H of area n without mutual overlapping, is not greater than the greatest possible area of one disc of perimeter $\leqq p$ lying in a regular hexagon of unit area, i.e.*

$$a \leqq a_6(p).$$

[2] Cf. L. FEJES TÓTH: *Lagerungen in der Ebene, auf der Kugel und im Raum.* Berlin—Göttingen—Heidelberg, 1953.

[3] We denote a domain and its area with the same symbol.

[4] In the original form of the above theorems the discs are supposed to be isoperimetric (Theorem A) and equiareal (Theorem P), respectively, but the original proofs remain valid without any modification for these slightly more general statements too.

[5] Hexagon in a wider sense: polygon having at most six vertices.

Theorem P. *The average perimeter p of n convex discs, each of area $\geqq a$ ($a \leqq 1$), lying in a convex hexagon*[5] *H of area n without mutual overlapping, is not smaller than the least possible perimeter of one disc of area $\geqq a$ lying in regular hexagon of unit area, i.e.*

$$p \geqq p_6(a) .$$

Equality holds in the following cases: (i) H is a regular hexagon containing one single disc, namely the corresponding smooth hexagon, (ii) the discs are congruent circles (in both theorems) and (iii) — in Theorem A — when they fill H without gaps. However these bounds can be approximated with an arbitrary exactitude for great values of n. Since for great values of n the special shape of the given domain is irrelevant these theorems inform us about the asymptotic behaviour of the extremal configurations of the discs in an arbitrary domain. It is interesting to observe that in spite of the fact that the arrengements to be compared were originally entirely irregular a single optimum-requirement implies the congruence of the discs as well as their regular shape and arrengement.

Now the question arises whether Theorems A and P remain valid without the restriction of the convexity of the discs.

We shall show that in the case of Theorem A the answer is affirmative. This is expressed in

Theorem A*. *The average area a of n discs, each of perimeter $\leqq p$, lying in a convex hexagon H of area n without mutual overlapping, is not greater than the greatest possible area of a disc of perimeter $\leqq p$ lying in a regular hexagon of unit area, i.e.*

$$a \leqq a_6(p) .$$

We also shall prove a variant of Theorem A*:

Theorem A.** *Let U be the union of n faces of a tessellation consisting of regular hexagons of unit area. The average area a of n discs, each of perimeter*[6] *$\leqq p \leqq \sqrt{8\sqrt{3}} = 3{,}72\ldots$, lying in U without mutual overlapping, attains its maximum in the case when all the discs are congruent, namely circles or smooth hexagons of perimeter p, inscribed in the faces of the tessellation contained in U, i.e.*

$$a \leqq a_6(p) .$$

The proofs of Theorem A* and Theorem A** are based on the proof of Theorem A and on three lemmas listed below.

We shall say that two closed convex domains C_1 and C_2 *intersect simply* if they satisfy one of the following conditions:

(i) C_1 and C_2 have no inner points in common,

(ii) one of them is completely covered by the other, or

(iii) C_1 and C_2 overlap and the boundary of their (convex) intersection can be split up into two non-overlapping connected arcs, one belonging to the boundary of C_1 and the other to that of C_2.

Lemma 1. *If two discs, D_1 and D_2 have no inner points in common then the convex hull of D_1 and that of D_2 intersect simply.*

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[6] The constant $\sqrt{8\sqrt{3}}$ is the perimeter of a hexagonal face of the tessallation.

Let C_1 and C_2 be the convex hulls of D_1 and D_2. In contrary to our statement we suppose that C_1 and C_2 do not intersect simply. Then they have inner points in common but none of the discs is completely covered by the other. Thus the boundary B of the union $C_1 \cup C_2$ consists of the common points of the two boundaries (single points or closed arcs) and of the open "proper boundary arcs" of C_1 and C_2, respectively, consisting of the boundary points of C_1 outside C_2 and the boundary points of C_2 outside C_1. Since C_1 and C_2 do not satisfy (iii) there exist (at least) two pairs of proper boundary arcs belonging to C_1 and C_2, respectively, and having the property that each of this pairs separates the arcs of the other pair on B. It is easy to see that to each proper boundary arc there exist a supporting line, which does not meet the other disc. On the other hand each supporting line of the convex hull of a connected bounded set contains at least one of the boundary points of the original set. Consequently B contains two pairs of points belonging alternately to the boundaries of the original discs D_1 and D_2. Thus both pairs of their four different points can be connected by a single arc through the interior of D_1 and D_2, respectively. But this contradicts to the fact that D_1 and D_2 have no inner points in common.

