# ON THE JORDAN-HÖLDER THEOREM FOR UNIVERSAL ALGEBRAS

by

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## Introduction

There are several ways to generalize the Jordan-Hölder theorem. One may consider this theorem as a result on equivalence relations on a set[1] (the set being the group or ring itself); or as a statement on the „subnormal" elements of lattices upon which a binary relation ("$a$ is normal in $b$") is defined,[2] the lattice being the lattice of all subgroups of the given group. But if we look at the theorem as it is then the most natural way of generalization is to get rid of the axioms of groups (rings) and prove the result for arbitrary universal algebras. This was done by A. W. GOLDIE [3] and my first aim is to give a variant of his proof. My proof is much shorter although no essentially new idea is used. The concepts are also different and, I hope, more natural. At the proper place (§ 4) I'll give my reasons for not following the old pattern.

Then we will point out that the proof employed makes us possible to give further generalizations. In the proofs we used the concept of homomorphism and subalgebra but the operations were very seldom taken into consideration. Therefore we consider universal algebras as sets among which certain mappings, called homomorphisms, are defined, satisfying certain axioms.[3] This axiomatic treatment makes us possible to extend the Jordan-Hölder theorem to multialgebras.

## § 1. Preliminaries

An *algebra* is a couple $(A; F)$ where $A$ is a set and the elements of $F$ are finitary operations on $A$, i.e. each $f \in F$ is a function of $n$-variables, $n$ depending on $f$, it associates with every $n$-tuple $(a_1, \ldots, a_n)$ of elements of $A$ an element $f(a_1, \ldots, a_n)$ of $A$. Let $F_n$ denote the set of all operations of $n$ variable, $F = \bigcup_{n=0}^{\infty} F_n$.

[1] See e.g. BIRKHOFF [1], pp. 87—89 and the references on p. 89.

[2] See e.g. ZASSENHAUS [7], pp. 190—198, where some references are also given.

[3] That this is the natural framework for the Jordan-Hölder theorem was first pointed out to me by E. FRIED who also gave an axiom system similar to I—VIII of § 5. I don't know what is the connection between his (still unpublished) axiom system and mine. Of course, this is not new. If we go one step further, we get the notion of categories.

In this note all algebras considered are of the same type, i.e. the set of operations can be denoted by the same letter $F$. For simplicity's sake we omit the letter $F$. Thus if we say that $A$ is an algebra then we have the algebra $(A; F)$ in mind.

A non-void subset $B$ of $A$ is a *subalgebra* if $b_1, \ldots, b_n \in B, f \in F_n$ imply $f(b_1, \ldots, b_n) \in B$. The intersection of two subalgebras is again a subalgebra, provided it is non-void.

The set of all equivalence relations over $A$ is denoted by $P(A)$. If $\varepsilon_\lambda \in P(A)$, $\lambda \in \Lambda$ then $\bigcup \varepsilon_\lambda$, defined by $x \equiv y(\bigcup \varepsilon_\lambda)$ if and only if there exists in $A$ a finite sequence $x = z_0, z_1, \ldots, z_n = y$ such that $z_{i-1} \equiv z_i(\varepsilon_{\lambda_i})$ for some $\lambda_i \in \Lambda$, $i = 1, 2, \ldots, n$ and $\bigcap \varepsilon_\lambda$ defined by $x \equiv y(\bigcap \varepsilon_\lambda)$ if and only if $x \equiv y(\varepsilon_\lambda)$ for all $\lambda \in \Lambda$ are again in $P(A)$. If we partially order $P(A)$ by $\varepsilon_1 \leq \varepsilon_2$ if and only if $\varepsilon_1 = \varepsilon_1 \cap \varepsilon_2$ then we see that $P(A)$ is a complete lattice in which $\bigcup \varepsilon_\lambda$ and $\bigcap \varepsilon_\lambda$ are the least upper bound, resp. greatest lower bound of the set $\{\varepsilon_\lambda; \lambda \in \Lambda\}$.

A *congruence relation* $\Theta$ is an equivalence relation satisfying the *substitution property:* if $a_i \equiv b_i\ (\Theta)$, $i = 1, 2, \ldots, n$ and $f \in F_n$ then $f(a_1, \ldots, a_n) \equiv$ $\equiv f(b_1, \ldots, b_n)\ (\Theta)$. The set of all congruence relations is denoted by $\Theta(A)$. Obviously, $\Theta(A) \subseteq P(A)$ and as it is known $\varepsilon_\lambda \in \Theta(A)$, $\lambda \in \Lambda$ imply $\bigcup \varepsilon_\lambda$, $\bigcap \varepsilon_\lambda \in \Theta(A)$. We denote by $\omega$ and $\iota$ the least, resp. greatest element of $P(A)$. Obviously $\omega, \iota \in \Theta(A)$.

