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Determination of the capacitance of capacitor with orthotropic dielectric material

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ORIGINAL RESEARCH PAPER

ABSTRACT

The paper deals with the capacitance of cylindrical two-dimensional capacitor which consists of Cartesian orthotropic dielectric material. The determination of the capacitance of capacitor with orthotropic dielectric material by a suitable coordinate transformation is reduced to the computation of capacitance of an isotropic capacitor. It is proven that the capacitance of a Cartesian orthotropic capacitor can be obtained in terms of an isotropic capacitor whose dielectric constant is the geometric mean of the dielectric constant of the orthotropic capacitor.

KEYWORDS

capacitance, two-dimensional capacitor, orthotropic dielectric material

1. INTRODUCTION

Combining different computation methods with analytical procedures is a common method for solving electrical problems. Paper [1] developes a computation method determining eddy currents effect in electrical steel sheets. Ecsedi and Baksa formulate a mathematical model to obtain upper and lower bounds for a three-dimensional hollow non-homogeneous body [2].

This study includes the determination of a cylindrical two-dimensional capacitor. The considered capacitor consists of homogeneous and orthotropic dielectric materials. The capacitance is the ability of a capacitor to store electric charge per unit voltage across its inner and outer surfaces. The capacitance is a function depending on the geometry of a capacitor and the permittivity of its dielectric material. The solution of the capacitance of capacitor with Cartesian orthotropic dielectric materials is reduced to the problem of capacitor with isotropic dielectric material with a suitable homogeneous linear coordinate transformation. Knowledge of electricity, which is necessary for formulating and solving the set task, can be found in detail in books [3–6]. This type of the presented approach was used in paper [7] to solve the Saint-Venant's torsion problem of Cartesian orthotropic elastic bars.

2. GOVERNING EQUATIONS OF ELECTROSTATIC FIELD FOR TWO-DIMENSIONAL CAPACITOR

Figure 1 shows a two-dimensional hollow plane domain A whose inner boundary curve is ∂A_1 and outer boundary curve is ∂A_2. The origin of the Cartesian coordinate system Oxy is an inner point of closed curve ∂A_1, and the unit vectors of the Oxy coordinate system are $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$ denotes the position vector of an arbitrary point P in $\overline{A} = A \cup \partial A_1 \cup \partial A_2$. To give the concept of capacitor for two-dimensional hollow domain shown in Fig. 1 the following boundary value problem is defined

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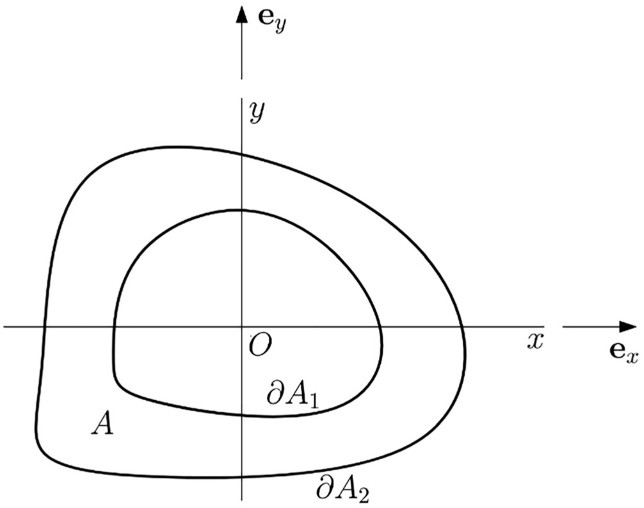

Fig. 1. Two-dimensional hollow plane domain in the plane *Oxy*

$$\nabla \cdot (\varepsilon \cdot \nabla U) = 0, \quad \boldsymbol{r} \in A\,, \tag{1}$$

$$\begin{aligned} U(\boldsymbol{r}) &= U_1, \quad \boldsymbol{r} \in \partial A_1, \\ U(\boldsymbol{r}) &= U_2, \quad \boldsymbol{r} \in \partial A_2, \end{aligned} \qquad U_1 \neq U_2\,. \tag{2}$$

