THE ASYMPTOTIC BEHAVIOUR OF A SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS

by

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Introduction

A great number of works deal with the stability and asymptotic behaviour of the solutions of the nonlinear differential equation (system)

$$(1) \qquad \frac{d\bar{x}}{dt} = \bar{A}\bar{x} + \bar{f}(t, \bar{x}), \qquad \bar{f}(t, 0) = 0$$

where $\bar{x} = (x_1, \ldots, x_n)$, $x_i = x_i(t)$, $\bar{f}(t, \bar{x}) = (f_1, \ldots, f_n)$, $f_i = f_i(t, x_1, \ldots, x_n)$, $i = 1, 2, \ldots, n$ and $\bar{A} = (a_{ik})$, $a_{ik} =$ const, $i, k = 1, 2, \ldots, n$. Their statements bring into connection the behaviour of the solutions of (1) and those of the linear approximate equation

$$(2) \qquad \frac{d\bar{y}}{dt} = \bar{A}\bar{y}\,.$$

First POINCARÉ and LIAPUNOV obtained results of this type. They assumed the real parts of the characteristic roots of $\bar{A}$ to be different, $\bar{x}$ and $\bar{f}(t, \bar{x})$ analytic, the series of the last function beginning with at least second power. PERRON [1] assuming the continuity of $\bar{f}(t, \bar{x})$ only, and the property[1] $\| \bar{f}(t, \bar{x}) \| = o(\| x \|)$ $(\bar{x} \to 0, t \to +\infty)$, weakened these hypotheses and restated the theorems of the above authors. As long as these results involved relations on a logarithmic scale, COTTON [2] found certain proper asymptotic connections, which hold between $\bar{x}$ and $\bar{y}$. He assured the „smallness" of the perturbation $\bar{f}(t, \bar{x}) = \bar{B}\bar{x}$ by the condition $\int^{\infty} \| \bar{B} \| \, dt < \infty$.

WINTNER ([3]—[4]) treated the contrary-case, where the characteristic roots of $\bar{A}$ have all equal (vanishing) real parts (equal roots permitted too), however he assumed that (2) possesses bounded (pure sinusoid) solutions only, i.e. the corresponding elementary divisors of $\bar{A}$ are linear. LEVINSON [5] did not assume the roots to be imaginary, but the boundedness of *every* solution of (2) and showed — having been restricted to linear $\bar{f}(t, \bar{x}) = \bar{B}\bar{x}$ — that every solution $\bar{x}(t)$ of (1) determines a solution $\bar{y}(t)$ of (2) containing pure sinusoid terms only, with the property $\bar{x}(t) - \bar{y}(t) \to 0$ as $t \to +\infty$. A special case of this theorem is involved in [4], and in certain papers of CESARI [6] and BELLMAN [7] previously appeared. The last two works pertain to homogeneous linear equations of n-th order. LEVINSON's theorem has been generalized by

[1] Throughout this paper $\| \bar{x} \| = \sum_i' | x_i |$, $\| \bar{A} \| = \sum_{i,k}' | a_{ik} |$

H. WEYL [8], namely in two respects. First he did not assume $\bar{f}(t, \bar{x})$ to be linear, instead, he imposed on $\bar{f}(t, \bar{x})$ the requirement to have a "linear majorant" in the sense

$$\|\bar{f}(t, \bar{x})\| \leq g(t)\, \|\bar{x}\|, \qquad \int^{\infty} g(t)\, dt < \infty .$$

On the other hand, he showed the converse of the theorem too, that there belongs to every solution $\bar{y}(t)$ of (2) a solution $\bar{x}(t)$ of (1) with $\bar{y} - \bar{x} \to 0$, $t \to +\infty$, provided that a *linear* condition of the type

$$\|\bar{f}(t, \bar{x}) - \bar{f}(t, \bar{x}^*)\| \leq g(t)\, \|\bar{x} - \bar{x}^*\|, \qquad \int^{\infty} g(t)\, dt < \infty \tag{3}$$

is satisfied ($t \geq 0$, $\bar{x}$, $\bar{x}^*$ arbitrary). This means, that there exists a one-to-one correspondence between the solutions of (1) and (2).

The author stipulated in [9] instead of the linear majorization (3) the nonlinear condition

$$\|\bar{f}(t, \bar{x})\| \leq g(t)\, \omega(\|\bar{x}\|), \qquad \int^{\infty} g(t)\, dt < \infty \tag{4}$$

($t \geq 0$, $\bar{x}$ arbitrary and $\omega(u)$ continuous, monotone increasing etc.) and concluded from the boundedness of the solutions of (2) to that of the solutions of (1) and to the stability of the solution $\bar{x} \equiv 0$.

The present paper gives the generalization of the LEVINSON—WEYL theorem under the condition (4) and

$$\|\bar{f}(t, \bar{x}) - \bar{f}(t, \bar{x}^*)\| \leq g(t)\, \omega(\|\bar{x} - \bar{x}^*\|), \qquad \int^{\infty} g(t)\, dt < \infty \tag{5}$$

(t, $\bar{x}$, $\bar{x}^*$ arbitrary)

In addition, it involves the extension of a result of WINTNER [10] concerning the convergence of successive approximations and of another [12] related to a result of the author [11].

1. Let us begin with the mentioned remark as to the successive approximations.

If $\bar{f}(t, \bar{x})$ is continuous in a certain domain of the space $(t, \bar{x})$, then there is a solution of (1) passing through every point of the domain and existing on an interval which includes the point. However, without any further conditions this solution cannot be obtained by successive approximation, i.e. the correspondent (usual) successive approximations do not converge. For an equation of the form

$$\frac{d\bar{x}}{dt} = \bar{g}(t, \bar{x}) \tag{6}$$

a Lipschitz condition assures the convergence and the uniqueness. This suggests, that perhaps the mere uniqueness of the solution is sufficient for the convergence, but some known examples refute this. Nevertheless, if the sufficient assumptions of the well-known uniqueness theorems are stipulated for $\bar{g}(t, \bar{x})$ then the successive approximations relative to (6) are to be convergent at least in a sufficiently short interval (s. [10], [13] p. 53, [14], [15]). The same will be shown here concerning (1), provided that $\bar{f}(t, \bar{x})$ satisfies an analogous condition which is not the most general one (it is more resp. less general than that of [10] resp. [13]), on the other hand the convergence will be established on the whole $t \geqq 0$ axis.

