ON THE CONNECTEDNESS OF BICHROMATIC RANDOM GRAPHS

by

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§ 1. Introduction

A graph is given by a set of labelled points (vertices) $P_1, P_2, \ldots, P_n$ and by a set of pairs (P_i, P_j) of its points, called the edges of the graph. (See [2], [3].) Let us suppose that $i \neq j$ (no loops are allowed).

A graph is bichromatic, if the set of its n points can be split into two subsets $P_1, P_2, \ldots, P_m$ and $Q_1, Q_2, \ldots, Q_{n-m}$ (we can imagine that all the points are coloured e.g. all the points P_i are red but all Q_j are blue), so that no vertices of the same colour are connected by an edge.

A graph is called a random graph if its edges are chosen at random so that each admitted choice has the same probability. (See in [1] and [4].)

P. ERDŐS and A. RÉNYI considered in the paper [1] the random graphs $\Gamma_{n,N}$ with n vertices and N edges, the latter chosen at random so that all possible $\binom{\binom{n}{2}}{N}$ choices have the same probability. They answered the question: what is the probability of the random graph obtained in such a way being connected. They showed that if the number of the edges is equal to N_c, where

$$N_c = \left[\frac{n}{2}\log n + cn\right] \tag{1}$$

and c is an arbitrary fixed real number ($[x]$ means the integral part of x), and if $\mathbf{P}_0(n, N_c)$ denotes the probability of the random graph Γ_{n,N_c} being connected, then

$$\lim_{n\to\infty} \mathbf{P}_0(n, N_c) = e^{-e^{-2c}}. \tag{2}$$

The outline of their proof is the following.

Let us call a graph to be of type A if it consists of a connected subgraph with $n - k$ vertices and of k isolated points ($k = 0, 1, \ldots$). Any graph which is not of type A is called to be of type $\bar{A}$. Let $\mathbf{P}(\bar{A}, n, N_c)$ denote the probability that the random graph be of type $\bar{A}$.

It has been shown in [1] that

$$\lim_{n\to\infty} \mathbf{P}(\bar{A}, n, N_c) = 0 \tag{3a}$$

holds if N_c is given by (1).

It follows from (3a) that if $\mathbf{P}_0^*(n, N_c)$ denotes the probability that the random graph Γ_{n,N_c} should contain no isolated points, then

$$\lim_{n\to\infty} (\mathbf{P}_0^*(n, N_c) - \mathbf{P}_0(n, N_c)) = 0\,. \tag{3b}$$

It remained only to prove that

(4) $$\lim_{n\to\infty} \mathbf{P}_0^*(n, N_c) = e^{-e^{-2c}},$$

which could be achieved relatively easily; (2) follows evidently from (3b) and (4).

The problem had to be treated in this way because no explicit formula is known for the number of connected graphs with n vertices and N edges which would admit asymptotic evaluation.

Our aim is to determine the probability of a bichromatic random graph being connected and to examine the asymptotical behaviour of these probabilities.

§ 2. Bichromatic random graphs

Let the bichromatic random graph $\Gamma_{m,n,N}$ have m given labelled points $P_1, P_2, \ldots, P_m$ of one colour (say red), n given labelled points $Q_1, Q_2, \ldots, Q_n$ of another colour (say blue) and N edges, each of which connecting a red point with a blue point, chosen at random in such a way that all possible choices have the same probability $1\Big/\binom{m\,n}{N}$ (see in [5], [6]).

A bichromatic graph G is connected, if any P_i can be connected with any Q_j by a path in G. (This implies that any two points can be connected by a path.)

We shall deal with the case when $m \sim \lambda n$ (where $\lambda > 0$ is a constant). First let be $\lambda = 1$, $m = n$ and let us prove the following

Theorem 1. *If* $\mathbf{P}(n, n, N_c)$ *denote the probability of the bichromatic random graph* Γ_{n,n,N_c} *being connected, assuming that*

(5) $$N_c = [n \log n + cn]$$

(*where* c *is an arbitrary fixed real number*), *then*

(6) $$\lim_{n\to\infty} \mathbf{P}(n, n, N_c) = e^{-2e^{-c}}.$$

Proof. Likewise to the considerations of P. Erdős and A. Rényi in [1] we shall call the bichromatic random graph to be of type A if it has a component with exactly $n - k$ red points and $n - l$ blue points, further k isolated red points ($k = 0, 1, \ldots$) and l isolated blue points ($l = 0, 1, \ldots$). All other graphs belong to the type $\bar{A}$.