In Eqs (1) and (2) $U = U(\boldsymbol{r})$ is the electric potential, ε is a two-dimensional second order positive definite tensor called the permittivity tensor of the Cartesian orthotropic dielectric material and ∇ is the two-dimensional Nabla operator

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y}. \tag{3}$$

In Eq. (1) the scalar product is denoted by dot. The representation of permittivity tensor in the coordinate system *Oxy* is as follows

$$\varepsilon = \varepsilon_x \boldsymbol{e}_x \circ \boldsymbol{e}_x + \varepsilon_y \boldsymbol{e}_y \circ \boldsymbol{e}_y, \tag{4}$$

where the circle between two vectors denotes their diadic product. The matrix representation of permittivity tensor will be used

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & 0 \\ 0 & \varepsilon_y \end{bmatrix}. \tag{5}$$

In the case of isotropic dielectric material

$$\boldsymbol{\varepsilon} = \varepsilon\left(\boldsymbol{e}_x \circ \boldsymbol{e}_x + \boldsymbol{e}_y \circ \boldsymbol{e}_y\right) = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \varepsilon \mathbf{1}, \tag{6}$$

where $\mathbf{1}$ is the two-dimensional second order unit tensor.

Denote C the capacitance of the two-dimensional capacitor. The unit of C is (F/m). The electric energy of the capacitor can be obtained as [3–6]

$$W = \frac{C}{2}(U_1 - U_2)^2. \tag{7}$$

Formula (7) is reformulated by the use of a new function $u = u(\boldsymbol{r})$. The connection between $U = U(\boldsymbol{r})$ and $u = u(\boldsymbol{r})$ is as follows

$$U(\boldsymbol{r}) = (U_1 - U_2)u(\boldsymbol{r}) + U_2. \tag{8}$$

It is evident that $u = u(\boldsymbol{r})$ satisfies the following Dirichlet type boundary value problem

$$\nabla \cdot (\varepsilon(\boldsymbol{r}) \cdot \nabla U) = 0, \quad \boldsymbol{r} \in A\,, \tag{9}$$

$$\begin{aligned} u(\boldsymbol{r}) &= 1, \quad \boldsymbol{r} \in \partial A_1, \\ u(\boldsymbol{r}) &= 0, \quad \boldsymbol{r} \in \partial A_2. \end{aligned} \tag{10}$$

The specific electric energy can be computed as [3–6].

$$\begin{aligned} w &= \frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{D} = \frac{1}{2}\boldsymbol{E} \cdot \varepsilon \cdot \boldsymbol{E} = \frac{1}{2}\nabla U \cdot \varepsilon \cdot \nabla U \\ &= \frac{1}{2}(U_1 - U_2)^2 \nabla u \cdot \varepsilon \cdot \nabla u, \end{aligned} \tag{11}$$

where $\boldsymbol{E}$ is the electric field vector and $\boldsymbol{D}$ is the electric displacement vector. The whole electric energy of the two-dimensional Cartesian orthotropic capacitor can be formulated as

$$W = \int_A w dA = \frac{1}{2}(U_1 - U_2)^2 \int_A \nabla u \cdot \varepsilon \cdot \nabla u \; dA. \tag{12}$$

Comparison of Eq. (7) with Eq. (12) gives an explicit formula for the capacitance C

$$C = \int_A \nabla u \cdot \varepsilon \cdot \nabla u \; dA. \tag{13}$$

It must be noted that the capacitance of a capacitor of width h is h times that which is given by formula (13), where h is the length of the capacitor.

3. TRANSFORMATION OF THE GOVERNING BOUNDARY VALUE PROBLEM

The law of the homogeneous linear transformation between the coordinates (X, Y) and (x, y) is defined as

$$X = a_x x, \quad Y = a_y y\,, \tag{14}$$

$$a_x = \sqrt[4]{\varepsilon_y / \varepsilon_x}, \quad a_y = \sqrt[4]{\varepsilon_x / \varepsilon_y}\,. \tag{15}$$

Let

$$\tilde{F}_i(X, Y) = 0, \quad (X, Y) \in \partial \tilde{A}_i \tag{16}$$

be the equation of boundary curve $\partial \tilde{A}_i$ $(i = 1, 2)$. The plane domain whose inner boundary is curve $\partial \tilde{A}_1$ and its outer boundary curve is $\partial \tilde{A}_2$ in the plane $\tilde{O}XY$ is denoted by $\tilde{A}$ (Fig. 2).

