

Oscillator Models of the Solar Cycle and the Waldmeier Effect

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Received 30 May 2005, accepted 11 Nov 2005

Published online later

Key words Sun: magnetic fields – sunspots

We study the behaviour of the van der Pol oscillator when either its damping parameter μ or its nonlinearity parameter ξ is subject to additive or multiplicative random noise. Assuming various power law exponents for the relation between the oscillating variable and the sunspot number, for each case we map the parameter plane defined by the amplitude and the correlation time of the perturbation and mark the parameter regime where the sunspot number displays solar-like behaviour. Solar-like behaviour is defined here as a good correlation between the rise rate and cycle amplitude *and* the lack of a good correlation between the decay rate and amplitude, together with significant ($\gtrsim 10\%$) r.m.s. variation in cycle lengths and cycle amplitudes. It is found that perturbing μ alone the perturbed van der Pol oscillator does not show solar-like behaviour. When the perturbed variable is ξ , solar-like behaviour is displayed for perturbations with a correlation time of about 3–4 years and significant amplitude. Such studies may provide useful constraints on solar dynamo models and their parameters.

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1 Introduction

A complete spatial truncation of the nonlinear $\alpha\Omega$ dynamo equations with a dimensional analysis of some of the terms is known to give rise to a nonlinear oscillator equation of the form

$$\ddot{B} = -\omega^2 B - \mu(\xi B^2 - 1)\dot{B} - \gamma B^3 \quad (1)$$

where B is the amplitude of the toroidal magnetic field, and the parameters μ , ξ and γ may be expressed by the dynamo parameters (dynamo number, meridional flow amplitude, nonlinearity parameters) (Mininni, Gómez, & Mindlin 2001; Passos & Lopes 2011). This is a combination of the van der Pol and Duffing oscillators, the two most widely studied nonlinear oscillator problems.

In the past decade, the possibility of representing the sunspot number series with such a nonlinear oscillator model was explored in a number of papers (e.g. Mininni, Gómez, & Mindlin 2000; Pontieri et al. 2003; Lopes & Passos 2009; Passos 2012). These studies demonstrated that with a suitable choice of parameters, the overall phase space structure of the sunspot number series can be well reproduced and cycle-to-cycle variations may also be qualitatively well modeled by admitting a stochastic perturbation to one or more of the model parameters.

A more quantitative study of this problem should, however, also examine whether the behaviour of a stochastically perturbed oscillator of the form (1) adheres to the known regularities in the cycle-to-cycle variation of solar activity. The most important such regularity is the Waldmeier effect. In its currently adopted formulation the effect consists in a good correlation (coefficient $r \simeq 0.85$) between

the rise rate of a cycle and its maximum amplitude (Lantos 2000; Cameron & Schüssler 2008). On the other hand the *lack* of a statistically significant correlation between the decay rate and the cycle amplitude forms an equally important quantitative constraint. Taking into account the length of the sunspot number series this implies that $|r| \lesssim 0.5$ for any such correlation.

In order to study this problem we have started a systematic investigation of the parameter space of stochastically perturbed nonlinear oscillators of the type (1). As a first step, here we consider a pure van der Pol oscillator, neglecting the last term in equation (1).

In the next section we present our stochastically perturbed van der Pol oscillator model in detail while results of the Monte Carlo simulations are presented and discussed in Section 3. Section 4 concludes the paper.

2 Objectives and Method

2.1 Objectives

The equation of the van der Pol oscillator used in the present study is

$$\ddot{x} = -\omega^2 x - \mu(\xi x^2 - 1)\dot{x}. \quad (2)$$

The parameters in the equation are: frequency ω , dumping parameter μ , and nonlinearity ξ . Of these, ω and ξ determine the cycle length and amplitude, respectively, while μ determines the degree of asymmetry in cycle profiles.

We assume that the oscillator variable x is proportional to the amplitude of the toroidal magnetic field and in turn it is related to the sunspot number as a power law $SSN = x^n$.

arXiv:1404.3668v1 [astro-ph.SR] 14 Apr 2014

For n the values 1, 1.5 and 2, often used in the literature (e.g. Bracewell 1988; Mininni, Gómez, & Mindlin 2000), were used in various cases.

Mininni, Gómez, & Mindlin (2000) fitted this oscillator to the entire sunspot number series registered between 1816 and 2000, assuming $n = 2$. In our simulations we arbitrarily chose these fits as the unperturbed values of the parameters: $\omega_0 = 0.2993$, $\mu_0 = 0.2044$, $\xi_0 = 0.0102$. (Note that this choice is clearly arbitrary for $n \neq 2$ and, due to our definition of ξ being different from that of Mininni et al., even for $n = 2$. However, different choices would only result in different cycle lengths and amplitudes which may be accounted for by proper normalization.)