Let us first prove the following lemma.

Lemma 1. *Let* $\mathbf{P}(\bar{A}, n, n, N_c)$ *denote the probability that the random graph* Γ_{n,n,N_c} *is of type* $\bar{A}$. *Then we have*

(7) $$\lim_{n\to\infty} \mathbf{P}(\bar{A}, n, n, N_c) = 0,$$

where N_c *is given by* (5). *Thus if* n *is large enough then „almost all" graphs* Γ_{n,n,N_c} *are of type* A, *assuming that* (5) *holds.*

Proof of Lemma 1. Let us put $U = \log\log n$. All graphs Γ_{n,n,N_c} consisting of the vertices $P_1, P_2, \ldots, P_m$ and $Q_1, Q_2, \ldots, Q_n$ and N_c edges belong to one of the following two classes: Let us define E_U as the class consisting of those graphs in which the greatest component (i.e. with the greatest number of points) contains not less than $n-U$ red points and not less than $n-U$ blue points. All other graphs belong to the class $\bar{E}_U$ (i.e. those graphs in which the greatest components consist of less than $n-U$ red or less than $n-U$ blue points).

Let r and s denote the number of points outside a greatest component of the first colour and the second colour resp.

If a graph consists of t components such that the i-th component consists of a_i red and b_i blue points where of course

$$\sum_{i=1}^{t} a_i = \sum_{i=1}^{t} b_i = n$$

and

$$\sum_{i=1}^{t} a_i b_i \geqq N_c$$

holds; then — according to the inequality concerning the arithmetic and geometric means — we obtain

$$\sum_{i=1}^{t} \left(\frac{a_i + b_i}{2}\right)^2 \geqq N_c$$

and thus

$$\max_i \, (a_i + b_i) \left(\sum_{i=1}^{t} (a_i + b_i)\right) \geqq 4 N_c ,$$

therefore

$$\max_i \, (a_i + b_i) \geqq \frac{2 N_c}{n} .$$

Accordingly if the greatest component consists of $n - r$ red and $n - s$ blue points, then

$$n - r + n - s \geqq \frac{2 N_c}{n} ,$$

whence

$$\max (n - r, n - s) \geqq \frac{N_c}{n} ,$$

i.e.

$$n - \min (r, s) \geqq \frac{N_c}{n} ,$$

thus

$$\min (r, s) \leqq n - \frac{N_c}{n} .$$

Let us fix the $n - r$ red points, and the $n - s$ blue points belonging to the greatest component; then $s(n - r) + r(n - s)$ edges could be established connecting these points with points outside, this component and these edges cannot belong to the graph; thus if $\mathcal{N}(\bar{E}_U, n, n, N_c)$ denotes the number of graphs not belonging to the class E_U, then

$$(8)\quad \mathcal{N}(\bar{E}_U, n, n, N_c) \leqq 2 \sum_{U<r<n} \sum_{\substack{0 \leqq s \leqq n - \frac{N_c}{n} \\ s \leqq r}} \binom{n}{r}\binom{n}{s}\binom{n^2 - s(n-r) - r(n-s)}{N_c}.$$

If $\mathbf{P}(\bar{E}_U, n, n, N_c)$ denotes the probability of the event that the graph G_{n,n,N_c} does not belong to the class E_U then

$$(9)\qquad \mathbf{P}(\bar{E}_U, n, n, N_c) \leqq \frac{\mathcal{N}(\bar{E}_U, n, n, N_c)}{\binom{n^2}{N_c}}.$$

Now we have (by the inequality $1 - x \leqq e^{-x}$)

$$(10)\qquad \binom{n}{r}\binom{n}{s} \frac{\binom{n^2 - s(n-r) - r(n-s)}{N_c}}{\binom{n^2}{N_c}} \leqq \frac{n^r}{r!}\frac{n^s}{s!} e^{-N_c\left(\frac{r}{n}+\frac{s}{n}\right)+N_c\frac{2rs}{n^2}}.$$