Let

$$F_i(x, y) = 0, \quad (i = 1, 2) \tag{17}$$

be the equation of the boundary curve ∂A_i $(i = 1, 2)$ (Fig. 1). It is evident that

$$F_i(x, y) = \tilde{F}_i\left(a_x x, a_y y\right), \quad (i = 1, 2)\,. \tag{18}$$

The one-to-one map given by Eq. (14) preserves the area. This fact follows from Eq. (19)

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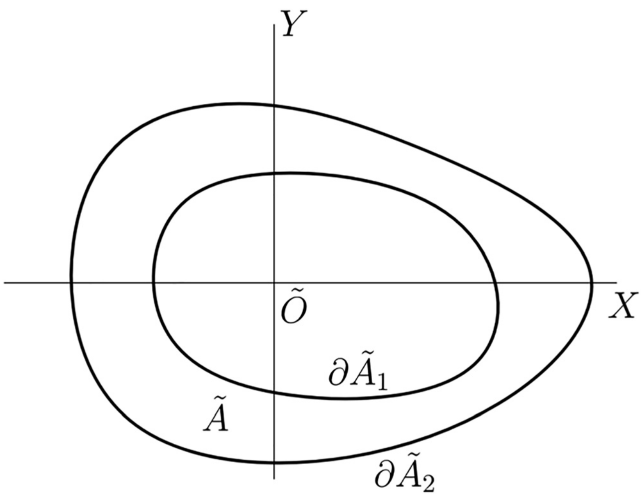

Fig. 2. Two-dimensional hollow plane domain in the plane $\tilde{O}XY$

$$d\tilde{A} = dXdY = \left|\frac{\partial(X,Y)}{\partial(x,y)}\right| \mathrm{d}x\mathrm{d}y = \begin{vmatrix} a_x & 0 \\ 0 & a_y \end{vmatrix} dxdy$$
$$= \begin{vmatrix} \sqrt[4]{\varepsilon_y/\varepsilon_x} & 0 \\ 0 & \sqrt[4]{\varepsilon_x/\varepsilon_y} \end{vmatrix} dA = dA. \tag{19}$$

Let $u = u(x,y)$ be defined as

$$u(x,y) = \tilde{u}(a_x x, a_y y), \tag{20}$$

which is the solution of the boundary value problem formulated by the following equations

$$\frac{\partial^2\tilde{u}}{\partial X^2} + \frac{\partial^2\tilde{u}}{\partial Y^2} = 0, \quad (X,Y) \in \tilde{A}\,, \tag{21}$$

$$\tilde{u}(X,Y) = 1, \quad (X,Y) \in \partial\tilde{A}_1\,, \tag{22}$$

$$\tilde{u}(X,Y) = 0, \quad (X,Y) \in \partial\tilde{A}_2\,. \tag{23}$$

Substituting the following expressions

$$\tilde{u} = \tilde{u}(X,Y), \quad \boldsymbol{\varepsilon} = \varepsilon\mathbf{1} \tag{24}$$

in Eq. (13) yields

$$C(\varepsilon) = \tilde{C} = \varepsilon\int_A \left[\left(\frac{\partial\tilde{u}}{\partial X}\right)^2 + \left(\frac{\partial\tilde{u}}{\partial Y}\right)^2\right]\mathrm{d}\tilde{A}. \tag{25}$$