There will be used the familiar formula

$$\bar{x}(t) = \bar{y}(t) + \int_0^t \bar{Y}(t-\tau)\bar{f}(\tau, \bar{x}(\tau))\, d\tau, \qquad \bar{y}(t) = \bar{Y}(t)\,\bar{a} \tag{7}$$

connecting the solutions $\bar{x}(t)$ and $\bar{y}(t)$ of (1) and (2) resp., where $\bar{Y}(t)$ denotes the solution of

$$\frac{d\bar{Y}}{dt} = \bar{A}\bar{Y}, \qquad \bar{Y}(0) = \bar{I} \qquad \text{(identity)} \tag{8}$$

($\bar{a}$ is an arbitrary vector).

Theorem 1. *Let the following conditions be satisfied:*

1. *every solution of* (2) *is bounded for* $t \geqq 0$, *i.e. the real parts of the characteristic roots of* $\bar{A}$ *are non-positive and the elementary divisors belonging to the roots with zero real parts are linear (e.g. these roots are simple),*

2. $\bar{f}(t, \bar{x})$ *is defined for* $t \geqq 0$ *and arbitrary* $\bar{x}$ *and satisfies* (5), *where* $\omega(u)$ *is positive, continuous, non-decreasing for* $u \geqq 0$, $\int_1^\infty \frac{du}{\omega(u)} = \infty$, $\int_0^1 \frac{du}{\omega(u)} = \infty$ (*of course* $\omega(0) = 0$), $\int^\infty g(t)\, dt < \infty$ *and* $g(t)$ *is bounded.*

Then the successive approximations

$$\begin{aligned} &\bar{x}_0(t) = \bar{y}(t), && \bar{y}(t) = \bar{Y}(t)\,\bar{a} \\ &\bar{x}_{n+1}(t) = \bar{y}(t) + \int_0^t \bar{Y}(t-\tau)\bar{f}(\tau, \bar{x}_n(\tau))\, d\tau, && (n = 0, 1, 2, \ldots) \end{aligned} \tag{9}$$

converge uniformly on $t \geqq 0$ *to the unique solution of* (1) *with* $\bar{x}(0) = \bar{a}$. (The starting point $\bar{x}_0(t)$ is here not as commonly a constant.)

Proof. First we show that the sequence $\{\bar{x}_n(t)\}$ is equicontinuous for $t \geqq 0$. Namely, if $t_1 > 0$, $t_2 > 0$ are arbitrary values, then by (9)

$$\bar{x}_{n+1}(t_1) - \bar{x}_{n+1}(t_2) = \bar{y}(t_1) - \bar{y}(t_2) + \bar{Y}(t_1)\int_0^{t_1} \bar{Y}(-\tau)\bar{f}_n\, d\tau - \bar{Y}(t_2)\int_0^{t_2} \bar{Y}(-\tau)\bar{f}_n\, d\tau,$$

where $\bar{f}_n = \bar{f}(\tau, \bar{x}_n(\tau))$. Making use of the fact that the sequence $\{\bar{x}_n(t)\}$ is uniformly bounded (see below where this bound is given explicitly) say $||\bar{x}_n(t)|| \leqq M$, $t \geqq 0$ we have (e.g. for $t_2 \leqq t_1$)

$$\begin{aligned} ||\bar{x}_{n+1}(t_1) - \bar{x}_{n+1}(t_2)|| &\leqq ||\bar{a}||\, ||\bar{Y}(t_1) - \bar{Y}(t_2)|| + \\ &+ ||\bar{Y}(t_1) - \bar{Y}(t_2)||\, \omega(M)\int_0^{t_1} ||\bar{Y}(-\tau)||\, g(\tau)\, d\tau + ||\bar{Y}(t_2)||\, \omega(M)\int_{t_2}^{t_1} ||\bar{Y}(-\tau||\, g(\tau)\, d\tau. \end{aligned}$$

Here the right member is independent of n and may be arbitrary small by choosing $t_1 - t_2$ small enough, which means exactly the mentioned equicontinuity.

In the second place the same sequence is uniformly bounded for $t \geqq 0$. According to cond. 1 $\| \bar{Y}(t) \| < c$ for some c.

We assert that

(10) $$r_n(t) = \|\bar{x}_n(t)\| \leqq \Omega^{-1}(\Omega(\alpha) + c \int_0^t g(\tau)\, d\tau) = K(t) \quad (n = 0, 1, 2, \ldots)$$

where $\alpha = c\,\|\bar{a}\|$, $\Omega(u) = \int_{u_0}^{u} \frac{dz}{\omega(z)}$ $(u_0 > 0)$. As $\int^{\infty} g(t) < \infty$ and $\int_1^{\infty} \frac{du}{\omega(u)} = \infty$, (10) assures the boundedness in question. Actually, we state somewhat more. From (9) we have

$$r_{n+1} \leqq \alpha + c \int_0^t \omega(r_n)\, g(\tau)\, d\tau \qquad (= V_n(t))$$

and a stronger assertion than (10) will be proved, namely that

(11) $$V_n(t) \leqq K(t)\,,\ t \geqq 0 \qquad (n = 0, 1, 2, \ldots)$$

In fact, suppose $V_{n-1}(t) \leqq K(t)$, $t \geqq 0$ and prove $V_n(t) \leqq K(t)$, $t \geqq 0$. But $V_n(0) = K(0)$, and if the last inequality failed to hold for all $t \geqq 0$, there would exist a first place $t_0 \geqq 0$, where