If $\Theta$ is a congruence relation on $A$ and $H$ is a subset of $A$ let $[H]\ \Theta$ denote the union of the congruence classes of $A$ represented by $H$, i.e. $[H]\ \Theta = \{x;\ x \equiv h(\Theta)$ for some $h \in H\}$. The algebra $A/\Theta$ is defined on the congruence classes $[x]\ \Theta$ in the following way:

$$f([x_1]\Theta, \ldots, [x_n]\Theta) = [f(x_1, \ldots, x_n)]\Theta\,.$$

The mapping $\varphi$: $x \to [x]\ \Theta$ is called a *homormorphism*.

It is easy to prove that if $B$ is a subalgebra then so is $[B]\Theta$. $B$ is called *closed* under $\Theta$ if $B = [B]\ \Theta$.

A subalgebra $B$ of $A$ is called *normal* if $A$ is a whole congruence class under some congruence relation $\Theta$, i.e. $B$ is closed under $\Theta$ and $x \equiv y(\Theta)$ for every $x, y \in B$.

Let $\Theta$ be a congruence relation and $B$ a subalgebra of $A$. Then we define $\Theta_B$, the *restriction* of $\Theta$ to $B$, as follows:

$x \equiv y(\Theta_B)$ if and only if $x, y \in B$ and $x \equiv y(\Theta)$. Obviously, $\Theta_B$ is a congruence relation of $B$.

Let $\Theta, \Phi \in \Theta(A)$ such that $B$ is closed under $\Theta$ and $\Phi$. Then $B$ is closed under $\Theta \cup \Phi$.

Let $\Theta$, $\Phi \in \Theta(A)$ and $B$ a subalgebra of $A$. We say (modifying the notion invented by GOLDIE [3]) that $\Theta$ and $\Phi$ are *weakly associable* over $B$ if

$$\big[[[B]\Theta]\Phi\big]\ \Theta = [[B]\Theta]\ \Phi\,,$$

or equivalently,

$$[[B]\Theta]\ \Phi = [[B]\Phi]\ \Theta\,,$$

or, a third equivalent form

$$[B]\Theta \cup \Phi = [[B]\Theta]\ \Phi\,.$$

This means, that if $b \equiv x(\Theta)$ and $x \equiv y(\Phi)$, $b \in B$, $x, y \in A$, then there exist $b' \in B$, $z \in A$ with $b' \equiv z(\Phi)$, $z \equiv y(\Theta)$.

Let $B, C$ and $D$ be subalgebras of $A$, $\Theta$ and $\Phi$ congruence relations on $B$ and $C$, respectively, $D \subseteq B \cap C$. $\Theta$ and $\Phi$ are said to be *weakly associable* over $D$ if $\Theta_{B\cap C}$ and $\Phi_{B\cap C}$ are weakly associable over $D$.

## § 2. The Zassenhaus lemma

The following statement is the essence of the Zassenhaus lemma:

**Lemma 1.** *Let $A$ be an algebra, $B$ a subalgebra of $A$, $\Theta$ a congruence relation of $A$, $\Phi$ a congruence relation of $B$ such that $\Theta_B \leq \Phi$. Define a relation $\Theta(\Phi)$ on $[B]\Theta$ by*

(1) $a \equiv b(\Theta(\Phi))$ *if and only if there exist $c, d \in B$ such that*
$$a \equiv c(\Theta),\ c \equiv d(\Phi),\ d \equiv b(\Theta).$$

*Then $\Theta(\Phi)$ is a congruence relation of $[B]\Theta$ and*

$$[B]\Theta/\Theta(\Phi) \cong B/\Phi\,, \tag{2}$$

*where an isomorphism is given by*

$$[x]\Theta(\Phi) \to [x]\Phi\,,\ x \in B\,. \tag{3}$$

The relation $\Theta(\Phi)$ is obviously reflexive and symmetric on $[B]\Theta$. The transitivity can be verified as follows: if $a \equiv b(\Theta(\Phi))$, $a' \equiv b'(\Theta(\Phi))$, $b = a'$ then there exist $c, d \in B$ and $c', d' \in B$ as required by (1). Since $d \equiv b(\Theta)$, $b = a' \equiv c'(\Theta)$ we get $d \equiv c'(\Theta)$, thus $d \equiv c'(\Phi)$. Therefore $c \equiv d \equiv c' \equiv$ $\equiv d'(\Phi)$ hence $c \equiv d'(\Phi)$ and $a \equiv b'(\Theta(\Phi))$ is verified.

