ON THE INTERCONNECTION BETWEEN THE REPRESENTATION THEOREMS OF CHARACTERISTIC FUNCTIONS OF UNIMODAL DISTRIBUTION FUNCTIONS AND OF CONVEX CHARACTERISTIC FUNCTIONS

by

PÁL MEDGYESSY

Dedicated to Professor G. Pólya on the occasion of his 75th birthday

1.

First we recall some definitions.

Following A. YA. KHINCHIN (see e.g. [1], p. 157)a distribution function $F(x)$ is called *unimodal* if there exists at least one value $x = a$ (called the *vertex* of the distribution function) such that $F(x)$ is convex for $x < a$ and concave for $x > a$. Let us notice that $F(x)$ is then continuous at every point $x = a$.

Following G. PÓLYA (see e.g. [2], p. 70), a function $\psi(t)$ defined for all real t is called a *convex characteristic function* if

(I)
a) $\psi(t)$ is real-valued and continuous,
b) for $t > 0$, $\psi(t)$ is convex,
c) $\lim\limits_{t\to\infty} \psi(t) = 0$,
d) $\psi(0) = 1$,
e) for $t < 0$, $\psi(t) = \psi(-t)$.

As proved by G. PÓLYA, such a function is in fact the characteristic function of a distribution function; moreover this distribution function is absolutely continuous.

2.

There exist representation theorems concerning the characteristic functions of unimodal distribution functions and convex characteristic functions, resp.

Theorem 1. *The function $\varphi(t)$ is the characteristic function of a unimodal distribution function $F(x)$ (with the vertex at $x = 0$) if and only if it can be represented in the form*

$$\varphi(t) = \frac{1}{t}\int_0^t \chi(u)\,du$$

where $\chi(u)$ is some characteristic function.

This theorem is due to A. YA. KHINCHIN (1938). For its proof see e. g. [1], pp. 157—160, supplemented by the corrections of K. L. CHUNG, *ibid.*, pp. 252—253.

This proof also involves that Theorem 1 is equivalent to the following

Theorem 1'. *The function $F(y)$ is a unimodal distribution function (with the vertex at $y = 0$) if and only if it can be represented in the following form:*

$$\text{(2.1)} \qquad \text{for } y < 0, \; F(y) = -\int_{-\infty}^{y}\int_{-\infty}^{u} \frac{dV(x)}{x}\,du = \int_{-\infty}^{y}\left(1-\frac{y}{x}\right)dV(x),$$

$$\text{(2.2)} \qquad \text{for } y > 0, \; F(y) = 1 - \int_{y}^{\infty}\int_{u}^{\infty} \frac{dV(x)}{x}\,du = 1 - \int_{y}^{\infty}\left(1-\frac{y}{x}\right)dV(x)$$

(see [1], pp. 158—160) *where $V(x)$ is some distribution function (exactly, this is the distribution function possessing the characteristic function $\chi(u)$).*

Evidently, $F(-0) = V(-0)$, $F(+0) = V(+0)$.

Our investigations will be based on this latter version of Theorem 1.

Theorem 2. *The function $\psi(t)$ is a convex characteristic function if and only if for $t > 0$ it can be represented in the form*

$$\text{(2.3)} \qquad \psi(t) = \int_{t}^{\infty}\left(1-\frac{t}{x}\right)dG(x) \qquad (t > 0)$$

where $G(x)$ is some distribution function for which $G(x) = 0$ if $x \leq 0$ and $G(+0) = 0$, and

$$\text{(.4)} \qquad \text{for } t < 0, \; \psi(t) = \psi(-t).$$

Evidently, $\psi(+0) = \psi(-0) = \psi(0) = 1$.

This is a simple consequence of a result due to D. Dugué (1955) and of some remarks of M. Girault (see [3], pp. 6—7 and [4], p. 292). The representation (2.3) is to be found in [3], p. 6.

3.

Our aim is to show that there is an intimate interconnection between Theorem 1' and Theorem 2 in **2**.

A. *Theorem 2 can be deduced by the aid of Theorem 1'.* Namely, let $\psi(t)$ be a convex characteristic function and let us consider the function $F(y)$ defined by

$$\text{(3.1)} \qquad F(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ 1-\psi(y) & \text{if } y > 0 \end{cases}$$

(see Fig. 1.). $F(y)$ will be a unimodal distribution function with vertex at $x = 0$; further, $F(+0) = 0$. Then, by (2.2), for $y > 0$ we have

$$\text{(3.2)} \qquad F(y) = 1 - \int_{y}^{\infty}\left(1-\frac{y}{x}\right)dV(x)$$

where $V(x)$ is some distribution function; further, $F(+0) = V(+0) = 0$. Now let

$$W(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ V(x) & \text{if} \quad x > 0\,. \end{cases} \tag{3.3}$$

Then, for $t > 0$, by (3.1), (3.2) and (3.3) we have

$$\psi(t) = \int_t^\infty \left(1 - \frac{t}{x}\right) dV(x) = \int_t^\infty \left(1 - \frac{t}{x}\right) dW(x)\,. \tag{3.4}$$

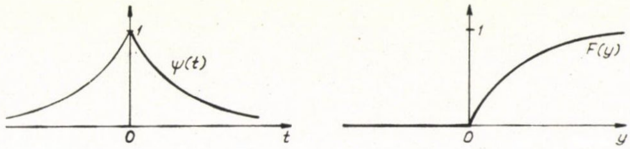

Fig. 1.