(12) $$V_n(t_0) = K(t_0) \quad \text{and} \quad V_n(t) > K(t)\,, \quad t_0 < t < t_1\,,$$

where $t_1 - t_0 > 0$ is small enough. Then

$$V_{n-1}(t) < V_n(t) \quad \text{or} \quad \omega(V_{n-1}) \leqq \omega(V_n)\,, \qquad t_0 < t < t_1\,,$$

whence being $r_n \leqq V_{n-1}$ resp. $\omega(r_n) \leqq \omega(V_{n-1})$ and $V_n \geqq \alpha > 0$, $V_{n-1} \geqq \alpha > 0$ we have

$$\frac{cg(t)\,\omega(r_n)}{\omega(V_n)} \leqq \frac{cg(t)\,\omega(r_n)}{\omega(V_{n-1})} \leqq cg(t)\,, \qquad t_0 < t < t_1$$

Hence by integration

$$\Omega(V_n(t)) \leqq \Omega(V_n(t_0)) + c \int_{t_0}^{t} g(\tau)\, d\tau\,.$$

However, by (12)

$$\Omega(V_n(t_0)) = \Omega(K(t_0)) = \Omega(\alpha) + c \int_0^{t_0} g(\tau)\, d\tau\,,$$

therefore

$$\Omega(V_n(t)) \leqq \Omega(\alpha) + c \int_0^t g(\tau)\, d\tau = \Omega(K(t))\,, \qquad t_0 < t < t_1$$

or $V_n(t) \leqq K(t)$ in contradiction with (12). This proves (11). But $r_n(t) \leqq V_{n-1}(t)$, consequently $r_n(t) \leqq K(t)$, $t \geqq 0$. It remains to ascertain whether $V_0(t) \leqq K(t)$ $t \geqq 0$ holds. Really we have

$$V_0(t) = \alpha + c \int_0^t \omega(\|\bar{y}\|)\, g(\tau)\, d\tau$$

and the relation

$$\Omega(V_0(t)) = \Omega\big(\alpha + c\int_0^t \omega(\|y\|)\, g(\tau)\, d\tau\big) \leqq \Omega(\alpha) + c\int_0^t g(\tau)\, d\tau = \Omega(K(t))$$

holds for $t \geqq 0$, since it holds for $t = 0$ and the derivative of the right member is not less for $t \geqq 0$ than that of the left one. In fact

$$\frac{d}{dt}\left[\Omega\left(\alpha + c\int_0^t \omega(\|y\|)\, g(\tau)\, d\tau\right)\right] = \frac{c\,\omega(\|y\|)\, g(t)}{\omega\big(\alpha + c\int_0^t \omega(\|y\|)\, g(\tau) d\tau\big)} \leqq \frac{c\,\omega(\alpha)\, g(t)}{\omega\big(\alpha + c\int_0^t \omega(\|y\|)\, g(\tau) d\tau\big)}$$

and

$$\frac{d}{dt}\,\Omega(K(t)) = cg(t)$$

and an immediate comparison verifies our assertion.

Thus the sequence (9) turned out to be uniformly bounded and equicontinuous. Therefore — corresponding to Arzela's theorem — it involves a uniformly convergent subsequence on every interval $0 \leqq t \leqq T$, the continuous limit function of which let be denoted by $x_T(t)$. An easy argumentation shows that T may be taken $T = \infty$ too. Viz., let be $T = n$ $(n = 1, 2, \ldots)$ and regard the corresponding subsequence for $[0, n_0]$ (n_0 is a fixed integer), then one of its convergent subsequences corresponding to $[0, n_0 + 1]$, etc. Now making use of the well-known diagonal method, we receive a subsequence uniformly converging for $t \geqq 0$. Denote this by $\{\bar{x}_{k_n}(t)\}$ $(n = 1, 2, \ldots)$ and its limit function by $\bar{x}(t)$. We assert $\bar{x}(t)$ to be the (unique) required solution, and the *total* successive approximation is converging to it. Namely

$$\|\bar{f}(\tau, \bar{x}(\tau)) - \bar{f}(\tau, \bar{x}_{k_n}(\tau))\| \leqq g(\tau)\, \omega(\|\bar{x} - \bar{x}_{k_n}\|)$$

Denote $\max\limits_{0 \leqq \tau < \infty} \|\bar{x} - \bar{x}_{k_n}\|$ by δ_{k_n}, then

$$\Big\|\int_0^t \bar{Y}(t-\tau)\, [\bar{f}(\tau, \bar{x}(\tau)) - \bar{f}(\tau, \bar{x}_{k_n}(\tau))]\, d\tau\Big\| \leqq c\,\omega(\delta_{k_n}) \int_0^t g(\tau)\, d\tau$$

which tends to 0 as $n \to \infty$. Therefore the sequence $\{\bar{x}_{k_n+1}\}$ $(n = 1, 2, \ldots)$ consisting of the terms subsequent to the terms of the sequence $\{\bar{x}_{k_n}\}$, is also uniformly convergent and its limit $\bar{x}^*(t)$ satisfies (according to (9))

$$\bar{x}^*(t) = \bar{y}(t) + \int_0^t \bar{Y}(t-\tau)\, \bar{f}(\tau, \bar{x}(\tau))\, d\tau$$

Therefore $\bar{x}(t)$ is a solution of (1) for $t \geqq 0$ if and only if

$$\bar{x}^*(t) = \bar{x}(t)\,, \quad t \geqq 0$$

To this end it suffices to prove that

$$u(t) = \limsup_{n \to \infty} \|\bar{x}_{n+1}(t) - \bar{x}_n(t)\| = 0\,, \qquad t \geqq 0\,.$$

This may be done by a slight modification of Wintner's proof in [10] which can be omitted here (s. there).[2] In order to demonstrate the convergence of the total sequence (9), it is enough to ascertain that (1) has a unique solution

[2] Otherwise this proof will be carried out later (in 4) in a more complicated case.

with $\bar{x}(0) = \bar{a}$, as in this case $\{\bar{x}_n(t)\}$ cannot have another cluster element (viz. this would be also a solution).