If $f \in F_n$, $a_i \equiv b_i(\Theta)$, $i = 1, 2, \ldots, n$ (with $c_i, d_i$ satisfying (1)) then $f(c_1, \ldots, c_n) \equiv f(d_1, \ldots, d_n)(\Phi)$, $f(a_1, \ldots, a_n) \equiv f(c_1, \ldots, c_n)(\Theta)$, $f(b_1, \ldots, b_n) \equiv$ $\equiv f(d_1, \ldots, d_n)(\Theta)$, thus $f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)(\Theta(\Phi))$, proving that $\Theta(\Phi)$ is a congruence relation on $[B]\Theta$.

By (1) every congruence class of $[B]\Theta$ modulo $\Theta(\Phi)$ can be represented by an element of $B$, thus we get that (3) maps the left side of (2) onto the right side of (2). Further $x \equiv y(\Theta(\Phi))$ is equivalent to $x \equiv y(\Phi)$ if $x, y \in B$ therefore (3) sets up an isomorphism.

**Corollary (Zassenhaus lemma).** *Let $D$ and $E$ be subalgebras of $A$ with non-void intersection, $\Theta$ and $\Phi$ congruence relations of $D$ and $E$, respectively. Put $\Psi = \Theta_{D\cap E} \cup \Phi_{D\cap E}$. Then*

$$[D\cap E]\Theta/\Theta(\Psi) \cong [D\cap E]\Phi/\Phi(\Psi)\,, \tag{4}$$

*an isomorphism is given by*

$$[x]\Theta(\Psi) \to [x]\Phi(\Psi)\,,\quad x \in D\cap E\,. \tag{5}$$

Indeed, Lemma 1 applied to the algebras $A = [D\cap E]\Theta$ and $B = D\cap E$ with the congruence relations $\Theta$ and $\Psi$ (obviously $\Theta_{D\cap E} \leq \Psi$) gives

$$[D\cap E]\Theta/\Theta(\Psi) \cong D\cap E/\Psi,$$
$$[x]\Theta(\Psi) \to [x]\Psi,\ x \in D\cap E.$$

This, and the similar result for $\Phi$ rather than $\Theta$ gives (4) and (5).

## § 3. The Jordan-Hölder-Schreier theorem

*Normal series*

A *normal series of* $A$ is a series of subalgebras

$$A = A_0 \supseteqq A_1 \supseteqq \ldots \supseteqq A_n \tag{6}$$

such that there exists a series of relations

$$\Theta_0, \Theta_1, \ldots, \Theta_n = \omega_{A_n}, \tag{7}$$

such that $\Theta_i$ is a congruence relation on $A_i$ and $A_i = [A_n]\Theta_{i-1}$, $i = 1, 2, \ldots, n$.

The algebras $A_i/\Theta_i$, $i = 0, 1, \ldots, n-1$ are called the *quotient algebras* of (6) (with respect to (7)).

Let

$$A = B_0 \supseteqq B_1 \supseteqq \ldots \supseteqq B_m \tag{8}$$

be also a normal series, accompanied by

$$\Phi_0, \Phi_1, \ldots, \Phi_m = \omega_{B_m}, \tag{9}$$

$$\Phi_i \in \Theta(B_i),\ i = 0, 1, \ldots, m,\ B_i = [B_m]\Phi_{i-1},\ i = 1, 2, \ldots, m.$$

The normal series (6) and (8) are called *isomorphic* if $n = m$, $A_n = B_m$, and (7), (9) *can be chosen* in such a way that $A_i/\Theta_i \cong B_{k_i}/\Phi_{k_i}$, for some permutation $k_0, k_1, \ldots, k_{n-1}$ of $0, 1, \ldots, n-1$.

(8) is a refinement of (6) if $A_n = B_m$ and every $A_i$ is a $B_j$.

**Theorem 1 (Schreier's theorem).** *The normal series* (6) *and* (8) *have isomorphic refinements if* $A_n = B_m$ *and* (7), (9) *can be chosen in such a way that* $\Theta_i$ *is weakly associable with* $\Phi_j$ *over* $A_n (= B_m)$, $i = 0, \ldots, n$, $j = 0, \ldots, m$.