Making use of the assumption (5), we obtain according to (9) and (10),

$$(11)\qquad \mathbf{P}(\bar{E}_U, n, n, N_c) \leqq 2 \sum_{U<r<n} \sum_{\substack{0 \leqq s \leqq n - \frac{N_c}{n} \\ s \leqq r}} a_{rs}$$

where

$$(12)\quad a_{rs} = \binom{n}{r}\binom{n}{s} \frac{\binom{n^2 - s(n-r) - r(n-s)}{N_c}}{\binom{n^2}{N_c}} \leqq \frac{e^{\frac{2rs}{n}(\log n + c) - cr - cs + 2}}{r!\,s!}.$$

Let us estimate the sums on the right hand side of (11).

Case 1. Let us write (12) in the following form

$$(12')\qquad a_{rs} \leqq \frac{e^{(2-c)r+(2-c)s+2}}{r!\,s!} e^{\frac{2rs}{n}(\log n + c) - 2r - 2s}$$

and let us consider first the values of r and s for which

$$\frac{rs}{n}(\log n + c) \leqq r + s,$$

that is

$$(13)\qquad \frac{\log n + c}{n} \leqq \frac{1}{r} + \frac{1}{s}.$$

(13) certainly holds, if

$$s \leqq \frac{n}{\log n + c}. \tag{14}$$

If s satisfies (14) we say that we have case 1. Thus

$$a_{rs} \leqq \frac{e^{(2-c)r + (2-c)s + 2}}{r!\, s!} \tag{15}$$

holds, if (14) is valid.

Case 2. Let us consider the terms in (11), for which (14) does not hold, but

$$r + s \leqq n. \tag{16}$$

Applying Stirling's formula we obtain that, for sufficiently large n, these terms are less than

$$\exp\left\{\frac{2rs}{n}(\log n + c) + (r+s)(1-c) - r\log r - s\log s\right\}. \tag{17}$$

Using the inequality

$$2rs \leqq \frac{(r+s)^2}{2}$$

the expression in brackets in (17) is less than

$$\frac{(r+s)^2}{2}(\log n + c) + (1-c)(r+s) - (r\log r + s\log s). \tag{18}$$

Since $x\log x$ is a convex function, we conclude by Jensen's inequality

$$-(r\log r - s\log s) \leqq -(r+s)\log\frac{r+s}{2}.$$

Thus (18) is less than $\varphi(x) + x\log 2$ where

$$\varphi(x) = \frac{x^2}{2n}(\log n + c) + (1-c)x - x\log x$$

and

$$x = r + s.$$

According to (16) $\frac{2n}{\log n + c} < x \leqq n$. Now

$$\varphi'(x) = \frac{x}{n}(\log n + c) - (\log x + c) < 0\,;\ \text{if}\ e^{1-c} < x \leqq n,$$

because $\frac{x}{c + \log x}$ is an increasing function of x if

$$e^{1-c} < x < n.$$

Thus it follows that

$$\varphi(x) \leq \varphi\left(\frac{2n}{\log n + c}\right) \leq -2n + \frac{K n \log\log n}{\log n} \tag{19}$$

where $K > 0$ is a constant. Thus for $n \geq n_0$ the sum of the terms on the right side of (11) for which (16) holds does not exced $n^2 e^{-n}$ and therefore tends to 0, if $n \to \infty$.

Case 3. Taking into account that $a_{rs} = a_{r's'}$ where $r' = n - s$, $s' = n - r$ the estimation of the terms a_{rs} with $r + s > n$ can be reduced to the estimation of the terms $a_{r's'}$ with $r' + s' \leq n$, regarding the fact that from $r + s > n$ there follows that $r' + s' < n$ further that from $s \leq n - \frac{N_c}{n}$ it follows that $r' \geq \frac{N_c}{n} > U = \log\log n$, if n is sufficiently large.