Suppose (1) has another solution $\bar{x}_1(t)$ for $t \geqq 0$ with $\bar{x}_1(0) = \bar{a}$, then by (7) we have for the function $\| \bar{x} - \bar{x}_1 \| = r(t)$

$$r(t) \leqq c \int_0^t g(\tau)\, \omega(r(\tau))\, d\tau \tag{13}$$

where $r(0) = 0$. Let $t_1 \geqq 0$ be the first place, where $r(t_1) = 0$, but $r(t) > 0$ for $t_1 < t < t_2$ with $t_2 - t_1$ small enough. Then by (13)

$$r(t) \leqq c \int_{t_1}^t g(\tau)\, \omega(r(\tau))\, d\tau\,. \tag{14}$$

Let the right member be denoted by $V(t)$, then $V(t) > O$ $(t > t_1)$ and $r(t) \leqq V(t)$, whence

$$\omega(r(t)) \leqq \omega(V(t)) \text{ resp. } \frac{cg(t)\ \omega(r(t))}{\omega(V)} \leqq cg(t)\,.$$

Hence by integration

$$\int_0^{V(t)} \frac{du}{\omega(u)} \leqq c \int_{t_1}^t g(\tau)\, d\tau\,, \qquad t > t_1\,. \tag{15}$$

According to cond. 2 the integral on the left is divergent, whereas that on the right is convergent, which involves a contradiction.

2. The "asymptotic initial value problem" may be treated in the same way. Here the relation $\bar{x}(\infty) = \bar{a} = \bar{y}(\infty)$ or $\bar{x}(t) - \bar{y}(t) \to 0$, $t \to +\infty$ will be prescribed (the latter when $\bar{y}(\infty)$ does not exist). Then in in (7) and (9) the sign $\int_0^t$ must be replaced by $-\int_t^\infty$ and instead of the estimate (10) we get

$$\|x_n(t)\| \leqq \Omega^{-1}(\Omega(\alpha) + c' \int_t^\infty g(\tau)\, d\tau)$$

where $c' = \sup_{t \leq 0} \| \bar{Y}(t) \|$, that is, now the boundedness of the solutions of (2) for $t \leq 0$ must be supposed. The corresponding successive approximations converge then too.

3. Now we turn to the generalization of the result of LEVINSON and WEYL.

Theorem 2. *Let the conditions* 1 *and* 2 *of Theorem* 1 *be satisfied and in addition the following one:*

Cond. 3:

$$(F_\omega) \quad \begin{cases} \Omega(m\varrho_2) - \Omega(m\varrho_1) = \displaystyle\int_{m\varrho_1}^{m\varrho_2} \frac{du}{\omega(u)} > \varrho_2 - \varrho_1\,, & c_2 > c_1 \\ \Omega(m\varrho_1) - \Omega(m\varrho_2) = \displaystyle\int_{m\varrho_2}^{m\varrho_1} \frac{du}{\omega(u)} > \varrho_1 - \varrho_2\,, & c_1 > c_2 \end{cases}$$

where $\varrho_i = c_i q (i=1,2), q = \int\limits_0^\infty g(t)\,dt,\ 0 < m \leqq \omega(2M)$ (*M and* $\overline{Y}_1(t), \overline{Y}_2(t)$ *will be defined later*) *and* $c_1 = \sup\limits_{t \geqq 0} \| \overline{Y}_1(t) \|,\ c_2 = \sup\limits_{t \leqq 0} \| \overline{Y}_2(t) \|$. *Summarising*

$$\int\limits_{m\lambda_1}^{m\lambda_2} \frac{du}{\omega(u)} > \lambda_2 - \lambda_1\,, \qquad \lambda_1 = \min(\varrho_1, \varrho_2)\,,\ \lambda_2 = \max(\varrho_1, \varrho_2)\,.$$

(This is involved by cond. 2 provided $\lambda_1 = 0$).

Then every solution $\overline{x}(t)$ *of (1) determines a solution* $\overline{y}(t)$ *of (2) with* $\overline{x}(t) - \overline{y}(t) \to 0$ *as* $t \to +\infty$ *and the converse statement holds too.*

In the case of a linear majorant — say case (W) — $\omega(u) \equiv u$ and condition 3 reads

$$(F_1) \qquad \frac{e^{\lambda_1}}{\lambda_1} > \frac{e^{\lambda_2}}{\lambda_2}\,.$$

Since the function $f(\varrho) = \dfrac{e^\varrho}{\varrho}$ has a minimum at $\varrho = 1$, condition (F_1) is satisfied if $\lambda_2 < 1$ — the only case observed by WEYL—, but obviously in other cases too (e.g. for $\lambda_2 = 1$ or $\lambda_1 = 0{,}1,\ \lambda_2 = 1{,}1$).

The first part of the *proof* differs hardly from that of WEYL.