Define

$$A_{ij} = [A_i \cap B_j]\,\Theta_i,\ B_{ij} = [A_i \cap B_j]\,\Phi_j, \tag{10}$$

$$\Theta_{ij} = \Theta_i(\Psi_{ij}),\ \Phi_{ij} = \Phi_j(\Psi_{ij}),\ \text{where } \Psi_{ij} = \Theta_{i_{A_i \cap B_j}} \cup \Phi_{j_{A_i \cap B_j}}.$$

Then $A_{ij}/\Theta_{ij} \cong B_{ij}/\Phi_{ij}$ by the Zassenhaus lemma.

Further $A_{i0} = A_i$, $A_{im} = [A_i \cap B_m]\,\Theta_i = [A_n]\Theta_i = A_{i+1}$, $A_{ij} \supseteqq A_{ij+1}$, hence to prove that

$$A = A_{00} \supseteqq A_{01} \supseteqq \ldots \supseteqq A_{0m} = A_1 \supseteqq \ldots \supseteqq A_n$$

and

$$A = B_{00} \supseteqq B_{10} \supseteqq \ldots \supseteqq B_{n0} = B_1 \supseteqq \ldots \supseteqq B_m$$

are isomorphic refinements it is enough to verify that

$$[A_n]\,\Theta_{ij} = A_{ij+1} \qquad (0 \leqq j < m) \tag{11}$$

and the similar statement for $\Phi_{ij}$. Indeed, $[A_n]\Theta_{ij} = [A_n]\Theta_i\,(\Psi_{ij}) = [[A_n]\,\Psi_{ij}]\,\Theta_i \supseteqq [[A_n]\,\Phi_{j_{A_i \cap B_j}}]\,\Theta_i = [A_i \cap B_{j+1}]\,\Theta_i = A_{ij+1} \supseteqq A_n$, hence $[A_n]\Theta_{ij} = [A_{ij+1}]\Theta_{ij}$, thus in order to show (11) it is enough to prove that

$$A_{ij+1} \text{ is closed under } \Theta_{ij}. \tag{12}$$

$A_{i\,j+1}$ is, by definition, closed under $\Theta_i$, hence it is closed under $\Theta_{ij} = \Theta_i(\Theta_{i_{A_i \cap B_j}} \cup \Phi_{j_{A_i \cap B_j}})$ if and only if $A_{i\,j+1} \cap (A_i \cap B_j) = A_{i\,j+1} \cap B_j$ is closed under $\Phi_{j_{A_i \cap B_j}}$. Since

$$A_i \cap B_{j+1} = A_i \cap [A_r]\,\Phi_j = [A_n]\,\Phi_{j_{A_i \cap B_j}}$$

and

$$A_{i\,j+1} \cap B_j = [A_i \cap B_{j+1}]\,\Theta_i \cap B_j = [A_i \cap B_{j+1}]\,\Theta_{i_{B_j}} = \left[[A_n]\,\Phi_{j_{A_i \cap B_j}}\right]\Theta_{i_{A_i \cap B_j}} =$$

(by the definition of weak associability of $\Theta_i$ and $\Phi_j$ over $A_n$) =

$$= \left[[A_n]\,\Theta_{i_{A_i \cap B_j}}\right]\Theta_{j_{A_i \cap B_j}}$$

which was to be proved.

*Principal series*

The normal series (6) is a *principal series* if $A_i$ $(i = 0, 1, \ldots, n)$ is a normal subalgebra and if in (7) every $\Theta_i$ is a congruence relation of $A$; then the factor algebras are $A_i/\Theta_{i_{A_i}}$, $i = 0, 1, \ldots, n-1$.

**Theorem 2 (Schreier's theorem).** *The principal series* (6) *and* (8) *have isomorphic refinements if $A_n = B_m$ and* (7), (9) *can be chosen in such a way that $\Theta_i$ and $\Phi_j$ are weakly associable over $A_n$.*

Since $A_i$ $(B_i)$ is a normal subalgebra, there exists a congruence relation $\eta_i$ $(\xi_i)$ of $A$ such that $A_i$ $(B_i)$ is a whole congruence class modulo $\eta_i$ $(\xi_i)$. We put

$$A_{ij} = [A_i \cap B_j]\,\Theta_i, \qquad B_{ij} = [A_i \cap B_j]\,\Phi_j,$$

$$\Theta_{ij} = (\Theta_i \cup \Phi_j) \cap \eta_{i-1}, \quad \Phi_{ij} = (\Theta_i \cup \Phi_j) \cap \xi_{j-1}.$$

Now $A_{ij}/\Theta_{ij} \cong B_{ij}/\Theta_i \cup \Phi_j$ follows from Lemma 1 (with $\Theta = \Theta_{ij}$, $\Phi = (\Theta_i \cup \Phi_j)_{A_i \cap B_j}$) therefore

$$A_{ij}/\Theta_{ij} \cong B_{ij}/\Phi_{ij},$$

hence again it remained only to verify (11), which is again reduced to (12). But $A_{ij} \subseteq A_i$ therefore it is in one class modulo $\eta_{i-1}$, thus the problem is reduced to $(\Theta_i \cup \Phi_j)_{A_i}$ and therefore the proof given at the end of Theorem 1 applies here too.