From Eq. (20) it follows that

$$\frac{\partial u}{\partial x} = a_x\frac{\partial\tilde{u}}{\partial X}, \quad \frac{\partial^2 u}{\partial x^2} = a_x^2\frac{\partial^2\tilde{u}}{\partial X^2}, \tag{26}$$

$$\frac{\partial u}{\partial y} = a_y\frac{\partial\tilde{u}}{\partial Y}, \quad \frac{\partial^2 u}{\partial y^2} = a_y^2\frac{\partial^2\tilde{u}}{\partial Y^2}. \tag{27}$$

A simple computation results that

$$\varepsilon_x\frac{\partial^2 u}{\partial x^2} + \varepsilon_y\frac{\partial^2 u}{\partial y^2} = \varepsilon_x a_x^2\frac{\partial^2\tilde{u}}{\partial X^2} + \varepsilon_y a_y^2\frac{\partial^2\tilde{u}}{\partial Y^2}$$
$$= \sqrt{\varepsilon_x\varepsilon_y}\left(\frac{\partial^2\tilde{u}}{\partial X^2} + \frac{\partial^2\tilde{u}}{\partial Y^2}\right), \quad (x,y) \in A, \quad (X,Y) \in \tilde{A} \tag{28}$$

4. DETERMINATION OF THE CAPACITANCE OF ANISOTROPIC CAPACITOR

According to Eqs (26) and (27) it follows that

$$C(\varepsilon_x, \varepsilon_y) = \int_A \nabla u \cdot \boldsymbol{\varepsilon} \cdot \nabla u \, dA = \int_{\tilde{A}} \left[\varepsilon_x a_x^2\left(\frac{\partial\tilde{u}}{\partial X}\right)^2 + \varepsilon_y a_y^2\left(\frac{\partial\tilde{u}}{\partial Y}\right)^2\right] d\tilde{A}$$
$$= \sqrt{\varepsilon_x\varepsilon_y}\int_A \left[\left(\frac{\partial\tilde{u}}{\partial X}\right)^2 + \left(\frac{\partial\tilde{u}}{\partial Y}\right)^2\right] d\tilde{A} = C\left(\sqrt{\varepsilon_x\varepsilon_y}\right). \tag{29}$$

Eq. (29) shows that the capacitance of a Cartesian orthotropic capacitor can be expressed in terms of an isotropic capacitor whose dielectric constant is the geometric mean of the dielectric constants of the orthotropic capacitor. Eq. (29) gives a possibility to obtain the capacitance of a Cartesian orthotropic two-dimensional capacitor in terms of an isotropic capacitor whose dielectric constant is

$$\varepsilon = \sqrt{\varepsilon_x\varepsilon_y}. \tag{30}$$

5. EXAMPLE

Let the isotropic two-dimensional capacitor be a cylindrical capacitor whose inner boundary circle is $\partial\tilde{A}_1$ and outer boundary circle is $\partial\tilde{A}_2$ and

$$X^2 + Y^2 - R_1^2 = 0, \quad (X,Y) \in \partial\tilde{A}_1\,, \tag{31}$$

$$X^2 + Y^2 - R_2^2 = 0, \quad (X,Y) \in \partial\tilde{A}_2\,, \tag{32}$$

and $0<R_1<R_2$ (Fig. 3). The capacitance of isotropic hollow circular capacitor [3–7] is

$$C(\varepsilon) = 2\pi\varepsilon\left(\ln\frac{R_2}{R_1}\right)^{-1}. \tag{33}$$

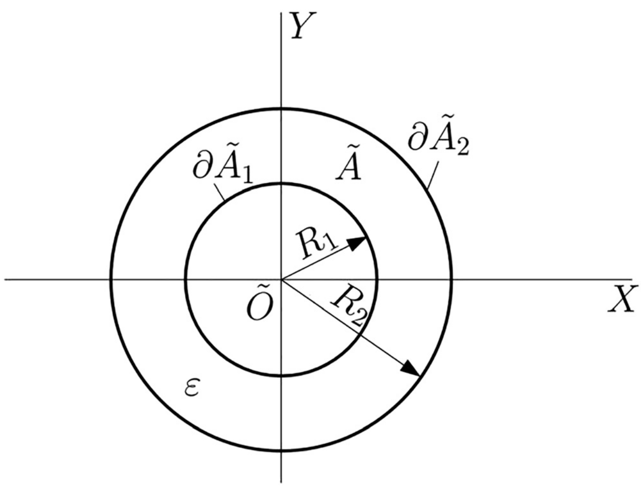

Fig. 3. Isotropic two-dimensional capacitor with cylindrical boundary curves

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In the present problem

$$F_i(x,y) = \tilde{F}_i(a_x x, a_y y) = 0, \quad (i = 1, 2). \tag{34}$$

A detailed form of the equation of boundary contour ∂A_i is as follows (Fig. 4)