In a suitable coordinate system (carrying out a non-degenerate transformation, if necessary) $\overline{Y}(t)$ consists of blocks (elementary divisors) of the form

$$\begin{bmatrix} e^{\lambda t}t_0 & 0 & \dots & 0 \\ e^{\lambda t}t_1 & e^{\lambda t}t_0 & & 0 \\ \vdots & \vdots & & \vdots \\ e^{\lambda t}t_{m-1} & e^{\lambda t}t_{m-2} & & e^{\lambda t}t_0 \end{bmatrix}, \quad t_i = \frac{t^i}{i!}$$

where λ means a characteristic root of $\overline{A}$. If $Re\,\lambda = 0$, $m = 1$ (s. cond. 1). Unifying every elementary divisor corresponding to $Re\,\lambda < 0$ in a block $\overline{Z}_1$ and those corresponding to $Re\,\lambda = 0$ in another block $\overline{Z}_2$ we obtain a decomposition of $\overline{Y}(t)$ as follows

$$\overline{Y} = \begin{bmatrix} \overline{Z}_1 & 0 \\ 0 & \overline{Z}_2 \end{bmatrix}$$

in which $\overline{Z}_1(t) \to 0$ $(t \to \infty)$ and $\overline{Z}_2(t)$ are bounded for *every* t (also for $t < 0$). Let the corresponding decomposition of the unit-matrix be

$$\overline{I} = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 \end{bmatrix} = \overline{I}_1 + \overline{I}_2$$

where $\overline{I}_1$ and $\overline{I}_2$ are $n \times n$ matrices. Then $\overline{Y}(t)$ dissociates in the form

$$\overline{Y} = \overline{Y}\overline{I}_1 + \overline{Y}\overline{I}_2 = \overline{Y}_1 + \overline{Y}_2$$

Obviously, $\bar{Y}_1(t) \to 0,\ t \to +\infty$ and $\bar{Y}_2(t)$ is bounded (for $t < 0$ too). Say $\| \bar{Y}_2(t) \| \leqq c_2\ (t < 0)$.

Regard the solution of (1) and (2) corresponding to $\bar{x}(0) = \bar{y}(0) = \bar{a}\,(= \text{const})$. Then $\bar{y}(t) = \bar{Y}(t)\bar{a}$. These solutions are connected by the relation (7).Conversely, if (7) holds and $x(t)$ satisfies (1), then $\bar{y}(t)$ fulfils (2) and $\bar{y}(0) = \bar{x}(0)$.

Corresponding to $\bar{Y}(t)$, $\bar{x}(t)$ will be decomposed as follows:

$$\bar{I}_1 \bar{x} + \bar{I}_2 \bar{x} = \bar{x}_1 + \bar{x}_2\,,$$

and similarly $\bar{y}(t)$ dissociates in the vectors $\bar{y}_1(t)$ and $\bar{y}_2(t)$, where $\bar{y}_1(t)$ is a damped oscillation and $\bar{y}_2(t)$ consists of pure sinusoid terms. Carrying out this decomposition in (7) too,

$$(16)\qquad \bar{x}(t) = \bar{y}(t) + \int_0^t \bar{Y}_1(t-\tau)\,\bar{f}\,d\tau + \int_0^t \bar{Y}_2(t-\tau)\,\bar{f}\,d\tau\,,\ \bar{f} = \bar{f}(\tau, \bar{x}(\tau))\,.$$

Transform the second integral in the following way

$$\int_0^t = \bar{Y}(t)\int_0^\infty \bar{Y}_2(-\tau)\,\bar{f}\,d\tau - \int_t^\infty \bar{Y}_2(t-\tau)\,\bar{f}\,d\tau$$

$$(\text{viz. } \bar{Y}_2(t-\tau) = \bar{Y}(t-\tau)\,\bar{I}_2 = \bar{Y}(t)\,\bar{Y}(-\tau)\,\bar{I}_2 = \bar{Y}(t)\,\bar{Y}_2(-\tau))\,.$$

Here the term $\int_0^\infty \bar{Y}_2(-\tau)\,\bar{f}d\tau$ is a constant vector $\bar{b}$, consequently (16) takes on the form

$$(17)\quad \bar{x}(t) = \bar{z}(t) + \int_0^t \bar{Y}_1(t-\tau)\,\bar{f}d\tau - \int_t^\infty \bar{Y}_2(t-\tau)\,\bar{f}d\tau\,, \qquad \bar{z}(t) = \bar{Y}(t)\,(\bar{a}+\bar{b})$$

where $\bar{z}(t)$ is a solution of (2), belonging to the initial condition $\bar{z}(0) = \bar{a} + \bar{b}$ (recall $\bar{a} = \bar{y}(0)$), and $\bar{x}(t)$ is a solution of (1) in the future too independently of the preliminaries, provided $\bar{z}(t)$ satisfies (2) and the second integral in (17) converges. But this converges, $\bar{x}(t)$ being bounded — say $\| \bar{x}(t) \| \leqq M,\ t \geqq 0$ — and

$$\|\bar{Y}_2(t-\tau)\,\bar{f}(\tau, \bar{x}(\tau))\| \leqq c_2\, g(\tau)\,\omega(\|\bar{x}\|) \leqq c_2\, g(\tau)\,\omega(M)\,,\quad \int^\infty g(t)\,dt < \infty$$

Conversely, if $\bar{x}(t)$ is a solution of (1) ,then

$$(17')\qquad \bar{z}(t) = \bar{x}(t) - \int_0^t \bar{Y}_1(t-\tau)\,\bar{f}d\tau + \int_t^\infty \bar{Y}_2(t-\tau)\,\bar{f}d\tau\,, \qquad \bar{f} = \bar{f}(\tau, \bar{x}(\tau))$$

is one of (2). Thus by (17') to all solutions of (1) there corresponds a unique solution of (2).

(17') gives after a multiplication by $\bar{I}_1$ and $\bar{I}_2$, resp.

$$\begin{cases} (18_1)\quad \bar{z}_1(t) = \bar{x}_1(t) - \int_0^t \bar{Y}_1(t-\tau)\,\bar{f}(\tau, \bar{x}(\tau))\,d\tau \\ (18_2)\quad \bar{z}_2(t) = \bar{x}_2(t) + \int_t^\infty \bar{Y}_2(t-\tau)\,\bar{f}(\tau, \bar{x}(\tau))\,d\tau \end{cases}$$

$$(\text{viz. } \bar{I}_1\bar{Y}_1 = \bar{Y}_1,\ \bar{I}_1\bar{Y}_2 = 0,\ \bar{I}_2\bar{Y}_1 = 0,\ \bar{I}_2\bar{Y}_2 = \bar{Y}_2)\,.$$

Now the relation $\bar{x}_2(t) - \bar{z}_2(t) \to 0$, $t \to +\infty$ follows from the convergence of the integral in (18_2). In order to prove that $\bar{x}_1(t) - \bar{z}_1(t) \to 0$, $t \to +\infty$, it is necessary to show that the integral in (18_1) tends to 0 as $t \to +\infty$, since $\bar{z}_1(t)$ behaves similarly.