## § 4. Some definitions of normal series

The situation in groups and rings is very simple compared to abstract algebras (due to the fact that every homomorphism of a group or a ring is determined by its kernel) therefore it is difficult to find the most natural generalization of normal (and principal) series. I want to compare here some definitions.

Goldie defined the notion of a homomorphic relation $R$ of an algebra $A$, which means a congruence relation of a subalgebra $D(R)$ of $A$. Further, he supposed the existence of a subalgebra $A_0$ contained in every subalgebra of $A$, and he put $\{R\} = [A_0]R$, which he called the kernel of $R$. Then he defined a normal series to be a sequence of homomorphic relations $\iota = R_0$, $R_1, \ldots, R_n = \omega_{A_0}$, such that $\{R_{i+1}\} \subseteqq \{R_i\} \subseteqq D(R_{i+1})$ and $\{R_i\}$ is closed under $R_{i+1}$. The quotient algebras are $\{R_i\}/R_{i+1}$. In this way, one might think, it is possible to get a stronger form of Theorem 1 since there we say nothing about the relations accompanying the refinements, while in this form every relation which was a member of a series will remain a member of the new series.

This is obviously true in the proof of Theorem 1 too. Simply, because $\Theta_{im} = \Theta_i$. But is it of any importance? In every application of the general theory (to groups, rings, loops, groupoids, semigroups or lattices) we consider only normal series in which $A_{i+1}$ is an entire class modulo $\Theta_i$ $(i = 0, \ldots, n-1)$. In this case $A_{im}/\Theta_{im} = A_{im}/\Theta_i$ is a one element algebra. Therefore if we omit from the refined series the superfluous terms $\Theta_i$ will be among the omitted ones.

Let us see an example. Let $B$ be a one element subalgebra of $A$ such that $A$ has two non-trivial congruence relations $\Theta, \Phi$ $(\Theta \neq \Phi)$ and $B$ is closed under $\Theta$ and $\Phi$. Then $\iota$, $\Theta$, $\omega_B$ and $\iota$, $\Phi$, $\omega_B$ are two normal series, the series of kernels of both is $A$, $B$, $B$. The refinements will be

$$\iota, \Theta \cup \Phi, \Theta, \omega_B, \quad \text{the kernels are} \quad A, B, B, B;$$

$$\iota, \Theta \cup \Phi, \Phi, \omega_B, \quad \text{the kernels are} \quad A, B, B, B.$$

And after omitting the superfluous terms, $\Theta$ and $\Phi$ drop out. In this situation this is inevitable.

STEINFELD [6] defines a normal sequences as a sequence of couples:

$$A = A_0(\Theta_0) \supseteqq \ldots \supseteqq A_n(\Theta_n),$$

where $A_i$ is a subalgebra of $A$ and $\Theta_i$ is a congruence relation of $A_i$. But fixing the congruence relation to the subalgebra makes it impossible to define the refinement, see the example above.

To sum up, even if in the definition of normal series the congruence relations are included, in the actual applications they have no well defined connection with the congruence relations of the refined sequence. Therefore there is no loss if we do not fix in the definition the accompanying sequence of congruence relations.

However, my definition has a disadvantage. It is not obvious whether or not the isomorphism of normal series is a transitive relation. It would be of interest to find an answer to this problem.

Of course, it is very easy the change the definition so as to make the isomorphism of normal series transitive. Let us say that (6) and (8) are isomorphic if to any accompanying series (7) there corresponds an accompanying series (9) such that $A_i/\Theta_i \cong B_{k_i}/\Phi_{k_i}$.

Then e.g. Theorem 1 has to be modified as follows: "(7) and (9) can be chosen in such a way" is to be replaced by "to any accompanying series (7) there corresponds an accompanying series (9) such".

## § 5. Classes of algebras

If we analyse the proofs in § 2 and the proof of the statements made in § 1 we arrive at the conslusion, that the operations very seldom played rôle in them. Therefore it is not surprising that we can generalize these results by considering „algebras" where besides the sets only mappings are considered.

A *class of algebras* is a class of sets **K** and a class of mappings **H**, called *homomorphisms*. A $\varphi \in \mathbf{H}$ is always a many-one mapping $\varphi\colon A \to B$ where $A, B \in \mathbf{K}$.

