

Weak and strong field approximations and circular orbits of Kehagias-Sfetsos space-time

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The Kehagias-Sfetsos asymptotically flat black hole and naked singularity solutions of Hořava-Lifshitz gravity are investigated both in the weak-field and strong-field regimes. In the weak-field limit the gravitational field generated by the Kehagias-Sfetsos spherically symmetric solution is weaker than in the case of the Schwarzschild black hole of general relativity. In the strong-field regime naked singularities with $\omega_0 \ll 1$ display an unusual distance dependence: gravity becomes weaker when approaching the singularity. The stability of circular orbits is also analyzed. While in the black hole case the square of the angular momentum should be larger than a certain finite, non-zero minimal value, in the naked singularity case there are stable circular orbits for any non-zero value of the angular momentum. In this regime we prove the existence of an infimum of the allowed radii of circular orbits (corresponding to vanishing angular momentum).

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1 Introduction

General relativity (GR) has been precisely tested on the Solar system scale, however the very small and very large distance behaviour of gravity is less well verified, leading to numerous proposed modifications of GR. Recently Hořava proposed a modification of GR at high energies, motivated by the Lifshitz scalar field theory in solid state physics. The Hořava-Lifshitz (HL) gravitational theory introduces anisotropy between space and time. A recent review of its Lorentz invariance violation, occurring at trans-Planckian energy scales is presented in (17). Among the several proposed versions of the HL theory, the infrared (IR)-modified Hořava gravity is the one which seems to be consistent with the current observational data (4, 5, 11).

The spherically symmetric space-time in vacuum HL gravity is characterized by the family of metrics (16)

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

with

$$f(r) = 1 + (\omega - \Lambda)r^2 - \sqrt{r[\omega(\omega - 2\Lambda)r^3 + \beta]}. \quad (2)$$

Here β is an integration constant, while Λ and ω are real parameters. Depending on the values of β , ω and Λ , special cases of the metric (2) arise.

The choice $\beta = -(\alpha^2)/(\Lambda)$ and $\omega = 0$ leads to the Lu-Mei-Pope black hole solution (13). With $\beta = 4\omega m$ and $\Lambda = 0$ the Kehagias-Sfetsos (KS) solution is obtained (10).

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This paper focuses on the latter space-time, for which the function $f(r)$ is

$$f(r) = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega m r}. \quad (3)$$

This solution is asymptotically flat and for large r or when $\omega \rightarrow \infty$ it approaches the Schwarzschild solution of GR. A slowly rotating modification of the Kehagias-Sfetsos solution was introduced in 12.

Beyond mass m , the space-time (3) is characterised by the Hořava-Lifshitz parameter $\omega > 0$. It is customary to employ the dimensionless parameter $\omega_0 = \omega m^2 > 0$ too. General relativity is recovered for $\omega_0 \rightarrow \infty$ but the black hole interpretation continues to hold for any $\omega_0 \geq 0.5$. Indeed, when $\omega_0 > 1/2$ there are two event horizons at

$$r_{\pm} = m \left(1 \pm \sqrt{1 - \frac{1}{2\omega_0}} \right). \quad (4)$$

The two event horizons coincide for $\omega_0 = 0.5$. The space-time (3) becomes a naked singularity whenever $\omega_0 < 0.5$.

The value of the parameter ω_0 has been constrained by various methods. The radar echo delay in the Solar system, analyzed in (6) gave the limit $\omega_{0,\min}^{(red)} = 2 \times 10^{-15}$. The analysis of the perihelion precession of Mercury and of the deflection of light by the Sun resulted in $\omega_{0,\min}^{(pp)} = 6.9 \times 10^{-16}$ and $\omega_{0,\min}^{(ld)} = 1.1 \times 10^{-15}$, respectively. Tighter constraints for ω_0 were presented in (9), based on the analysis of the range-residuals of the planet Mercury: $\omega_{0,\min}^{(residual)} = 7.2 \times 10^{-10}$. A slightly stronger constraint $\omega_{0,\min}^{(Sag)} = 8 \times 10^{-10}$ arises from the observation of the S2 star orbiting the Supermassive Black Hole (Sagittarius A*) in the center of our

Galaxy. It has been shown in (8) that the forthcoming Large Synoptic Survey Telescope will be able to constrain ω_0 up to 10^{-1} from strong gravitational lensing. We remark that neither of these observations could render the limiting value of ω_0 into the regime where the Kehagias-Sfetsos solution describes a black hole.