Really, according to the definition of $\bar{Y}_1(t)$ the relation $\|\bar{Y}_1(t)\| \leqq c_1 e^{-kt}$ $(k > 0)$ holds, consequently

$$\int_0^t \|\bar{Y}_1(t-\tau)\bar{f}(\tau, \bar{x}(\tau))\| d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \leqq c_1 e^{-k\frac{t}{2}} \omega(M) \int_0^{\frac{t}{2}} g(\tau)\, d\tau +$$

$$+ c_1 \omega(M) \int_{\frac{t}{2}}^t g(\tau)\, d\tau .$$

Here both terms tend to zero as $t \to +\infty$. The first because of the exponential factor, the second in virtue of the convergence of $\int^\infty g(t)\,dt$. According to (18), $\bar{z}_1(0) = \bar{x}_1(0)$, $\bar{z}_2(t) - x_2(t) \to 0$, $t \to +\infty$.

4. In order to prove the converse assertion, we apply successive approximation to solve the integral-equation (17′) for a given $\bar{z}(t)$. Namely, let it be defined by

$$(19) \quad \begin{cases} \bar{x}_0(t) = \bar{z}(t) \\ \bar{x}_{n+1}(t) = \bar{z}(t) + \int_0^t \bar{Y}_1(t-\tau)\bar{f}(\tau, \bar{x}_n(\tau))\, d\tau - \int_t^\infty \bar{Y}_2(t-\tau)\bar{f}(\tau, \bar{x}_n(\tau))\, d\tau \end{cases}$$

$$(n = 0, 1, 2, \dots)$$

Hence $\|\bar{x}_0(t)\| \leqq c_1 \|\bar{c}\| = \gamma$ and let us suppose that e.g. $c_2 > c_1$ and

$$(20) \qquad r_n(t) = \|\bar{x}_n(t)\| \leqq K(t) = \Omega^{-1}\Big(\Omega(\gamma + a) + (c_2 - c_1)\int_t^\infty g(\tau)\, d\tau\Big), \qquad t \geqq 0$$

then we prove $r_{n+1}(t) \leqq K(t)$, $t \geqq 0$. Here a is defined — if possible — by

$$(20') \qquad \varrho_2 - \varrho_1 = \Omega\left(\gamma + \frac{\varrho_2}{\varrho_1} a\right) - \Omega(\gamma + a) = \int_{\gamma+a}^{\gamma + \frac{\varrho_2}{\varrho_1} a} \frac{du}{\omega(u)}.$$[3]

In the case (W) $a = \gamma\varrho_1 \dfrac{e^{\varrho_1} - e^{\varrho_2}}{\varrho_1 e^{\varrho_2} - \varrho_2 e^{\varrho_1}}$.

We have from (20)

$$\Omega(K(t)) = \Omega(\gamma + a) + (c_2 - c_1)\int_t^\infty g(\tau)\, d\tau$$

[3] This is a restriction concerning $\omega(u), \gamma, \varrho_1, \varrho_2$. It is unchanged for $c_1 > c_2$.

whence

$$K'(t) = -(c_2 - c_1)\, g(t)\, \omega(K(t))\,. \tag{21}$$

Hence by (19), (20) and (21)

$$r_{n+1}(t) \leqq \gamma + c_1 \int_0^t g(\tau)\, \omega(r_n(\tau))\, d\tau + c_2 \int_t^\infty g(\tau)\, \omega(r_n(\tau))\, d\tau \leqq$$

$$\leqq \gamma + c_1 \int_0^t g(\tau)\, \omega(K(\tau))\, d\tau + c_2 \int_t^\infty g(\tau)\, \omega(K(\tau))\, d\tau =$$

$$= \gamma - \frac{c_1}{c_2 - c_1} \int_0^t K'(\tau)\, d\tau - \frac{c_2}{c_2 - c_1} \int_t^\infty K'(\tau)\, d\tau = \gamma + K(t) + \frac{c_1 K(0) - c_2 K(\infty)}{c_2 - c_1}.$$

Here it holds $K(\infty) = \gamma + a$ and by (20')

$$K(0) = \Omega^{-1}(\Omega(\gamma + a) + \varrho_2 - \varrho_1) = \gamma + \frac{\varrho_2}{\varrho_1} a$$

Therefore

$$r_{r+n}(t) \leq K(t), \quad t \geqq 0\,.$$

But

$$r_0(t) = \|\bar{z}(t)\| \leqq \gamma < K(t)\,.$$

thus (20) is proven by induction, i.e.

$$r_n(t) \leqq K(t) \leqq K(0) = \gamma + \frac{\varrho_2}{\varrho_1} a = M\,, \quad t \geqq 0\,, \qquad (n = 0, 1, 2, \ldots),$$

e.g. in the case (W)

$$M = \gamma\, e^{\varrho_2} \frac{\varrho_1 - \varrho_2}{\varrho_1 e^{\varrho_2} - \varrho_2 e^{\varrho_1}}\,. \tag{22}$$[4]

Therefore the sequence $\{\bar{x}_n(t)\}$ $(n = 0, 1, 2, \ldots)$ is uniformly bounded. It is also equicontinuous, what can be easily shown along the lines of 1.