(3.4) is identical with the representation (2.3), since $W(x)$ is evidently of the same type as $G(x)$ in (2.3). Since, *per definitionem*, $\psi(t) = \psi(-t)$ for $t < 0$, all these facts involve that the conditions (2.3), (2.4) are necessary for that $\psi(t)$ be a convex characteristic function. — Conversely, if $\psi(t)$ is any function satisfying the conditions (2.3) — (2.4) then it is easily seen that $\psi(t)$ satisfies also the conditions under (I).

Thus the conditions (2.3) — (2.4) are also sufficient for that $\psi(t)$ be a convex characteristic function and, consequently, the deduction of Theorem 2 from Theorem 1′ is completed.

B. As to Theorem 1′, for

$$F(y) \equiv F_0(y) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

Theorem 1′ is obvious from the identities

$$F_0(y) = \int_{\infty}^{y} \left(1 - \frac{y}{x}\right) dF_0(x) \qquad (y < 0) \tag{3.5}$$

$$F_0(y) = 1 - \int_y^\infty \left(1 - \frac{y}{x}\right) dF_0(x) \qquad (y > 0)\,. \tag{3.6}$$

If $F(y) \not\equiv F_0(y)$ *i.e. if* $F(y)$ *is non-degenerate, Theorem* 1′ *can be deduced from Theorem* 2. The way of showing this consists in reverting the sequence of ideas in A in some sense, — having to distinguish, at any rate, three particular cases.

a) Let $F_1(x)$ be a non-degenerate unimodal distribution function with vertex at $x = 0$ for which $F_1(x) = 1$ if $x > 0$. Then, *per definitionem*, the function

$$\varphi_1(t) = \begin{cases} \dfrac{F_1(t)}{F_1(-0)} & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t > 0 \end{cases} \tag{3.7}$$

may be regarded over $(-\infty, 0)$ as the part lying over $(-\infty, 0)$ of a convex characteristic function and, by (2.3) and (2.4), we have

$$\varphi_1(t) = \int_{-t}^{\infty} \left(1 + \frac{t}{x}\right) dG_1(x) \qquad (t < 0) \tag{3.8}$$

where $G_1(x)$ is a distribution function for which $G_1(x) = 0$ if $x \leqq 0$ and $G_1(+0) = 0$. Then, by (3.7) and (3.8)

$$F_1(y) = -F_1(-0) \int_{-\infty}^{y} \left(1 - \frac{y}{x}\right) dG_1(-x) \qquad (y < 0). \tag{3.9}$$

Upon introducing the distribution function

$$R_1(x) = \begin{cases} F_1(-0)\,[1 - G_1(-x)] & \text{if} \quad x \leqq 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

$(R_1(-0) = F_1(-0),\ R_1(+0) = 1)$ we then have [cf. (3.9) and (3.6)]:

$$F_1(y) = \int_{-\infty}^{y} \left(1 - \frac{y}{x}\right) dR_1(x) \qquad \text{if} \quad y < 0 \tag{3.10}$$

$$F_1(y) = 1 - \int_{y}^{\infty} \left(1 - \frac{y}{x}\right) dR_1(x) \qquad \text{if} \quad y > 0. \tag{3.11}$$

Consequently, (3.10) and (3.11) are necessary for that $F_1(x)$ be a unimodal distribution function.

b) Let $F_2(x)$ be a non-degenerate unimodal distribution function with vertex at $x = 0$ for which $F_2(x) = 0$ if $x \leq 0$. Then, *per definitionem*, the function

$$\varphi_2(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ \dfrac{1 - F_2(t)}{1 - F_2(+0)} & \text{if} \quad t > 0 \end{cases} \tag{3.12}$$

may be regarded over $(0, \infty)$ as the part lying over $(0, \infty)$ of a convex characteristic function and, by (2.3), we have

$$\psi_2(t) = \int_t^\infty \left(1 - \frac{t}{x}\right) dG_2(x) \qquad (t > 0) \tag{3.13}$$

where $G_2(x)$ is a distribution function for which $G_2(x) = 0$ if $x \leqq 0$ and $G_2(+0) = 0$. Then, by (3.12) and (3.13)

$$F_2(y) = 1 - [1 - F_2(+0)] \int_y^\infty \left(1 - \frac{y}{x}\right) dG_2(x) \qquad (y > 0). \tag{3.14}$$