Charged particles in orbital motion about a compact astrophysical body form an accretion disk. The simplest accretion disk model is the steady-state thin disk, based on several simplifying assumptions (3). For accretion disks with sub-Eddington luminosities the inner edge of the accretion disk is located at the innermost stable circular orbit (ISCO) (15). However, for larger accretion disk luminosities, there is no uniquely defined inner edge. Different definitions lead to different edges, the differences increasing with the luminosity. In (2) six possible definitions of the inner edge have been listed.

Accretion characteristics in the IR limit of HL gravity, based on the thin disk model were explored both for spherically symmetric (7) and for slowly rotating (6) black holes. In the spherically symmetric case the energy flux, the temperature distribution of the disk and the spectrum of the emitted black body radiation all significantly differ from the GR predictions. Also, the intensity of the flux emerging from the disk surface is larger for the slowly rotating KS solution than for the rotating Kerr black hole of GR.

This paper revisits various aspects related to accretion in a KS space-time and is organized as follows. In Sections 2 and 3 we investigate the KS metric in the weak-field and strong-field limits, respectively. In Section 4 we analyze the properties of the stable circular orbits in both the black hole and the naked singularity parameter regimes. We summarize our results in Section 5.

2 Weak-field regime: weakened gravity

The smallness of the post-Newtonian parameter $\varepsilon = m/r$ (in units $G = 1 = c$) characterizes the weak-field regime. Even with $\varepsilon \ll 1$, the relative magnitude of the two parameters ε and ω_0 leads to three different limits.

When $\omega_0 \gg \varepsilon^3$ then $\omega_0^{-1}\varepsilon^3 \ll 1$ and the Kehagias-Sfetsos metric function can be rewritten as

$$\begin{aligned} f &= 1 + \omega_0 \left(\frac{r}{m}\right)^2 \left[1 - \left(1 + \frac{4m^3}{\omega_0 r^3}\right)^{1/2} \right] \\ &= 1 + \omega_0 \varepsilon^{-2} \left[1 - \left(1 + \frac{4}{\omega_0} \varepsilon^3\right)^{1/2} \right]. \end{aligned} \quad (5)$$

Expanding the expression in the bracket as

$$\left(1 + \frac{4}{\omega_0} \varepsilon^3\right)^{1/2} \approx 1 + \frac{2}{\omega_0} \varepsilon^3, \quad (6)$$

the metric function becomes

$$f \simeq 1 - 2\varepsilon, \quad (7)$$

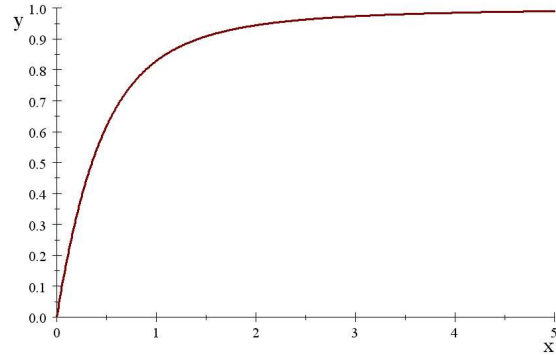


Fig. 1 The effective mass parameter in the $\varepsilon \approx \omega_0^{1/3} \ll 1$ regime as function of $x = (\omega_0 \varepsilon^{-3})^{1/2} / 2$.

thus in this parameter regime the KS metric approximates the Schwarzschild metric.