Previous authors have found that adding stochastic perturbations to the oscillator parameters, the phase space structure of the solar activity cycle can be reproduced qualitatively well. However, it has not been studied whether and under what conditions such stochastically perturbed oscillator models give rise to cycle variations that compare favorably with the variations in the observed solar cycle also in a quantitative manner. The single most important observed quantitative constraint on solar cycle variations is the Waldmeier rule; in its current formulation (Cameron & Schüssler 2008) this consists in a strong ($r \gtrsim 0.8$) positive correlation between the rise rate of solar cycles and their maximal amplitudes. However, an equally important constraint is the *lack* of a statistically significant correlation between the decay rate of cycles and their amplitudes; taking into account the number of cycles on which this conclusion is based, this implies that any such correlation should have $|r| \lesssim 0.5$. Finally, we know that there is a significant (~ 20 – 30%) cycle to cycle variation in both cycle amplitudes and cycle lengths.

In summary, for a perturbed oscillator to qualify as solar-like, it is required to adhere to the following criteria. (1) $r_{\text{rise}} > 0.8$; (2) $|r_{\text{decay}}| < 0.5$; (3) r.m.s. variation of cycle amplitudes (rms_A) and periods (rms_T) should exceed 10%

2.2 Method

For a systematic study of the effect of various types of perturbations of the oscillator parameters on the resulting sunspot number series, a large number of Monte Carlo simulations were performed. Each of these simulations consisted in a numerical integration of equation (2) with random noise acting on one of its parameters.

For stochastic fluctuations acting on model parameters two cases were considered. One was additive noise:

$$\mu(t) = \mu_0 + d(t) \quad \text{or} \quad \xi(t) = \xi_0 + d(t) \quad (3)$$

and the other was multiplicative noise:

$$\mu(t) = \mu_0 e^{d(t)} \quad \text{or} \quad \xi(t) = \xi_0 e^{d(t)}. \quad (4)$$

Here the noise $d(t)$ was modeled as a Gaussian random process. For the time dependence of $d(t)$ two different approaches were applied again. In one case, $d(t)$ is a piecewise

constant function keeping a constant value for a time T_{corr} , then taking another random value R . In the other case, in order to produce a more realistic, continuously varying noise, $d(t)$ was modelled as an Ornstein–Uhlenbeck process (Gillespie 1996, Longtin 2000) according to

$$d(t) = d(t - dt) - Kd(t - dt)dt + \Delta R\sqrt{dt} \quad (5)$$

In this case the inverse of the relaxation constant K plays the role of a time constant, taking the place of T_{corr} . In both cases, the random value R is taken out of a Gaussian distribution of half width 1 and the amplitude of the perturbation is set by the parameter Δ .

The effects of time dependence in μ and ξ were considered separately, i.e. the equation used was either

$$\ddot{x} = -\omega_0^2 x - \mu(t) (\xi_0 x^2 - 1) \dot{x}, \quad (6)$$

or

$$\ddot{x} = -\omega_0^2 x - \mu_0 [\xi(t)x^2 - 1] \dot{x}. \quad (7)$$

For both cases, four combinations of perturbation types are possible: additive or multiplicative and piecewise constant or continuous. Combined with the 3 possible values of n mentioned above, this gives rise to 24 model families. In each family the parameter plane defined by the amplitude and timescale of the perturbation (Δ – T_{corr} plane or Δ – K plane) was covered by a grid of models (order of a hundred models in each family). The part of the parameter plane covered corresponds to correlation times of 1 to 20 years and perturbation amplitudes from 0.03 to 0.67. Occasionally, when the randomly varying perturbation leads to excessive values of x , the simulations would become unstable. To avoid this, further restrictions on the values of the fluctuation parameters were introduced. In the additive case, the value of the perturbed parameter was not allowed to exceed twice its unperturbed value, while in the multiplicative case it was restricted to be within one order of magnitude from the unperturbed value. In order to obtain statistically meaningful results, in each of the cases considered the oscillator equation was integrated for 2000 years.

3 Results and discussion

Having generated the model grids as described above, adherence to the criteria set of solar-like behaviour (see end of Sect. 2.1) was checked for each model. The following results were obtained.

3.1 The perturbed parameter is μ

If the perturbed parameter is μ , the model could only approximate the expected behaviour. The van der Pol oscillator (Eq. 6) reproduced the Waldmeier effect with several input parameter pairs (Δ – K and Δ – T_{corr}); however, none of the models did simultaneously satisfy all the criteria of solar-like behaviour.