Thus we have for (11)

$$\mathbf{P}(\bar{E}_U, n, n, N_c) \leq 4e^2 \sum_{0 \leq r \leq n - \frac{N_c}{n}} \frac{e^{(2-c)r}}{r!} \sum_{U < s < n} \frac{e^{(2-c)s}}{s!} + o(1) \leq$$

$$\leq 4e^{2+e^{2-c}} \left(\sum_{U < s} \frac{e^{(2-c)s}}{s!}\right) + o(1)\,. \tag{20}$$

As we have chosen $U = \log\log n$, we obtain

$$\lim_{n \to \infty} \mathbf{P}(\bar{E}_{\log\log n}, n, n, N_c) = 0\,. \tag{21}$$

Now we only need to show that the probability of obtaining a random graph not being of the type A, but nevertheless belonging to the class $E_{\log\log n}$, tends to zero. That is we have to show that

$$\lim_{n \to \infty} \mathbf{P}(\bar{A} E_{\log\log n}, n, n, N_c) = 0\,. \tag{22}$$

Since in these graphs the greatest component consists of $n - r$ red and $n - s$ blue points, therefore $q \geq 1$ denoting the number of edges connecting some of the r and s outside points, these outside edges can be chosen in $\binom{rs}{q}$ different ways; thus the remaining $N_c - q$ inner edges must be selected from the $(n - r)(n - s)$ possibilities, i.e.

$$\mathbf{P}(\bar{A} E_{\log\log n}, n, n, N_c) \leq \sum_{r=1}^{U} \sum_{s=1}^{U} \binom{n}{r} \binom{n}{s} \sum_{q=1}^{rs} \binom{rs}{q} \frac{\binom{(n-r)(n-s)}{N_c - q}}{\binom{n^2}{N_c}}\,. \tag{23}$$

Taking into account the inequalities

$$\binom{n}{r}\binom{n}{s} < \frac{n^r n^s}{r!\,s!}\,; \qquad \sum_{q=1}^{rs} \binom{rs}{q} = 2^{rs} - 1 < 2^{rs}\,,$$

and

$$\frac{\binom{(n-r)(n-s)}{N_c-q}}{\binom{n^2}{N_c}} \leq \frac{N_c^q}{(n^2-q)^q}\left(\frac{(n-r)(n-s)}{n^2-q}\right)^{N_c-q},$$

we obtain

$$\mathbf{P}(\bar{A}E_{\log\log n}, n, n, N_c) \leq \tag{24}$$

$$\leq \frac{\log n}{n}\sum_{r=1}^{U}\sum_{s=1}^{U}\frac{2^{rs}e^{-(r+s)c}}{r!\,s!} = O\left(\frac{2^{(\log\log n)^2}\log n}{n}\right) = o(1).$$

Thus (22) holds and therefore the proof of Lemma 1 is completed.

The proof of Theorem 1. Denoting by $\mathscr{N}'(n, n, N_c)$ the number of bichromatic random graphs without isolated points, according to the sieve method we have evidently

$$\mathscr{N}'(n, n, N_c) = \sum_{k=0}^{n}\sum_{l=0}^{n}(-1)^{k+l}\binom{n}{k}\binom{n}{l}\binom{(n-k)(n-l)}{N_c}. \tag{25}$$

Putting into (25) $k + l = h$, we obtain a more often used form:

$$\mathscr{N}'(n, n, N_c) = \sum_{h=0}^{2n}(-1)^h\mathscr{A}_h \tag{26}$$

where

$$\mathscr{A}_h = \sum_{k=0}^{h}\binom{n}{k}\binom{n}{h-k}\binom{n(n-h)+k(h-k)}{N_c}.$$

Using the following inequalities (similarly as was done in [1], p. 295):

$$\sum_{h=0}^{2H+1}(-1)^h\mathscr{A}_h \leq \mathscr{N}'(n, n, N_c) \leq \sum_{h=0}^{2H}(-1)^h\mathscr{A}_h \tag{27}$$

and taking into account that for any fixed value of h

$$\lim_{n\to\infty}\frac{\mathscr{A}_h}{\binom{n^2}{N_c}} = \sum_{k=0}^{h}\frac{e^{-ch}}{k!\,(h-k)!},$$

we obtain

$$\overline{\lim_{n\to\infty}}\frac{\mathscr{N}'(n, n, N_c)}{\binom{n^2}{N_c}} \leq \sum_{h=0}^{2H}(-1)^h\sum_{k=0}^{h}\frac{e^{-ch}}{k!\,(h-k)!}$$

and

$$\lim_{n\to\infty} \frac{\mathcal{N}'(n, n, N_c)}{\binom{n^2}{N_c}} \geqq \sum_{h=0}^{2H+1} (-1)^h \sum_{k=0}^{h} \frac{e^{-ch}}{k!\,(h-k)!}.$$