When $\omega_0 \approx \varepsilon^3$ then $\omega_0^{-1}\varepsilon^3 = \mathcal{O}(1)$, thus the Taylor series expansion cannot be performed as in the previous case. Nevertheless $\omega_0 \varepsilon^{-2} = \omega_0 \varepsilon^{-3} \varepsilon \simeq \varepsilon$ and the metric function can be rewritten as

$$\begin{aligned} f &= 1 + \varepsilon \left\{ \omega_0 \varepsilon^{-3} - \left[(\omega_0 \varepsilon^{-3})^2 + 4\omega_0 \varepsilon^{-3} \right]^{1/2} \right\} \\ &= 1 - 2\varepsilon \left\{ \left[\omega_0 \varepsilon^{-3} + \left(\frac{\omega_0 \varepsilon^{-3}}{2} \right)^2 \right]^{1/2} - \frac{\omega_0 \varepsilon^{-3}}{2} \right\} \\ &= 1 - 2\varepsilon \mathcal{O}(1). \end{aligned} \quad (8)$$

The curly bracket in the second line (to be denoted as y) can be rewritten in terms of $x = (\omega_0 \varepsilon^{-3})^{1/2} / 2$ as

$$y = 2x \left[(1 + x^2)^{1/2} - x \right] \quad (9)$$

and shown on Fig. 1 to take values in the interval $(0, 1)$ only. Therefore the weak-field approximation of the KS solution in this parameter range is again the Schwarzschild solution, nevertheless with an effective mass parameter which is smaller than the mass.

Finally when $\omega_0 \ll \varepsilon^3$ then $\omega_0^{-1}\varepsilon^3 \gg 1$, and we get by another series expansion

$$\begin{aligned} f &= 1 + (\omega_0 \varepsilon^{-3}) \varepsilon \left[1 - \left(1 + \frac{4}{\omega_0} \varepsilon^3\right)^{1/2} \right] \\ &\simeq 1 - 2\varepsilon (\omega_0 \varepsilon^{-3})^{1/2}. \end{aligned} \quad (10)$$

This is another Schwarzschild regime with a very small effective mass parameter.

In summary in the weak field limit the metric function becomes

$$f = 1 - 2\varepsilon y, \quad (11)$$

with the positive parameter

$$\begin{aligned} y &= 1, \text{ for } \omega_0 \gg \varepsilon^3, \\ y &< 1, \text{ for } \omega_0 \approx \varepsilon^3, \\ y &\ll 1, \text{ for } \omega_0 \ll \varepsilon^3. \end{aligned} \quad (12)$$

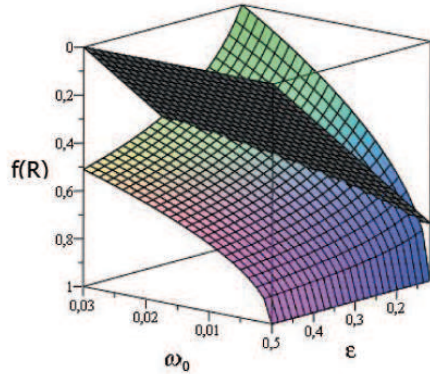


Fig. 2 The behaviour of the metric function f (lighter surface) as compared to the respecting metric function of the Schwarzschild black hole (dark surface). For large values of ε and small values of ω_0 the $1/r$ dependence (linear dependence on ε) of f of the Schwarzschild space-time is replaced by an $r^{1/2}$ dependence ($\varepsilon^{-1/2}$ dependence) in the KS space-time.

In spite of the fact that for all $\omega_0 < 0.5$ the KS space-time represents a naked singularity, its weak-field regime is Schwarzschild with a positive effective mass

$$m_{eff} = my. \quad (13)$$

As $m_{eff} \leq m$, gravity is weaker in the weak-field limit of the KS space-time compared to the Schwarzschild case.

3 Strong-field regime: unusual distance dependence

Approaching the center the value of the post-Newtonian parameter increases, at the Schwarzschild radius becoming 0.5. Hence in this strong-field regime $\varepsilon = \mathcal{O}(1)$. When $\omega_0 \gg 1$, a series expansion of Eq. (5) in ω_0^{-1} leads to the Schwarzschild metric function $f = 1 - 2\varepsilon$. The regime $\omega_0 = \mathcal{O}(1)$ has been studied numerically (7) for the black hole case $\omega_0 > 0.5$.