To simplify the axioms we define the basic notions.

If $A \in \mathbf{K}$ and $B \subseteq A$, then $B$ is called a *subalgebra* if there exists a $C \in \mathbf{K}$ and a $\varphi \in \mathbf{H}$, $\varphi\colon C \to A$ such that $C\varphi = B$.

An equivalence relation $\Theta$ on $A$ ($\in \mathbf{K}$) is a congruence relation if there exists a $B \in \mathbf{K}$ and a $\varphi \in \mathbf{H}$, $\varphi\colon A \to B$ such that $a \equiv b(\Theta)$ $(a, b \in A)$ if and only if $a\,\varphi = b\,\varphi$.

The set of all subalgebras, resp. congruence relation on $A$ ($\in \mathbf{K}$) is denoted by $S(A)$ and $\Theta(A)$, respectively.

The notations we use (e.g. $[B]\ \Theta$) are the same as those defined in § 1.

Now we list the axioms:

I. if $\varphi_1\colon A \to B$, $\varphi_2\colon B \to C$ and $\varphi_1, \varphi_2 \in \mathbf{H}$, $A, B, C \in \mathbf{K}$, then $\varphi_1\varphi_2 \in \mathbf{H}$; further if $A = C$, $B \in \mathbf{K}$ and $\varphi_1, \varphi_2$ are one-to-one and onto then $\varphi_1 \in \mathbf{H}$ implies $\varphi_2 \in \mathbf{H}$;

II. $\omega, \iota \in \Theta(A)$;

III. $B \in S(A)$ and $\Theta \in \Theta(A)$ imply $\Theta_B \in \Theta(B)$;

IV. if $B, C \in S(A)$ and $B \cap C$ is not void then $B \cap C \in S(A)$;

V. $\Theta(A)$ is a $\cup$-semilattice, i.e. $\Theta \cup \Phi$ exists[4] for every $\Theta, \Phi \in \Theta(A)$;

VI. $B \in S(A)$ and $\Theta \in \Theta(A)$ imply $[B]\Theta \in S(A)$;

VII. $B \in S(A)$, $\Theta, \Phi \in \Theta(A)$, $[B]\Theta = B$ and $[B]\Phi = B$ imply[5] $[B]\Theta \cup \Phi = B$;

VIII. $B \in S(A)$, $\Theta \in \Theta(A)$, $\Phi \in \Theta(B)$ and $\Theta_B \leq \Phi$ imply[6] $\Theta(\Phi) \in \Theta([B]\Theta)$.

Since the notions of weakly associable congruence relations, normal series are defined in terms of congruence relations and subalgebras therefore these definitions apply in the general situation as well.

**Theorem 1'.** *Schreier's theorem for normal series holds for algebras satisfying axioms* I—VIII.

To verify this statement one has to observe that nothing else but axioms I—VIII were used in the proof of Theorem 1.

To prove Theorem 2 we used another axiom:

IX. if $\Theta, \Phi \in \Theta(A)$, and we form $\Theta \cap \Phi$ in $P(A)$ then $\Theta \cap \Phi \in \Theta(A)$.

**Theorem 2'.** *Schreier's theorem for principal series holds for algebras satisfying axioms* I—IX.

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[4] We do *not* require that $\Theta \cup \Phi$ should be the same union as defined in § 1.

[5] In other words, if $B$ is closed under $\Theta$ and $\Phi$ it be closed under $\Theta \cup \Phi$.

[6] $\Theta(\Phi)$ was defined in Lemma 1. Axiom VIII is the first statement of Lemma 1.

## § 6. Multi algebras

The notion of groups was generalized to multi groups[7] by defining the product of two elements as a subset rather than an element. In the same way we define *multi algebras* as a couple $(A; F)$ where $A$ is a set, $F = \bigcup\limits_{i=0}^{\infty} F_i$ and $F_n$ is the set of *multi operations* of $n$ variables; $f \in F_n$ if. $f(a_1, \ldots, a_n)$ is a unique *subset* of $A$ for every $n$-tuple $(a_1, \ldots, a_n)$ of elements of $A$. Instead of $(A; F)$ we use again the notation $A$.