Lemma 2. *Suppose that any two of the convex discs $C_1, \dots, C_m$ intersect simply and, that none of them is completely covered by the others. Then the discs can be contracted into non-overlapping discs $\overline{C}_1, \dots, \overline{C}_m$ ($\overline{C}_i \subset C_i$, $i = 1, \dots, m$), the union of which equals the union of $C_1, \dots, C_m$.*

The proof rests on the following lemma of Bambah and Rogers (see [4] Lemma 1) which we quote without proof:

Lemma 3. *Let S and T be two convex discs which intersects simply. Suppose that a segment divides S into two sets $S^{(1)}$ and $S^{(2)}$. Then $S^{(1)}$ and T intersect simply; exept possibly when the segment divides T into sets $T^{(1)}$ and $T^{(2)}$, one of which is contained in $S^{(2)}$. In this latter case $S^{(1)}$ and $T^{(1)}$ intersect simply.*

Lemma 2 is trivial if no pair of the discs have inner points in common. Therefore we suppose that there exist two discs, say C_i and C_j, which overlap. Since the discs intersect simply we can find (according to (iii)) two points P_1 and P_2 on the intersection of the boundaries of C_i and C_j, which split the boundary of the union $C_i \cup C_j$ into two non-overlapping connected arcs B_i and B_j belonging to the boundary of C_i and C_j, respectively. Thus the segment P_1P_2 divides $C_i \cup C_j$ into the non-overlapping discs $C_i^{(1)}$ and $C_j^{(1)}$ ($C_i^{(1)} \subset C_i$ and $C_j^{(1)} \subset C_j$). Replacing now C_i by $C_i^{(1)}$ and C_j by $C_j^{(1)}$ it may happen that some of the new discs do not intersect simply (Fig. 2). For instance, let C_k be a disc which does not intersect $C_j^{(1)}$ simply. Then, referring to Lemma 3, P_1P_2 divides C_k into two parts, one of which is contained in $C_j^{(1)}$ and the other, $C_k^{(1)}$, intersects $C_i^{(1)}$ (and $C_j^{(1)}$) simply. In this case we replace C_k by $C_k^{(1)}$. Proceeding in this way step by step, we can construct a new system of convex discs, intersecting simply one another. In each step of this process the union of the discs remains unchanged, but the number of the overlapping pairs of discs decreases. Thus in at most $\binom{m}{2}$ steps we obtain the desired system $\overline{C}_1, \dots, \overline{C}_m$. (Since, by assumption, none of the original discs were completely covered by the others, all the discs $\overline{C}_1, \dots, \overline{C}_m$ really occur.)

After these preparations we can easily prove Theorem A*. Let $D_1, \dots, D_n$ be n non-overlapping discs, each of perimeter $\leq p$, contained in the convex hexagon H. Instead of $D_1, \ldots, D_n$, consider their convex hulls $C_1, \ldots, C_n$. In consequence of Lemma 1, $C_1, \ldots, C_n$ intersect simply. If some of them are covered by the others we cancel them one after another. Finally we obtain a subset of the C_i's consisting of $m \leqq n$ discs, none of which is covered by the others and having the same union as originally. Then, using Lemma 2, we can construct, a system of m non-overlapping convex discs, each of perimeter

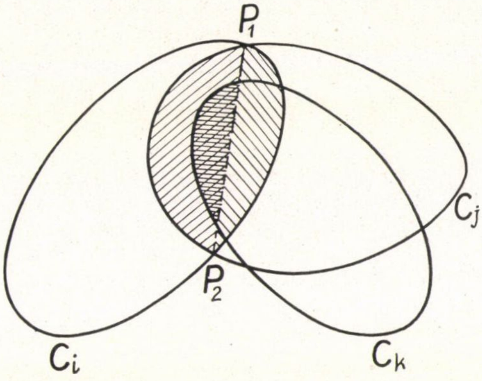

Fig. 2.