Since H can be chosen arbitrarily large, we obtain

$$\lim_{n\to\infty} \frac{\mathcal{N}'(n, n, N_c)}{\binom{n^2}{N_c}} = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-ck}}{k!} \sum_{l=0}^{\infty} (-1)^l \frac{e^{-cl}}{l!} = e^{-2e^{-c}}. \tag{28}$$

It is however evident that if $\mathcal{N}(n, n, N_c)$ denotes the number of the connected graphs, then

$$0 \leqq \frac{\mathcal{N}'(n, n, N_c) - \mathcal{N}(n, n, N_c)}{\binom{n^2}{N_c}} \leqq \mathbf{P}(\bar{A}, n, n, N_c). \tag{29}$$

Applying Lemma 1, Theorem 1 follows immediately.

Let us suppose now that $\lambda \neq 1$.

Theorem 2. *Let us denote by* $\mathbf{P}(m, n, N_{c,\lambda})$ *the probability that the bichromatic random graph* $\Gamma_{m,n,N_{c,\lambda}}$ *be connected, assuming that* $m \sim \lambda n$ *and*

$$N_{c,\lambda} = [m \log m + cm] \tag{30}$$

(*where* $\lambda > 1$ *and* c *are constants*); *then*

$$\lim_{n\to\infty} \mathbf{P}(m, n, N_{c,\lambda}) = e^{-e^{-c}} \tag{31}$$

holds.

Proof. In this case we shall call the graphs consisting of a component with $m - k$ red and n blue vertices and of k isolated red points ($k = 0, 1, \ldots$) (that is one component contains all blue points) to be of type B. Any graph $\Gamma_{m,n,N_{c,\lambda}}$ which is not of type B shall be called to be of type $\bar{B}$. We shall prove that the following lemma is valid.

Lemma 2. *Let* $\mathbf{P}(\bar{B}, m, n, N_{c,\lambda})$ *denote the probability that the bichromatic random graph* $\Gamma_{m,n,N_{c,\lambda}}$ *is of the type* $\bar{B}$; *then*

$$\lim_{n\to\infty} \mathbf{P}(\bar{B}, m, n, N_{c,\lambda}) = 0. \tag{32}$$

Thus in case n is sufficiently large and $N_{c,\lambda}$ *is the same as in* (30), *then „almost all" graphs* $\Gamma_{m,n,N_{c,\lambda}}$ *will be of type* B.

Proof of Lemma 2. The proof of Lemma 2 is similar to that of Lemma 1, therefore we give only the outlines of the proof.

Let us denote by $\mathcal{N}(\bar{B}, m, n, N_{c,\lambda})$ the number of bichromatic graphs, with m red and n blue points and $N_{c,\lambda}$ edges, which are of type $\bar{B}$. Then we have clearly

$$\mathcal{N}(\bar{B}, m, n, N_{c,\lambda}) \leqq \sum_{r=0}^{m} \sum_{s=1}^{n-1} \binom{m}{r}\binom{n}{s}\binom{mn - s(m-r) - r(n-s)}{N_{c,\lambda}}. \tag{33}$$

The probability of the random graph $\Gamma_{m,n,N_{c,\lambda}}$ being of the type $\bar{B}$ is equal to

$$\mathbf{P}(\bar{B}, m, n, N_{c,\lambda}) = \frac{\mathcal{N}(\bar{B}, m, n, N_{c,\lambda})}{\binom{mn}{N_{c,\lambda}}} \tag{34}$$

and thus

$$\mathbf{P}(\bar{B}, m, n, N_{c,\lambda}) \leqq \sum_{r=0}^{m} \sum_{s=1}^{n-1} b_{rs} \tag{35}$$

where

$$b_{rs} = \binom{m}{r}\binom{n}{s}\binom{mn - s(m-r) - r(n-s)}{N_{c,\lambda}} \tag{36}$$

and thus

$$b_{rs} \leqq \frac{m^r n^s}{r!\, s!} e^{-\left(\frac{r}{m} + \frac{s}{n} - \frac{2rs}{mn}\right) N_{c,\lambda}}. \tag{37}$$