A subset $B$ of $A$ is a *subalgebra* if $b_1, \ldots, b_n \in B, f \in F_n$ imply $f(b_1, \ldots \ldots, b_n) \subseteq B$. If $A$ and $B$ are multi algebras then a many-one mapping $\varphi\colon A \to B$ is called a *homomorphism* if $f(a_1, \ldots, a_n)\,\varphi = f(a_1\,\varphi, \ldots, a_n\,\varphi)$ for every $f \in F_n$, where for a subset $C$ of $A, C\,\varphi$ denotes the set of all $c\,\varphi, c \in C$. Accordingly, an equivalence relation $\Theta$ is called a congruence relation if $a_i \equiv b_i(\Theta)$, $i = 1, 2, \ldots, n, f \in F_n$ imply that to every $c \in f(a_1, \ldots, a_n)$ there exists a $d \in f(b_1, \ldots, b_n)$ such that $c \equiv d(\Theta)$.

We are going to verify that the class of multi algebras **K** and the class of homomorphisms **H** satisfy axioms I—IX.

Axioms I—IV obviously hold true. Axioms V and VII and IX follow, as usual, from the following statement:

$\Theta(A)$ is a complete sublattice of $P(A)$. This can be verified in the same way as for algebras by properly using the characterization of congruence relations, as given above.

To verify Axiom VI let $B$ be a subalgebra and $\Theta$ a congruence relation of $A$. If $c_1, \ldots, c_n \in [B]\Theta$, $f \in F_n$ then there exist $b_1, \ldots, b_n \in B$ such that $c_i \equiv b_i(\Theta)$, $i = 1, 2, \ldots, n$. Then $c \in f(c_1, \ldots, c_n)$ implies the existence of a $d \in f(b_1, \ldots, b_n)$ such that $d \equiv c(\Theta)$. But $f(b_1, \ldots, b_n) \subseteq B$ therefore $d \in B$, thus $c \in [B]\,\Theta$ and $f(c_1, \ldots, c_n) \subseteq [B]\,\Theta$ follows.

The proof of Lemma 1 applies to multi algebras as well excepting the part where we proved that $\Theta(\Phi)$ satisfies the substitution property. Arguments, as the one used in the above paragraph, can be applied to modify the proof. Therefore axiom VIII is valid. We get

**Theorems 1″ and 2″.** *Theorems* 1 *and* 2 *hold true for multi algebras.*

Even more is true. We do not have to require that the *operations are finitary* and everything remains true. However, we do not go into the details.

*Applications*

Several applications of Theorems 1 and 2 are found in GOLDIE's paper [3]. In case of groups, rings any two congruence relations are weakly associable therefore Theorems 1 and 2 apply.

In case of semigroups (or even without the associativity of multiplication) we can get Schreier's theorems if the ideals take the place of subsemigroups and $\Theta$ is a congruence relation if $x \equiv y(\Theta)$ if and only if $x = y$ or $x$ and $y$ are elements of a fixed ideal. But in this case it is not enough to verify that any two congruence relations are permutable but one has to verify axioms I—IX as well. This is easy since all these statements are consequences of the fact that the set theoretical union and intersection of two ideals is again an ideal.

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[7] See e.g. BRUCK [2] and the bibliography therein.

A new application is the case of multi groups. In the paper [4] I Introduced the notion of standard ideals in lattices (the English version of this paper is GRÄTZER and SCHMIDT [5], where a more detailed theory of standard ideals was given). Given an ideal $S$ of a lattice $L$ we can define a relation $\Theta_s$ on $L$ as follows:

$x \equiv y(\Theta_s)$ if and only if $(x \cap y) \cup s = x \cup y$ for some $s \in S$. If $\Theta_s$ is a congruence relation then $S$ is called a *standard ideal* and $\Theta_s$ is called a *standard congruence relation* (in [4] this is not the definition of a standard ideal, but an equivalent condition, see condition $(\gamma)$ of Theorem 1, also the same in [5]). The set $\Theta_s(L)$ of all standard congruence relations is a sublattice of $\Theta(L)$ and $S(L)$ the set of all standard ideals form a sublattice of the lattice of all ideals of $L$. (This is Lemma 1 of [4], see Theorem 3 in [5] as well.)

Let $\mathbf{L}$ be the class of all lattices, $L_1, L_2 \in \mathbf{L}$. A many one mapping $\varphi: L_1 \to L_2$ is called an *S-homomorphism* if there exists a standard ideal $S$ in $L_1$ such that $a\varphi = b\varphi$ if and only if $a \equiv b(\Theta_s)$, further, every *isomorphism* is also called an $S$-homomorphism; let $\mathbf{S}$ denote the class of $S$-homomorphisms. The class $\mathbf{L}$, $\mathbf{S}$ just defined satisfies the axioms I—IX.

All axioms but axiom VIII are trivial or consequences of statements about standard ideals and congruences mentioned above.