$\leqq p$, lying in H and having a union U_m, which contains each of the original discs. Applying Theorem A to these convex discs, we have

$$\frac{\sum_{i=1}^{n} D_i}{n} \leqq \frac{U_m}{n} \leqq a_6(p).$$

Theorem A** is trivial in the case, when all the discs are circles of perimeter p or, more generally, if $p \leqq \frac{2\pi}{\sqrt[4]{12}}$. Thus we restrict ourselves to the case $p > \frac{2\pi}{\sqrt[4]{12}}$. We shall show that the problem may be reduced to the case of cellaggregates consisting of convex cells. The rest of the proof is similar to the proof of Theorem A.

Let $\mathcal{T}$ be a tessellation consisting of regular hexagons of unit area and U the union of n cells of $\mathcal{T}$. Without loss of generality we may suppose that U is connected. Let U' denote the union of U and the cells of $\mathcal{T}$ adjacent to U (Fig. 3). Being given n discs $D_1, D_2, \ldots, D_n$, each of perimeter $\leqq p$, lying in U without mutual overlapping, we have to show that the average area of these discs is not greater than $a_6(p)$. For this purpose we place into each cell of U' not contained in U a smooth hexagon of perimeter p and area $a_6(p)$. We denote these new discs with $C_{n+1}, \ldots, C_{n+k}$. We have only to show that the average area of the discs $D_1, \ldots, D_n, C_{n+1}, \ldots, C_{n+k}$ is not greater than $a_6(p)$.

Denote the convex hulls of the discs $D_1, \ldots, D_n$ with $C_1, \ldots, C_n$. It is not difficult to show that each disc C_i $(i \leq n)$ lies in U': Let c_1 be the circumcircle of a cell of $\mathcal{T}$ not contained in U' and c_2 the concentric circle of double radius (Fig. 4). Since D_i lies outside of c_2 and its perimeter is not greater than that of a cell of $\mathcal{T}$ $\left(p \leq \sqrt{8\sqrt{3}}\right)$, the convex hull C_i of D_i can not contain a chord of c_2 longer than $\frac{\sqrt{8\sqrt{3}}}{2}$, and therefore C_i can not intersect the cell lying in c_1 $(i = 1, \ldots, n)$.

In view of Lemma 1 any pair of $C_1, \ldots, C_{n+k}$ intersect simply. Then, referring to Lemma 2 and using similar considerations as in the proof of Theorem A*, we can construct a new system of $m \leq n + k$ non-overlapping convex discs $C'_1, \ldots, C'_m$, each of perimeter $\leq p$, contained in U' and covering together the original discs $C_1, \ldots, C_{n+k}$.

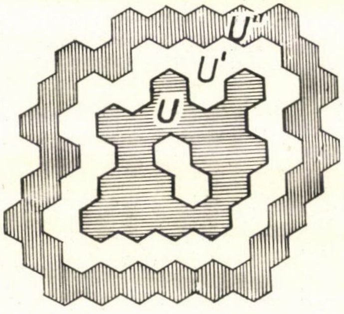

Fig. 3.

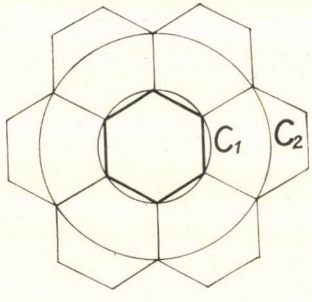

Fig. 4.