When $\omega_0 \ll 1$ (which is in the naked singularity regime) a series expansion of Eq. (5) in ω_0^{-1} leads to

$$f \simeq 1 - 2\varepsilon^{-1/2}\omega_0^{1/2} = 1 - 2\left(\frac{\omega_0}{m}\right)^{1/2}r^{1/2}. \quad (14)$$

(Note that f stays positive as $\varepsilon > 4\omega_0$ is obeyed in the chosen parameter range.) Hence gravity decreases when approaching the singularity. This behaviour is radically different from the GR prediction and is presented in Fig. 2.

4 Circular orbits about Kehagias-Sfetsos black holes

The observational tests on the limiting $\omega_0 \approx 10^{-(9 \div 16)}$ confirm that in the $\omega_0 = \mathcal{O}(1)$ regime the geodetic motion of free particles is a valid approximation (14). In this section we focus to this parameter range, studying the timelike geodetic motion in the KS space-time and determining the radius of the innermost stable circular orbit (the inner edge of the accretion disk).

Circular motions in the equatorial plane ($\theta = \pi/2$) are defined by a constant normalized radial coordinate $R := r/m$. The t , r and φ components of the geodesic equation reduce to

$$\left(\frac{d^2\varphi(\tau)}{d\tau^2}\right) = 0, \quad (15)$$

$$\left(\frac{d^2t(\tau)}{d\tau^2}\right) = 0, \quad (16)$$

$$\frac{\omega_0(R\Xi - \omega_0 R^3 - 1)}{m^2\Xi} \left(\frac{dt(\tau)}{d\tau}\right)^2 = R \left(\frac{d\varphi(\tau)}{d\tau}\right)^2, \quad (17)$$

respectively, where $\Xi = \sqrt{R\omega_0(R^3\omega_0 + 4)}$ and τ is the proper time. Eqs. (15) and (16) imply that $\dot{\varphi}(\tau) = d\varphi(\tau)/d\tau$ and $\dot{t}(\tau) = dt(\tau)/d\tau$ are constants, leading to conserved specific orbital angular momentum $L = m^2 R^2 \dot{\varphi}$ and specific energy $E = f\dot{t}$ of the particle. Eq. (17), through the positivity of ω_0 implies

$$R > \left(\frac{1}{2\omega_0}\right)^{1/3}. \quad (18)$$

Both for stable or unstable circular orbits, regardless whether the KS space-time represents a black hole or a naked singularity, Eq. (18) holds as a strict inequality. Note that the equality would imply $L = 0$, while smaller values of R would lead to the forbidden range $L^2 < 0$.

The effective potential can be expressed as 7:

$$V_{eff}(L, R, \omega_0) =$$

$$\left[1 + \omega_0 R^2 \left(1 - \sqrt{1 + \frac{4}{\omega_0 R^3}}\right)\right] \left(1 + \frac{L^2}{R^2}\right), \quad (19)$$

a constant of motion itself ($dV_{eff}/dR = 0$), as it depends only on L , R and ω_0 . Stable circular orbits occur at the local minima of the potential, while the local maxima in the potential are the locations of unstable circular orbits.

4.1 Black hole parameter range ($\omega_0 \geq 0.5$)

For each $\omega_0 \geq 0.5$ a value of the angular momentum L can be found for which there is one extremum of the effective potential. At this radius the conditions $dV_{eff}/dR = 0$ and $d^2V_{eff}/dR^2 = 0$ are obeyed. These equations can be solved numerically as follows. From the extremum condition $dV_{eff}/dR = 0$ one can express $L = L(R, \omega_0)$, which when inserted in the marginal stability condition