$$\frac{x^2}{\alpha_i^2} + \frac{y^2}{\beta_i^2} - 1 = 0, \quad (i = 1, 2), \tag{35}$$

$$\alpha_i = R_i \sqrt[4]{\varepsilon_x/\varepsilon_y}, \quad \beta_i = R_i \sqrt[4]{\varepsilon_y/\varepsilon_x}. \tag{36}$$

The capacitance of anisotropic capacitor with elliptical boundary curves is

$$C(\varepsilon_x, \varepsilon_y) = 2\pi\sqrt{\varepsilon_x \varepsilon_y}\left(\ln\frac{R_2}{R_1}\right)^{-1}. \tag{37}$$

The solution of the boundary value problem

$$\varepsilon_x \frac{\partial^2 u}{\partial x^2} + \varepsilon_y \frac{\partial^2 u}{\partial y^2} = 0 \quad \boldsymbol{r} \in A, \tag{38}$$

$$\begin{aligned} u(\boldsymbol{r}) &= 1, \quad \boldsymbol{r} \in \partial A_1, \\ u(\boldsymbol{r}) &= 0, \quad \boldsymbol{r} \in \partial A_2, \end{aligned} \tag{39}$$

is as follows

$$u(x,y) = k \ln\left(\left(a_x^2 x^2 + a_y^2 y^2\right)/R_2^2\right), \tag{40}$$

where $k = \left(\ln\left(\frac{R_1}{R_2}\right)^2\right)^{-1}$.

The equation of the boundary curve in polar coordinates

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan\,(y/x) \tag{41}$$

is

$$\begin{aligned} \left|\overline{OP_i}\right| &= \rho_i(\varphi) \qquad\qquad 0 \le \varphi \le 2\pi, \\ &= \alpha_i \beta_i \left(\alpha_i^2 \sin^2\varphi + \beta_i^2 \cos^2\varphi\right)^{-0.5}, \quad (i = 1, 2). \end{aligned} \tag{42}$$

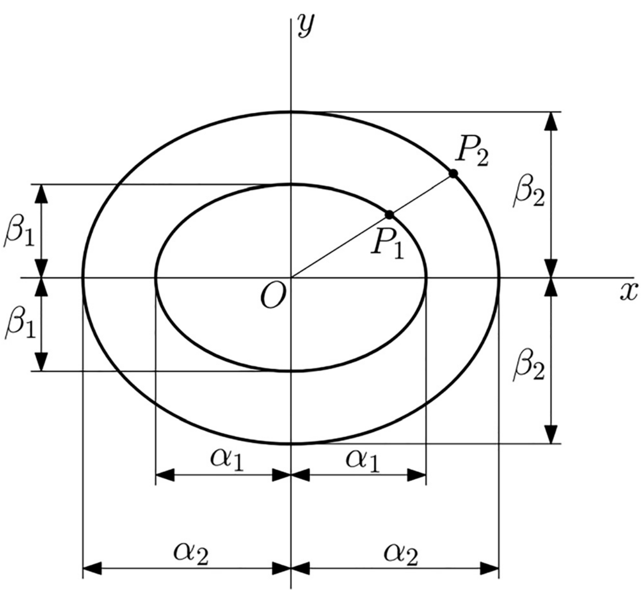

Fig. 4. Anisotropic two-dimensional capacitor with elliptical boundary curves

The function $u = u(x,y)$ can be represented in polar coordinates as

$$\begin{aligned} v(r,\varphi) &= u(r\cos\varphi, r\sin\varphi) \\ &= k\ln\left[\frac{r^2}{R_2^2}\left(a_x^2\cos^2\varphi + a_y^2\sin^2\varphi\right)\right]. \end{aligned} \tag{43}$$