Corresponding to Arzela's theorem these two properties imply the existence of a for $t \geqq 0$ uniformly convergent subsequence $\{\bar{x}_{n_k}(t)\}$ $(k = 1, 2, \ldots)$ of the previous one (as in 1.; see the proof there), the continuous limit function $\bar{x}(t)$ of which and that of the also uniformly convergent subsequence $\{\bar{x}_{n_k+1}(t)\}$ $(k = 1, 2, \ldots)$ — denoting its limit by $\bar{x}^*(t)$ — satisfy together

$$\bar{x}^*(t) = \bar{z}(t) + \int_0^t \bar{Y}_1(t - \tau) \bar{f}(\tau, \bar{x}(\tau))\, d\tau - \int_t^\infty \bar{Y}_2(t - \tau) \bar{f}(\tau, \bar{x}(\tau))\, d\tau$$

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[4] If $c_1 > c_2$, $M = K(\infty) = \gamma + a$ and in case (W) this reads as $M = \gamma e^{\varrho_1} \dfrac{\varrho_1 - \varrho_2}{\varrho_1 e^{\varrho_2} - \varrho_2 e^{\varrho_1}}$. The solution $\bar{x}(t)$ of (17) is stable *relative to* (17) at least in the case (W). This is obvious by (22).

Therefore $\overline{x}(t)$ is a solution of (17) if and only if

$$\overline{x}^*(t) = \overline{x}(t)\,, \quad t \geqq 0$$

In order to see this it is enough to prove that

$$r(t) = \limsup_{n=\infty} r_n(t) = 0\,, \qquad r_n(t) = \|\overline{x}_n(t) - \overline{x}_{n-1}(t)\|$$

furthermore that (17) has a unique solution (see 1.). Then the total successive approximation (19) converges too.

Now (19) gives

$$r_{n+1}(t) \leqq c_1 \int_0^t g(\tau)\,\omega(r_n(\tau))\,d\tau + c_2 \int_t^\infty g(\tau)\,\omega(r_n(\tau))\,d\tau\,, \qquad (n = 1, 2, \ldots)$$

whence by a lemma of Fatou (cf. [10] p. 17)

$$r(t) \leqq c_1 \int_0^t g(\tau) \limsup_{n=\infty} \omega(r_n(\tau))\,d\tau + c_2 \int_t^\infty g(\tau) \limsup_{n=\infty} \omega(r_n(\tau))\,d\tau$$

but $\limsup_{n=\infty} \omega(r_n(t)) \leqq \omega(r(t))$. Viz., for $n > N$ with a certain integer $N > 0$, $r_n(t) < r(t) + \varepsilon$ $(\varepsilon > 0)$ uniformly in t (i.e. for $t \geqq 0$; see [13] p. 55),[5] consequently $\omega(r_n(t)) \leqq \omega(r(t) + \varepsilon)$ or $\limsup_{n=\infty} \omega(r_n(t)) \leqq \omega(r(t) + \varepsilon)$ what involves our assertion, since $\varepsilon > 0$ is arbitrary. Hence $r(t)$ satisfies

$$(23) \qquad r(t) \leqq c_1 \int_0^t g(\tau)\,\omega(r(\tau))\,d\tau + c_2 \int_t^\infty g(\tau)\,\omega(r(t))\,d\tau \qquad (= V(t)),\, t \geqq 0\,.$$

If (17) had two solutions — say $\overline{x}(t)$ and $\overline{x}^0(t)$ — —, then $\varrho(t) = \|\overline{x}(t) - \overline{x}^0(t)\|$ also satisfies (23) with $\varrho(t)$ instead of $r(t)$. Therefore the proof of the identical vanishing of $r(t)$ and $\varrho(t)$ takes place simultaneously.

If e. g. $r(t) \not\equiv 0$, $t \geqq 0$, then in (23) $V(t) > 0$, $t \geqq 0$, i.e. $\omega(r(t)) \leqq \omega(V(t))$ or

$$\frac{(c_2 - c_1)\,\omega(r(t))\,g(t)}{\omega(V(t))} \leqq (c_2 - c_1)\,g(t) \qquad (c_2 > c_1)$$

and by the substitution $u = V(t)$, $du = -(c_2 - c_1)\,g(t)\,\omega(r(t))\,dt$

$$\int_{V(\infty)}^{V(0)} \frac{du}{\omega(u)} \leqq (c_2 - c_1) \int_0^\infty g(t)\,dt = (c_2 - c_1)\,q = \varrho_2 - \varrho_1\,.$$

Here we have

$$\left.\begin{array}{l} V(0) = c_2 \int_0^\infty g(\tau)\,\omega(r(\tau))\,d\tau = c_2 mq \\ V(\infty) = c_1 \int_0^\infty g(\tau)\,\omega(r(\tau))\,d\tau = c_1 mq \end{array}\right\} \quad 0 < m \leqq \omega(2M)$$

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[5] First this holds for a finite interval $(0, T)$, but also for $t > T$ provided T is sufficiently large, since $r(t) \to 0$, $r_n(t) \to 0$ uniformly in n as $t \to \infty$.

thus $\int\limits_{\varrho_1 m}^{\varrho_2 m} \frac{du}{\omega(u)} \leqq \varrho_2 - \varrho_1$ in contradiction to the condition 3. The case $c_1 > c_2$ may be treated in the same way.

For $c_1 \neq c_2$ $\omega(u)$ can be chosen as follows (suppose e.g. $c_2 > c_1$)

$$\omega(u) = u \log \frac{1}{u} = - u \log u \qquad \left(0 < u \leqq \frac{1}{e}\right)$$

and in a suitable manner for $u > \frac{1}{e}$.[6] The condition 3 reads now

$$- \int\limits_{\varrho_1 m}^{\varrho_2 m} \frac{du}{u \log u} = \log \frac{\log(\varrho_1 m)}{\log(\varrho_2 m)} > \varrho_2 - \varrho_1$$

provided that $\varrho_2 m \leqq \frac{1}{e}$ or

$$\text{(24)} \qquad e^{\varrho_1} \log(\varrho_1 m) < e^{\varrho_2} \log(\varrho_2 m), \quad \text{or} \quad (\varrho_1 m)^{e^{\varrho_1}} < (\varrho_2 m)^{e^{\varrho_2}}.$$