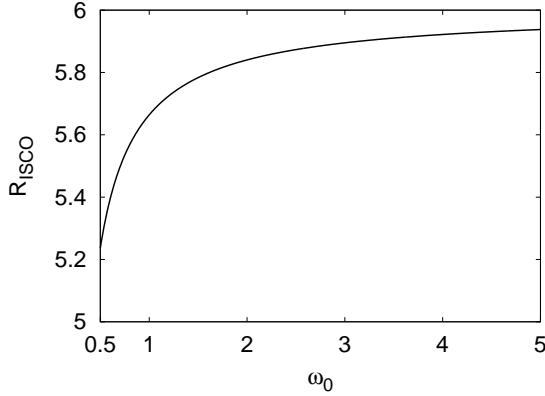


Fig. 3 The ISCO radius as a function of the dimensionless parameter ω_0 in the black hole regime. For large ω_0 values the radius of the ISCO approaches $R_{ISCO} = 6$, characteristic for the Schwarzschild black hole.

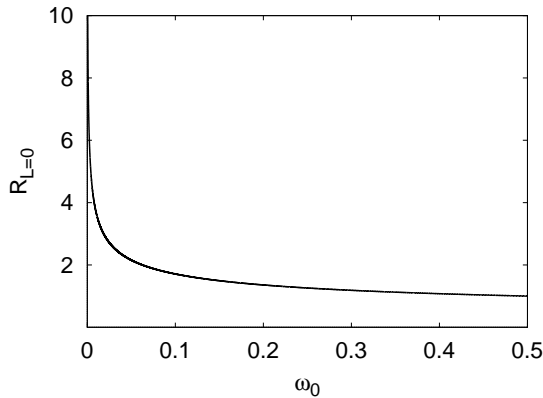


Fig. 4 The $(2\omega_0)^{-1/3}$ curve, where $L^2 = 0$. Stable circular orbits exist for radii lying above this curve.

$d^2V_{eff}/dR^2 = 0$ gives an implicit relation $F(R, \omega_0) = 0$, numerically solved as $R = R(\omega_0)$. This marginally stable orbit is the ISCO, the radius of which is represented as function of the parameter ω_0 on Fig 3.

4.2 Naked singularity parameter range ($\omega_0 < 0.5$)

The numerical study shows that stable circular orbits can exist for any non-zero angular momentum, as long as the inequality (18) holds. This means that the set of the allowed stable circular orbit radii have an infimum (a value which can be approached arbitrarily close, but cannot be reached exactly) at $(2\omega_0)^{-1/3}$. The location of the infimum as function of ω_0 is shown on Fig 4.

We note here that the ISCO radius represented on Fig. 4 of Ref. 1 seems to run below our curve, in the forbidden range $L^2 < 0$.

5 Conclusions

We investigated the Kehagias-Sfetsos asymptotically flat and spherically symmetric solution of Hořava-Lifshitz gravity both in the weak and strong-field regimes. This solution reduces to the Schwarzschild solution of general relativity in the $\omega_0 \rightarrow \infty$ limit of the Hořava-Lifshitz parameter ω_0 . For $\omega_0 \geq 0.5$ it represents a black hole, while for $0 < \omega_0 < 0.5$ it is a naked singularity.

We analyzed the weak-field regime (characterized by the smallness of the post-Newtonian parameter $\varepsilon = m/r$) for generic ω_0 , finding that gravity is weaker in the Kehagias-Sfetsos space-time than in the Schwarzschild space-time. Then we have shown that in the strong-field regime (close to the central singularity, where $\varepsilon = \mathcal{O}(1)$) of Kehagias-Sfetsos naked singularities with $\omega_0 \ll 1$ gravity surprisingly decreases as $r^{1/2}$ while approaching the center.

We also studied the timelike geodesics in the black hole and naked singularity regimes. In the black hole parameter range the ISCO radius as function of ω_0 has been studied in Ref. 7. We have extended the discussion into the naked singularity parameter range, finding that stable circular orbits always exist if the angular momentum is non-vanishing, implying that only an infimum of the stable circular orbit radii can be defined whenever $\omega_0 < 0.5$.

Based on these results, in a subsequent work we will investigate the energy flux of the accretion disk in the naked singularity case.

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