Let now E_1 denote the set of those pairs (r, s) for which

$$0 \leqq r \leqq \alpha m, \qquad 1 \leqq s \leqq n-1 \tag{38}$$

where

$$0 < \alpha < \frac{\lambda - 1 - \delta}{2\lambda} \qquad (0 < \delta < \lambda - 1). \tag{39}$$

Then we obtain easily

$$\sum_{(r,s)\in E_1} b_{rs} = O\left(\frac{1}{n^{\delta}}\right). \tag{40}$$

Let now E_2 denote the set of those pairs (r, s) for which

$$\alpha m < r < m, \quad 1 \leqq s \leqq \frac{n}{2}. \tag{41}$$

For these terms we get

$$\sum_{(r,s)\in E_2} b_{rs} = O\left(\frac{1}{n^{\lambda-1}}\right). \tag{42}$$

Finally if E_3 denotes the set of those pairs (r, s) for which

$$0 \leqq r \leqq m, \quad \frac{n}{2} < s \leqq n-1 \tag{43}$$

we get, in view of

$$b_{rs} = b_{m-r,n-s} \tag{44}$$

$$\sum_{(r,s)\in E_3} b_{rs} \leqq \sum_{(r,s)\in E_1} b_{rs} + \sum_{(r,s)\in E_2} b_{rs}. \tag{45}$$

Thus it follows from (35), (40), (42) and (45) that (32) holds.

Thus Lemma 2 has been proved.

Proof of Theorem 2. In this case we denote by $\mathcal{N}'(m, n, N_{c,\lambda})$ the number of those graphs, which do not contain isolated red points. We obtain

$$\mathcal{N}'(m,n,N_{c,\lambda}) = \sum_{k=}^{m} (-1)^k \binom{m}{k} \binom{(m-k)\,n}{N_{c,\lambda}}. \tag{46}$$

Since $\mathcal{N}'(m, n, N_{c,\lambda})$ lies between any two consecutive partial sums of the right hand side of (46), in the case of any fixed k,

$$\binom{m}{k} \frac{\binom{(m-k)\,n}{N_{c,\lambda}}}{\binom{mn}{N_{c,\lambda}}} \sim \frac{m^k}{k!}\left(1-\frac{k}{m}\right)^{N_{c,\lambda}}. \tag{47}$$

Thus we obtain

$$\lim_{n\to\infty} \frac{\mathcal{N}'(m,n,N_{c,\lambda})}{\binom{mn}{N_{c,\lambda}}} = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-ck}}{k!} = e^{-e^{-c}}. \tag{48}$$

From the inequality where $\mathcal{N}(m, n, N_{c,\lambda})$ denotes the number of connected graphs

$$0 \le \frac{\mathcal{N}'(m,n,N_{c,\lambda}) - \mathcal{N}(m,n,N_{c,\lambda})}{\binom{mn}{N_{c,\lambda}}} \le \mathbf{P}(\bar{B}, m, n, N_{c,\lambda}). \tag{49}$$

Theorem 2 follows.

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О СВЯЗНОСТИ ДВУХЦВЕТНЫХ СЛУЧАЙНЫХ ГРАФОВ

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Резюме

Пусть двухцветный случайный граф $\Gamma_{m,n,N}$ состоит из m пронумерованных вершин $P_1, P_2, \ldots, P_m$, которые окрашены первой краской, из n пронумерованных вершин $Q_1, Q_1, \ldots, Q_n$, которые окрашены второй краской, и из N случайно выбранных граней. Точки одинакового цвета нельзя соединять гранью. В работе показывается, что в случае $m = \lambda n$ (где $\lambda > 1$ константа) вероятность того, что двухцветный случайный граф $\Gamma_{m,n,N_{c,\lambda}}$ будет связным, стремится при $n \to \infty$ к $e^{-e^{-c}}$ (c — произвольная константа) при условии, что число граней $N_{c,\lambda} = [\, m \log m + + cm]$ ($[x]$ обозначает целую часть числа x) и $\lambda > 1$. В случае $\lambda < 1$ вероятность связности также стремится к пределу $e^{-e^{-c}}$ при условии, что число граней $N_c = [n \log n + + cn]$. В случае $m = n$ предельная вероятность связности двухцветных случайных графов равна $e^{-2e^{-c}}$ при условии, что число выбранных граней $N_c = [n \log n + cn]$.