First of all we join to the domain U' the neigbouring cells of $\mathcal{T}$ obtaining the domain U''. Just as above, we place into each new adjoined hexagon a smooth hexagon of perimeter p (and consequently of area $a_6(p)$). We denote these new discs with $C'_{m+1}, \ldots, C'_{m+l}$.

Let us now "blow up" the discs $C'_1, \ldots, C'_{m+l}$, preserving their convexity and the property of neither overlapping nor streching out of U'', obtaining $m + l$ convex polygons $P_1, \ldots, P_{m+l}$ having $\nu_1, \ldots, \nu_{m+l}$ sides, respectively. Although, in general, these polygons do not fill out U'' without gaps, they may be considered from a combinatorial point of view as to form a „polygonal decomposition" of U''[7]. We proceed to prove that the average number of sides $\bar{\nu} = \frac{\nu_1 + \ldots + \nu_{m+l}}{m + l}$ of these polygons does not exceed 6. For this purpose we shall compare the irregular decomposition with the regular hexagonal decomposition of U''. In the hexagonal decomposition let e denote the number of the edges, e_b the number of the edges on the boundary of U'', v the number of the vertices and v_2 the number of the vertices in which only two edges meet. Let e', e'_b, v' and v'_2 denote the corresponding

[7] Cf. L. Fejes Tóth [2].

data of the irregular polygonal decomposition. Counting the edges and vertices in both decompositions we obtain

$$6(n+k+l) = 2e - e_b, \quad \bar{\nu}(m+l) = 2e' - e'_b,$$

$$3v - v_2 = 2e \quad \text{and} \quad 3v' - v'_2 \leqq 2e'.$$

As a consequence of the construction of U'' the polygons, lying along the boundary of U'', of the irregular decomposition coincide with the cells of the regular decomposition. It follows from this that $e_b = e'_b$ and $v_2 = v'_2$. Using Euler's formula

$$(n+k+l) + v = e + 1 \quad \text{and} \quad (m+l) + v' = e' + 1$$

we obtain from the above relations that

$$e_b + 6 = 2v_2 \quad \text{and} \quad e_b + 6 \leqq 2v_2 + (6 - \bar{\nu})(m+l)$$

which involves the desired inequality $6 \geqq \bar{\nu}$.

Now we make use of a known inequality. Let C be a disc of perimeter $\leqq p$ contained in a convex polygon of given area P and given number of side ν. Than $C \leqq F(P, \nu)$, where the function $F(P, \nu)$ is defined by

$$F(P,\nu) = \begin{cases} P & \text{for} \quad P < \dfrac{p^2}{4\nu \operatorname{tg}\frac{\pi}{\nu}} \\[2ex] \dfrac{p\sqrt{P\nu \operatorname{tg}\frac{\pi}{\nu} - \frac{1}{4}p^2 - \pi P}}{\nu \operatorname{tg}\frac{\pi}{\nu} - \pi} & \text{for} \quad \dfrac{p^2}{4\nu \operatorname{tg}\frac{\pi}{\nu}} \leqq P \leqq \dfrac{p^2}{4\pi^2}\nu \operatorname{tg}\dfrac{\pi}{\nu} \\[2ex] \dfrac{p^2}{4\pi} & \text{for} \quad \dfrac{p^2}{4\pi^2}\nu \operatorname{tg}\dfrac{\pi}{\nu} < P. \end{cases}$$

In the middle interval $F(P, \nu)$ equals the area of a smooth polygon of perimeter p lying in a regular ν-gon of area P. Since $F(P, \nu)$ is a non-decreasing function both of P and ν, and as a function of two variables, it is concave,[8] we have, in view of Jensen's inequality

$$\frac{\sum_{i=1}^{N} C'_i}{N} \leqq \frac{\sum_{i=1}^{N} F(P_i, \nu_i)}{N} \leqq F\left(\frac{\sum_{i=1}^{N} P_i}{N}, \frac{\sum_{i=1}^{N} \nu_i}{N}\right) \leqq F(1, 6) = a_6(p),$$

where, for $i = m + l + 1, \ldots, n + k + l = N$, $C'_i = 0$, $P_i = 0$ and $\nu_i = 6$. Equality holds only when $C'_1, \ldots, C'_N$ are regular hexagons lying in the cells of $\mathcal{T}$.