The function $f(\varrho) = e^{\varrho} \log(\varrho m)$ has a maximum at $\varrho = \varrho_0$ provided $m \leqq \frac{1}{e}$ resp. $2M \leqq \frac{1}{e}$ where ϱ_0 is the solution of the equation $\varrho \log(\varrho m) + 1 = 0$, i.e. (24) is fullfilled for $\varrho_2 \leqq \varrho_0$, but in other cases as well. Condition $\varrho_2 m \leqq \frac{1}{e}$ is surely satisfied, when $\varrho_2 \omega(2M) \leqq \frac{1}{e}$ and $2M \leqq \frac{1}{e}$, i.e.

$$- \varrho_2 (2M) \log(2M) \leqq \frac{1}{e} \quad \text{or} \quad (2M)^{2M} \geqq e^{-\frac{1}{\varrho_2 e}}$$

This holds e.g. for $\varrho_2 \leqq 1$, since the minimum of $h(x) = x^x$ is $e^{-\frac{1}{e}}$. Equation (20′) requires

$$-\int\limits_{\gamma + a}^{\gamma + \frac{\varrho_2}{\varrho_1} a} \frac{du}{u \log u} = \varrho_2 - \varrho_1, \text{ resp. } e^{\varrho_1} \log\left(\gamma + \frac{\varrho_1}{\varrho_1} a\right) = e^{\varrho_2} \log\left(\gamma + \frac{\varrho_2}{\varrho_1} a\right)$$

$$\left(\text{provided } M = \gamma + \frac{\varrho_2}{\varrho_1} a \leqq \frac{1}{2e}\right).$$

For certain $\gamma, \varrho_1, \varrho_2, a$ this may be satisfied. Then all requirements are satisfied if ϱ_2 is small enough. However the assumption $M \leqq \frac{1}{2e}$ is not necessary. Per-

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[6] E.g. $\omega(u) = \frac{2}{e} + u \log u$ suits for $u > \frac{1}{e}$.

haps in the opposite case the function $f(\varrho) = e^{\varrho} \log (\varrho m)$ has no maximum — viz. if $m > \frac{1}{e}$ — and then ϱ_2 can be arbitrary large.

The functions $\omega(u) = u \log \frac{1}{u} \log \log \frac{1}{u}, \ldots$ and $u \left(\log \frac{1}{u}\right)^k, \ldots (0 < k < 1)$ can also be applied here.

The case $c_1 = c_2 = c$ may be settled in a straightforward way or by the limit process $\varrho_2 \to \varrho_1 = \varrho$ carried out in cond. 3 and (20'). Obtaining

$$\text{(25)} \qquad \begin{cases} \omega(\varrho m) < m, & 0 < m \leqq \omega(2M) \\ M = \varrho\omega(M) + \gamma, & M = \gamma + a. \end{cases}$$

In the case (W) this leads to

$$\varrho < 1, \quad M = \frac{\gamma}{1 - \varrho}.$$

In the present case for an actual nonlinear $\omega(u)$ it seems to be impossible to obtain a reasonable result.

Remark. Theorem 1 may be easily extended to a variable matrix $\bar{A}(t)$ too provided that it is periodic or

$$\varliminf_{t \to \infty} \int_0^t tr(\bar{A})\, dt > -\infty$$

holds. Theorem 2 seems also to be capable to an extension for a periodic $\bar{A}(t)$.

5. As an *application* let us regard the equation (see [11])

$$\text{(26)} \qquad u'' + u + \varrho(t)\, h(u, u') = 0, \qquad h(0, v) = 0$$

for the scalar function $u = u(t)$ with the following conditions to be satisfied.

1. $\varrho(t)$ is continuous for $t \geqq 0$ and $\int^{\infty} | \varrho(t) | \, dt < \infty$,
2. $| h(u, v) - h(u^*, v^*) \leqq \omega(| u - u^* | + | v - v^* |)$ where $\omega(z)$ is as before.

Then corresponding to every solution of (26) there exist two constants $\alpha_\infty \neq 0$ and δ_∞ such that

$$u(t) - \alpha_\infty \sin (t + \delta_\infty) \to 0, \quad u'(t) - \alpha_\infty \cos (t + \delta_\infty) \to 0, \quad t \to +\infty$$

holds and the converse assertion is valid as well.

This statement is closely related to the result of [11] (where the converse statement fails), and is a generalization of [12], p. 388. In particular, the estimate given in [11] may be obtained also here. In the present case as $\bar{Y}_1 \equiv 0$, cond. 3 of 3 is superfluous. Strange enough, neither of the conditions $h(\lambda u, \lambda v) = \lambda h(u, v)$, $\operatorname{sg} h(u, v) = \operatorname{sg} u$ (for arbitrary λ, u, v) of [11] is here necessary.

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АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ РЕШЕНИЙ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Резюме

LEVINSON [5] и WEYL [8] показали, что если все решения уравнения (2) ограничены, тогда каждому решению $\bar{x}(t)$ уравнения (1) приподлежит одно решение $\bar{y}(t)$ уравнения (2) такое, что

$$\bar{x}(t) - \bar{y}(t) \to 0, \qquad \text{если} \quad t \to \infty,$$

и что обратное утверждение также имеет силу. Результат LEVINSONа относится только к линейным системам, результат WEYLа имеет силу и для нелинейных систем, однако только в том случае, если функция $\bar{f}(t, \bar{x})$ в (1) имеет «линейный мажорант». В настоящей статье автор распространяет эту теорему на нелинейные системы при условии (4), соотв. (5), далее он дает обобщение одного результата WINTNERА [10], относящегося к сходимости последовательных приближений. Он показывает сходимость последовательных приближений (9), относящихся к уравнению (1) при условии (5). Наконец, в качестве приложения он дает обобщение одной теоремы WINTNER из [12], которое очень схоже с одним результатом автора из [11].

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