Let $V_j = V_j(r)$ be defined as

$$V_j(r) = v\left(r, \varphi_j\right), \quad \rho_1\left(\varphi_j\right) \le r \le \rho_2\left(\varphi_j\right). \tag{44}$$

The graphs of $V_j(r)$ for $\rho_1(\varphi_j) \le r \le \rho_2(\varphi_j)$ are shown in Figs 5 and 6 for $\varphi_j = 0$ and $\varphi_j = \pi/2$. The contour lines of the function $u = u(x,y)$ are presented in Fig. 7. The following numerical data were used $\varepsilon_x = 8\cdot 10^{-12}$ (F m$^{-1)}$), $\varepsilon_y = 8.5\cdot 10^{-12}$ (F m$^{-1)}$), $R_1 = 0.025$ m, $R_2 = 0.04$ m.

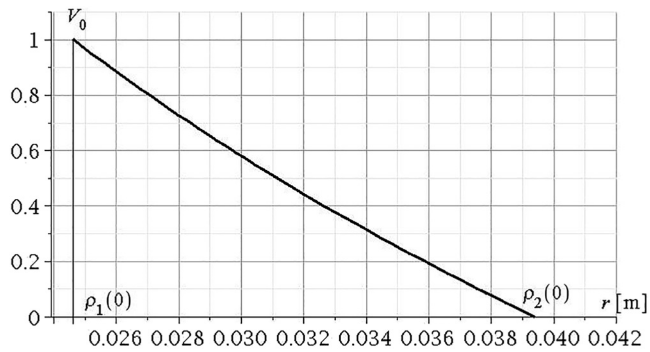

Fig. 5. The graph of $V_0(r)$ for $\rho_1(0) \le r \le \rho_2(0)$

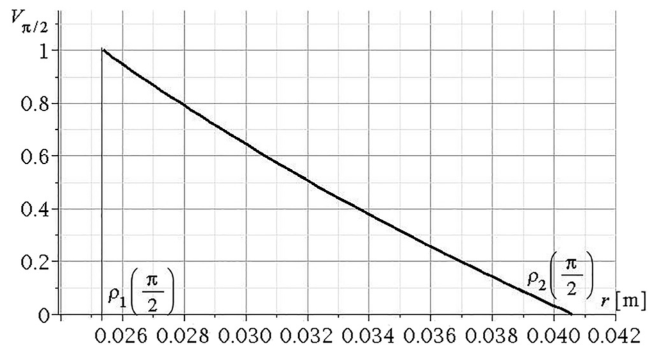

Fig. 6. The graph of $V_{\pi/2}(r)$ for $\rho_1(\pi/2) \le r \le \rho_2(\pi/2)$

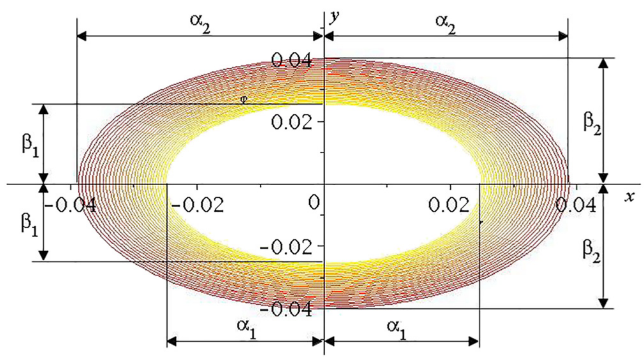

Fig. 7. The contour lines of the function $u = u(x,y)$

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6. CONCLUSIONS

The considered two-dimensional cylindrical capacitor consists of homogeneous Cartesian orthotropic dielectric material. The capacitance of orthotropic capacitor is expressed in terms of a homogeneous capacitor whose permittivity is the geometrical mean of the principle values of the Cartesian orthotropic capacitor.

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