To verify axiom VIII let $B$ be a sublattice and $S$ a standard ideal of $L$, $T$ be a standard ideal of $B$ such that[8] $\Theta_T \leq (\Theta_s)_B$. We extend $L$ by defining a zero element 0. Let $L_1 = \{L, 0\}$, $B_1 = \{B, 0\}$, $S_1 = \{S, 0\}$, $T_1 = \{T, 0\}$. If we verify axiom VIII for $L_1$, $B_1$, $\Theta_{S_1}$ and $\Theta_{T_1}$, then it implies that it holds for $L$, $B$, $\Theta_S$ and $\Theta_T$. Therefore we may suppose that $L$ has a 0 and $0 \in B$.

We state that

1. $[B]\Theta_S = \{b \cup s; b \in B, s \in S\}$;
2. *$S$ is a standard ideal of $[B]\Theta_S$;*
3. *$[T]\Theta_S$ is a standard ideal of $[B]\Theta_S$.*

If $\Theta = \Theta_S$, $\Phi = \Theta_T$ in $B$, $\Theta_B \leq \Phi$, then $\Theta(\Phi) = \Theta_{[T]\Theta_S}$ is an easy consequence of statement 3, therefore it is enough to prove statements 1—3.

**Proof of 1.** The relation

$$[B]\,\Theta_S \supseteq \{b \cup s;\ b \in B, s \in S\}$$

is obvious, further the right side is a $\cup$-semilattice containing $B$. Hence it is enough to prove that $\{b \cup s; b \in B, s \in S\}$ is a $\cap$-semilattice. Let $t_1 = b_1 \cup s_1$, $t_2 = b_2 \cup s_2$ $(b_1, b_2 \in B, s_1, s_2 \in S)$, then $b_1 \equiv t_1(\Theta_S)$ and $b_2 \equiv t_2(\Theta_S)$; therefore $b_1 \cap b_2 \equiv t_1 \cap t_2(\Theta_S)$ implying the existence of an $s \in S$ with $t_1 \cap t_2 = (b_1 \cap b_2) \cup s$ which means $t_1 \cap t_2 \in \{b \cup s;\ b \in B, s \in S\}$ since $b_1 \cap b_2 \in B$.

**Proof of 2.** $[B]\Theta_S$ is a sublattice of $L$ containing $S$ therefore $S$ is a standard ideal of $[B]\Theta_S$.

**Proof of 3.** We put $I = [T]\Theta_S$. Then $T \supseteq S$. We apply Theorem 9 of [4] (see also Theorem 14 of [5]) which says that $I$ is standard in $[B]\Theta_S$ if and only if $I/S$ ($I/S$ denotes $I/\Theta_S$) is standard in $[B]\Theta_S/\Theta_S$. But $x \to [x]\Theta_S$ is an isomorphism between $B$ and $[B]\Theta_S/S$ carrying $T$ into $I/S$. Therefore $I$ is standard in $[B]\Theta_S/S$ if and only if $I/S$ is standard in $[B]\Theta_S/S$ which in turn is equivalent to the fact that $T$ is standard in $B$, which in fact holds true.

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[8] $(\Theta_S)_B$ is the restriction of $\Theta_S$ to the sublattice $B$.

Thus we get that we can apply Theorems 1′ and 2′ for lattices. Theorem 1′ in this special case gives a generalization of the Jordan—Hölder—Schreier theorem of [4]. In the mentioned theorem we require that in a normal series $L = S_0 \supseteqq S_1 \supseteqq \ldots \supseteqq S_n$, $S_i$ is a standard ideal of $S_{i-1}$. As an application of Theorem 1′ we require only that $S_i$ is a sublattice of $S_{i-1}$ and that $S_{i-1}$ as a lattice contains a standard ideal $T_{i-1}$ such that

$$S_i = [S_n]\,\Theta_{T_{i-1}}, \quad i = 1, 2, \ldots, n\,.$$

If, as an application we want to get only the original results we can define in **L** the notion of homomorphism in the following way. $\varphi$ is a homomorphism of $L_1$ into $L_2$ if $\varphi$ is an $S$-homomorphism and $L_1\varphi$ is an ideal in $L_2$. In this case the verification of axioms I—IX is simpler.

The application of Theorem 2′ to standard ideals gives a new result contained neither in [4], nor in [5].

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## О ТЕОРЕМЕ JORDAN-HÖLDER ДЛЯ УНИВЕРСАЛЬНЫХ АЛГЕБР

G. GRÄTZER

Автор предлагает более простое доказательство и дальнейшее распространения обобщения Goldie теоремы Jordan—Hölder.