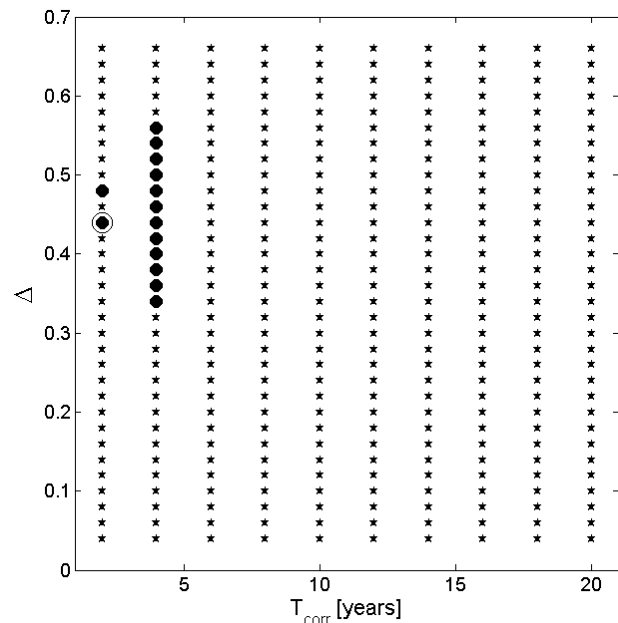


Fig. 1 Our model grid (small asterisks) on the parameter plane in the case of a piecewise constant perturbation applied multiplicatively to ξ , with $n = 2$. Larger dots (●) indicate the runs displaying solar-like behaviour as defined in the text (Sect. 2.1). The encircled “representative” model has $\Delta = 0.44$, $T_{corr} = 2$ years.

3.2 The perturbed parameter is ξ

If the time dependent parameter is ξ , the model (Eq. 7) shows all the expected features, detailed in Section 2.1, in several cases. Optimal results were found in the case of multiplicative perturbations. The form of the time dependence of the perturbation (piecewise constant or continuous) made no major difference to the results. As an example, in Figure 1 we present the map of the parameter plane for the case of a piecewise constant, multiplicative perturbation with $n = 2$.

The Waldmeier effect is displayed by all the models in the grid presented in Figure 1. However, all criteria of solar-like behaviour are only satisfied in a minority of cases, with correlation times of ~ 2 –4 years and an appropriately chosen perturbation amplitudes ($\sim \pm 50\%$). The cycles displayed by one such model are presented in Figure 2 while Table 1 presents the attributes of the model compared to the observed characteristics of the solar cycle. It is apparent from the figure that episodes of sustained positive fluctuations in ξ , for example after $t = 1100$, lead to solar cycles of a reduced amplitude. Similarly, stronger cycles are caused by episodes of sustained negative fluctuations in ξ values, for instance in the period between $t = 1051$ and 1100.

We note that the observed sunspot number series seems to display sustained periods of low or high activity lasting for up to three to six cycles. This seems to indicate a memory effect extending over several solar cycles; however, evidence for this is still controversial, so it was not included

among the criteria for solar-like behaviour. In any case, such behaviour is not seen in our models.

The occasional occurrence of extended grand minima like the Maunder minimum is also an empirical characteristic of solar activity. Again, this was not included among our criteria as reproducing grand minima was beyond the scope of the present study which focuses on reproducing the “normal” state of solar activity, outside grand minima, as evidenced in the official sunspot number series. Nevertheless, a little experimentation showed that grand minima may be obtained in our models with a time dependent ξ if the restrictions on the values of perturbed parameter were relaxed and $\xi(t)$ was allowed to reach higher values ($\xi(t) < 50\xi_0$). However, with this new limit, the model either became unstable at some point or, in addition to the grand minima, extremely long “supercycles” of ~ 100 years also appeared.

The amplitude and cycle length distributions of the models were also studied by performing a run for a 11000 year long interval for one representative case in each model family. For models that were stable and lacked the extremely long supercycles mentioned above, the histogram of cycle amplitudes was close to Gaussian, without an excess at low amplitudes. (Such an excess is the hallmark of grand minima as a separate state of activity, cf. Usoskin, Solanki & Kovaltsov 2007.)

The histogram of cycle lengths shows a maximum at ~ 11 years, the total range going from ~ 8 years up to ~ 20 years. Such unusually long periods occur once or twice during the 11000 year long time interval.

4 Conclusion

The present study is only the first step in a more extensive systematic investigation of the parameter space of the stochastically perturbed nonlinear oscillators potentially representing the sunspot number series.

Here we focused on the behaviour of stochastically perturbed van der Pol oscillator when only one of its parameters (either the damping μ or the nonlinearity ξ) is varied. We focused on reproducing the Waldmeier effect and other known features of the solar cycle. It was found that in certain parameter ranges the stochastically perturbed van der Pol oscillator does display all the expected characteristics simultaneously but only when the perturbed oscillator parameter is ξ .