Let us return now to Problem P. It admits of no doubt that also Theorem P continues to hold for not necessarily convex discs, but the proof of this conjecture seems to involve considerable difficulties. These difficulties are

[8] For the details of the proof of this statement see L. FEJES TÓTH [2] II and III.

implied in the fact that the best arrengement generally contains also not convex discs. Let us divide, for instance, a regular triangle into two parts of equal area by a shortest arc. It is easy to show[9] that this arc is an arc of circle centered at a vertex of the triangle. Thus one of the parts is not convex.

More generally, it is not difficult to show that the shortest net, dividing a (plane or spherical) domain into partial domains each of given area, consists of arcs of circle (of finite or infinite radius) meeting another at an angle equal to $\frac{2\pi}{3}$ and meeting the boundary of the domain at an angle not less than $\frac{\pi}{2}$.[10]

This necessary condition yields

Theorem S. *For* $n \neq 2, 3, 4, 6, 12$, *the shortest net dividing the sphere into n parts of equal area contains a non-convex mesh.*

Suppose that for $n = k > 1$ the shortest net consists of convex meshes i.e. convex spherical polygons. Then, in view of the equality of the angles, the area of a polygon depends only on the number of its sides. Thus each polygon must be a $\left(6 - \frac{12}{k}\right)$-gon, and, consequently, $\frac{12}{k}$ must be an integer. It is almost trivial that for $n = 2$ and $n = 3$ the extremal net consists of a great circle and of three half great circles meeting another at equal angles. It would be interesting to show that for $n = 4, 6$ and 12 the best net is the spherical net of a regular tetrahedron, hexahedron and dodekahedron, respectively. These nets play an important role in an analogous problem discussed by L. FEJES TÓTH [5]. He has given an estimation for the length of a spherical net consisting of *n convex* meshes of equal area. His estimation is exact in the cases $n = 2, 3, 4, 6, 12$ for the nets listed above.

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[9] Consider the regular hexagon $A_1, A_2, \ldots A_6$ with center A_0. Let α and β two arcs joining the sides $A_0 A_1$ and $A_0 A_2$ of the triangle $A_0 A_1 A_2$ and dividing $A_0A_1A_2$ into two parts of equal area; let α be a circular arc centered at A_0 and β an arbitrary one. Successive reflections in the lines $A_0A_2, A_0A_3, \ldots, A_0A_6$ complete α to a circle and β to a closed curve, both bounding a region the area of which equals the half area of the hexagon $A_0A_1 \ldots A_6$. Since the circle has less perimeter, α is shorter than β. This ingenious proof due to E. MOLNÁR.

[10] Hier we do not intend to discuss the problem of the existence of the shortest net.

ЗАПОЛНЕНИЕ ОБЛАСТИ ПЛОСКИМИ ФИГУРАМИ

A. HEPPES

Резюме

В качестве продолжения исследований, начатых несколько лет назад ([2], [3]) автор доказывает следующую теорему, а также другие теоремы.

Теорема. *Если в области, построенной как соединение n регулярных шестиугольников, являющихся составляющими мозаиками, помещены n друг друга не перекрывающих плоских фигур, причём периметр каждой из них не более p, тогда среднее значение площадей этих плоских фигур не может быть больше максимума площадей плоских фигур с периметрами не более p, помещаемыми в одном составляющем (шестиугольнике) мозаики.*

Эту оценку, очевидно, нельзя улучшить; ведь максимума можно достить, если в каждом составляющем мозаики поместить плоскую фигуру возможно наибольшей площади. Следовательно, экстремальная система состоит — соответственно значениям p — из конгруентных окружений, или же из шестиугольников, окруженных посредством конгруентных дуг окружностей, помещенных в первоначальных составляющих мозаики.