Upon introducing the distribution function

$$R_2(x) = \begin{cases} 0 & \text{if } x \leqq 0 \\ [1 - F_2(+0)]\, G_2(x) + F_2(+0) & \text{if } x > 0 \end{cases}$$

$(R_2(-0) = 0,\ R_2(+0) = F_2(+0))$ we then have [cf. (3.14) and (3.5)]:

$$F_2(y) = \int_{-\infty}^y \left(1 - \frac{y}{x}\right) dR_2(x) \qquad \text{if } y < 0 \tag{3.15}$$

$$F_2(y) = 1 - \int_y^\infty \left(1 - \frac{y}{x}\right) dR_2(x) \qquad \text{if } y > 0. \tag{3.16}$$

Consequently, (3.15) and (3.16) are necessary for that $F_2(x)$ be a unimodal distribution function.

c) Let $F(y)$ be a non-degenerate unimodal distribution function, $F(y) \not\equiv F_1(y)$, $F(y) \not\equiv F_2(y)$. Then, over $(-\infty, 0)$, it is of the same type as $F_1(x)$ over $(-\infty, 0)$ and, over $(0, \infty)$, it is of the same type as $F_2(x)$ over $(0, \infty)$. Then we have, with respect to (3.10) and (3.16):

$$F(y) = \begin{cases} \displaystyle\int_{-\infty}^y \left(1 - \frac{y}{x}\right) dS_1(x) & \text{if } y < 0 \\ \displaystyle 1 - \int_y^\infty \left(1 - \frac{y}{x}\right) dS_2(x) & \text{if } y > 0 \end{cases}$$

where $S_1(x)$ and $S_2(x)$ are distribution functions of the type of $R_1(x)$ and $R_2(x)$, resp. [hence $S_1(-0) = F_1(-0)$, $S_1(+0) = 1$, $S_2(-0) = 0$, $S_2(+0) = F(+0)$]. Defining the distribution function $R(x)$ by

$$R(x) = \begin{cases} S_1(x) & \text{if } x \leqq 0 \\ S_2(x) & \text{if } x > 0 \end{cases}$$

we finally have

$$F(y) = \int_{-\infty}^{y} \left(1 - \frac{y}{x}\right) dR(x) \quad \text{if} \quad y < 0 \tag{3.17}$$

$$F(y) = 1 - \int_{y}^{\infty} \left(1 - \frac{y}{x}\right) dR(x) \quad \text{if} \quad y > 0 \tag{3.18}$$

$(F(-0) = R(-0),\ F(+0) = R(+0))$. Consequently, (3.17) and (3.18) are necessary for that $F(x)$ be a unimodal distribution function.

As to the converse considerations, if $F_1(y)$, $F_2(y)$, $F(y)$ are functions satisfying the conditions (3.10) — (3.11), (3.15) — (3.16), (3.17) — (3.18), resp. then it is easily shown that they are unimodal distribution functions (with the vertex at $y = 0$). Thus the validity of (3.10) — (3.11), (3.15) — (3.16), (3.17) — (3.18) is also sufficient for that $F_1(y)$, $F_2(y)$, $F(y)$ be unimodal distribution functions (with the vertex at $y = 0$). Consequently, the deduction of Theorem 1′ from Theorem 2 is completed.

This way of deducing Theorem 1′ (and, implicitely, Theorem 1) by the aid of Theorem 2, whose proof is relatively simple, seems to have some advantages with respect to the original one presented in [1], pp. 157—160, which is, in view of the remarks of K. L. Chung referring hereto (see [1], pp. 252—253), rather complicated.

(Received January 21, 1963; in revised form September 9, 1963)

REFERENCES

[1] Gnedenko, B. V.—Kolmogorov, A. N.: *Limit distributions for sums of independent random variables*. Addison-Wesley Publishing Co. Inc., Cambridge, Mass., 1954.
[2] Lukács, E.: *Characteristic functions*. Hafner Publishing Co., New York, 1960.
[3] Dugué, D.—Girault, M.: "Fonctions convexes de Pólya". *Publications de l'Institut de Statistique de l'Université de Paris* **4** (1955) 3—10.
[4] Girault, M.: "Les fonctions caractéristiques et leurs transformations". *Publications de l'Institut de Statistique de l'Université de Paris* **4** (1955) 221—299.

О СВЯЗИ МЕЖДУ ТЕОРЕМАМИ ПРЕДСТАВЛЕНИЯ ХАРАКТЕРИСТИЧЕСКОЙ ФУНКЦИИ ОДНОВЕРШИННОГО ЗАКОНА РАСПРЕДЕЛЕНИЯ И ВЫПУКЛОЙ ХАРАКТЕРИСТИЧЕСКОЙ ФУНКЦИИ

P. MEDGYESSY

Резюме

В 1938 г. А. Я. Хинчин дал известное представление характеристической функции одноверщинного закона распределения. В 1955 г. D. Dugué сообщил формулу представления выпуклой характеристической функции. В работе устанавливается тесная связь упомянутых двух теорем представления.