Another interesting conclusion from these studies is that a good correlation between cycle amplitude and rise rate (Waldmeier effect) is displayed by a fairly wide class of nonlinear oscillators; at the same time the *lack* of a similar good correlation between cycle amplitudes and decay rates is a potentially even more stringent condition for solar-like behaviour that is in fact often harder to reproduce than the Waldmeier effect itself.

Our finding that random perturbations of μ alone do not lead to solar-like behaviour is of some interest as in at least one earlier study (Passos & Lopes 2008) cycle variations in

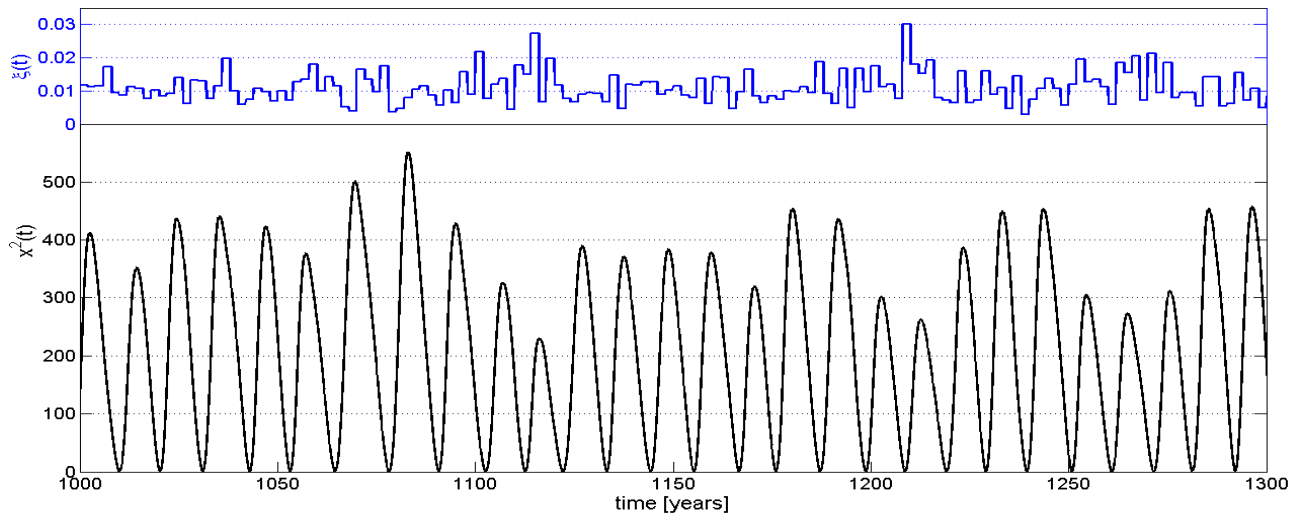


Fig. 2 A 300 year long segment of the “representative” model (encircled in Fig. 1) as an example for the behaviour of a stochastically perturbed van der Pol oscillator displaying solar-like behaviour. Shown are the temporal variations of the oscillator parameter ξ (top), and of the sunspot number (bottom), assumed to scale with $x^2(t)$ in this case. The properties of the model are summarized in Table 1.

Table 1 Comparison of the observed attributes of the solar cycle and of the van der Pol oscillator model, shown in Figure 2. The attributes were obtained by analyzing the entire 2000 year long $x^2(t)$ function of the oscillator.

	r_{rise}	r_{decay}	Rise time (year)	Period (year)	rms_T (%)	rms_A (%)
Solar cycle	0.85	0.35	3.5	11.02	11	30
van der Pol	0.84	-0.49	4.2	10.92	10	20

the observed SSN series were interpreted in terms of variations in the speed of meridional circulation which, in the truncated flux transport dynamo considered there, mainly influence the oscillator through the damping parameter. This underlines that while fitting the observed cycles with a non-linear oscillator with varying parameters results in parameter variations that may seem to be random at first sight, the random nature of such variations is actually not demonstrated and a systematic study of an ensemble of oscillators with truly random parameter variations is needed to confirm or discard solar-like behaviour.

It should be stressed, however, that this conclusion is still of a preliminary nature and it may change if e.g. the perturbations in μ and ξ are assumed to be interrelated, as expected for a flux transport dynamo.

Possibilities for future extensions of this study therefore include a joint perturbation of μ and ξ and the inclusion of a (possibly also perturbed) third order term in the oscillator equation, as in equation (1).

Acknowledgements. This research was supported by the Hungarian Science Research Fund (OTKA grants no. K83133 and 81421).

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